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SPACES NOT DISTINGUISHING IDEAL CONVERGENCES OF REAL-VALUED FUNCTIONS, II

Abstract

In [13] we gave combinatorial characterizations of non(P) for various properties P of spaces expressing non-distinguishability of some ideal convergences and semi-convergences of sequences of continuous functions. In the present paper we study three of these invariants: non((I, JQN)-space), non(($I, \leq_{\mathrm{K}} J$QN)-space), and non(w(I, JQN)-space). We study them in connection with partial orderings of ${}^{\omega}\omega$ restricted to relations between I-to-one functions and J-to-one functions. In particular we prove that non(w(I, JQN)-space) $\leq \mathfrak{b}$ for every capacitous ideal J on ω. This generalizes the same result of Kwela for ideals J contained in an F_σ-ideal. If J is a capacitous P-ideal, then non((I, JQN)-space) $=$ non(($I, \leq_{\mathrm{K}} J$QN)-space) $= \mathfrak{b}$ for every ideal $I \subseteq J$ and non(w(I, JQN)-space) $= \mathfrak{b}$ for every ideal I below J in the Katětov partial quasi-ordering of ideals.

Introduction

For an ideal I on ω, a sequence of reals $\xi = \langle \xi_n : n \in \omega \rangle$ is said to I-converge to 0, we write $\xi \xrightarrow{I} 0$, if for every $\varepsilon > 0$, $\{n \in \omega : |\xi_n| \geq \varepsilon\} \in I$ (we will not use a convergence to a non-zero value).

For ideals I, J, K on ω and for a sequence of real-valued functions $f = \langle f_n : n \in \omega \rangle$ defined on a set X we consider the following ideal convergences:

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$f \xrightarrow{I} 0$, if for every $x \in X$, $\langle f_n(x) : n \in \omega \rangle \xrightarrow{I} 0$ (the I-pointwise convergence).

$f \xrightarrow{JK\text{QN}} 0$, if there exists an $\varepsilon \in {}^{\omega}[0,\infty)$ such that $\varepsilon \xrightarrow{K} 0$ and for every $x \in X$, $\{n \in \omega : |f_n(x)| \geq \varepsilon_n\} \in J$ (the JK-quasi-normal convergence).

$f \xrightarrow{J\text{QN}} 0$, if $f \xrightarrow{JJ\text{QN}} 0$; i.e., if there exists an $\varepsilon \in {}^{\omega}[0,\infty)$ such that $\varepsilon \xrightarrow{J} 0$ and for every $x \in X$, $\{n \in \omega : |f_n(x)| \geq \varepsilon_n\} \in J$ (the J-quasi-normal convergence).

The I-convergence and the JQN-convergence are the same as in [4, 6]. The JKQN-convergence was introduced in [9] under the name "the (J,K)-equal convergence". This two-ideal convergence was introduced because different authors defined the "J-equal convergence" meaning the (J,K)-equal convergence with $K = \text{Fin}$ or with $K = J$.

Let X be a topological space and let $C(X)$ be the family of continuous real-valued functions on X. We define ([13]):

(A) X is an $(I, J\text{QN})$-space, if for every $f \in {}^{\omega}C(X)$, if $f \xrightarrow{I} 0$, then $f \xrightarrow{J\text{QN}} 0$.

(B) X is a $\text{w}(I, J\text{QN})$-space, if for every $f \in {}^{\omega}C(X)$, if $f \xrightarrow{I} 0$, then there is $\varphi \in {}^{\omega}\omega$ such that $\langle f_{\varphi(n)} : n \in \omega \rangle \xrightarrow{J\text{QN}} 0$.

These properties are generalizations of the notions of a QN-space and a wQN-space introduced in [5]: a QN-space means a $(\text{Fin}, \text{Fin}\text{QN})$-space and a wQN-space means a $\text{w}(\text{Fin}, \text{Fin}\text{QN})$-space, where Fin denotes the ideal of finite sets. The definitions of an $(I, J\text{QN})$-space and a $\text{w}(I, J\text{QN})$-space coincide with the definitions of an (I,J)QN-space and an (I,J)wQN-space in [4], respectively. Some authors (see [6, 10, 16]) use a similar definition to (B) with the requirement that φ is strictly increasing.

We use (A) and (B) as general schemes which can be used for arbitrary pair of convergences. For example:

– X is an $(I, JK\text{QN})$-space, if for every $f \in {}^{\omega}C(X)$, if $f \xrightarrow{I} 0$, then $f \xrightarrow{JK\text{QN}} 0$ (hence an $(I, J\text{QN})$-space is an $(I, JJ\text{QN})$-space).

– X is a $\text{w}(I, JK\text{QN})$-space, if for every $f \in {}^{\omega}C(X)$, if $f \xrightarrow{I} 0$, then there is $\varphi \in {}^{\omega}\omega$ such that $\langle f_{\varphi(n)} : n \in \omega \rangle \xrightarrow{JK\text{QN}} 0$ (hence a $\text{w}(I, J\text{QN})$-space is a $\text{w}(I, JJ\text{QN})$-space).

– X is a $\text{w}(I,J)$-space, if for every $f \in {}^{\omega}C(X)$, if $f \xrightarrow{I} 0$, then there is $\varphi \in {}^{\omega}\omega$ such that $\langle f_{\varphi(n)} : n \in \omega \rangle \xrightarrow{J} 0$.

The question whether the property of a $\text{w}(I, J\text{QN})$-space is the right generalization of a wQN-space led us in [13] to this definition:

(C) X is an $(I, \leq_{\mathrm{K}} J\mathrm{QN})$-space, if for every $f \in {}^{\omega}C(X)$, if $f \xrightarrow{I} 0$, then there exists an ideal $J' \leq_{\mathrm{K}} J$ such that $f \xrightarrow{J'\mathrm{QN}} 0$.

Recall that $\leq_{\mathrm{K}}$ is the Katětov partial quasi-ordering of ideals defined by $J' \leq_{\mathrm{K}} J$, if $(\exists \varphi \in {}^{\omega}\omega)(\forall a \in J')\ \varphi^{-1}(a) \in J$. For the non-tall ideals J, an $(I, \leq_{\mathrm{K}} J\mathrm{QN})$-space means the same as a $\mathrm{w}(I, J\mathrm{QN})$-space (see [13, Corollary 4.5 (a2) and (c1)]). For $f \in {}^{\omega}({}^{X}\mathbb{R})$ we define the following two semi-convergences:

$$f \xrightarrow{\leq_{\mathrm{K}} J\mathrm{QN}} 0 \Leftrightarrow (\exists J' \leq_{\mathrm{K}} J)\ f \xrightarrow{J'\mathrm{QN}} 0, \qquad f \xrightarrow{\leq_{\mathrm{K}} J} 0 \Leftrightarrow (\exists J' \leq_{\mathrm{K}} J)\ f \xrightarrow{J'} 0.$$

Definition (C) can be obtained from the scheme (A) by substituting the semi-convergence $f \xrightarrow{\leq_{\mathrm{K}} J\mathrm{QN}} 0$ for $f \xrightarrow{J\mathrm{QN}} 0$. In [13] we classified 2×16 modifications of (A) and (B) for all pairs of semi-convergences $\xrightarrow{J}$, $\xrightarrow{J\mathrm{QN}}$, $\xrightarrow{\leq_{\mathrm{K}} J}$, $\xrightarrow{\leq_{\mathrm{K}} J\mathrm{QN}}$. Each of these properties is equivalent to one of the following 9 properties:

$$\begin{array}{ccccccc}
(\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN}) & \Rightarrow & (I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN}) & \Rightarrow & \mathrm{w}(I\mathrm{QN}, J\mathrm{QN}) & \Rightarrow & \mathrm{w}(I\mathrm{QN}, J) \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN}) & \Rightarrow & (I, \leq_{\mathrm{K}} J\mathrm{QN}) & \Rightarrow & \mathrm{w}(I, J\mathrm{QN}) & \Rightarrow & \mathrm{w}(I, J) \\
 & & \Uparrow & & & & \\
 & & (I, J\mathrm{QN}) & & & &
\end{array}$$

Recall that $\mathrm{non}(P)$ denotes the minimal cardinality of a space not having the property P. The cardinal invariants $\mathrm{non}(P)$ for the above properties P are reduced to 5 cardinals because $\mathrm{non}(P) = \mathrm{non}(\mathrm{w}(I, J)\text{-space}) = \mathfrak{k}_{I,J}$ for P in the top row (see [13, Theorem 3.11]), where

$$\mathfrak{k}_{I,J} = \min\{|X| : X \subseteq I \text{ and } (\forall \varphi \in {}^{\omega}\omega)(\exists a \in X)\ \varphi^{-1}(a) \notin J\}, \quad \text{if } I \not\leq_{\mathrm{K}} J,$$

and $\mathfrak{k}_{I,J} = \infty$, if $I \leq_{\mathrm{K}} J$. By the convention, $\mathrm{non}(P) = \infty$ means that every space has the property P. For every property P in the diagram, if $\mathrm{non}(P) \neq \infty$, the value of $\mathrm{non}(P)$ is witnessed by a set of reals not having the property P.

There is no complete characterization of pairs of ideals for which the above properties are different. There are only some isolated examples: There is an ideal I such that every space is a $\mathrm{w}(I, I\mathrm{QN})$-space but there is a space that is not an $(I, I\mathrm{QN})$-space (a consequence of [13, Theorem 3.4 (a), (b) and Example 3.3 (2b)]). Assuming $\mathfrak{p} = \mathfrak{c}$, Šupina ([16, Theorem 1.5]) proved that there is an ideal I such that not every space is a $(\mathrm{Fin}, I\mathrm{QN})$-space and there is a $(\mathrm{Fin}, I\mathrm{QN})$-space which is not a QN-space. Kwela ([10, Theorem 2.11]) proved that consistently there is an ideal I such that not every space is a w(Fin, IQN)-space and there is a w(Fin, IQN)-space which is not a wQN-space. The ideal I in the last two results must be a weak P-ideal (Definition 1.2).

In the present paper we continue investigation of the cardinal invariants

$$\text{non}((I, J\text{QN})\text{-space}),\ \ \text{non}((I, \leq_{\text{K}} J\text{QN})\text{-space}),\ \ \text{non}(\text{w}(I, J\text{QN})\text{-space}).\quad (*)$$

In Section 2, we express the cardinals $(*)$ as bounding numbers of relations connected with restrictions of the partial ordering $\leq_J$ of ${}^\omega\omega$ defined by $f \leq_J g$ if $\|f < g\| \in J$ for $f, g \in {}^\omega\omega$ where $\|f < g\| = \{k \in \omega : f(k) < g(k)\}$. In particular, $\leq_J$ is considered between the set $F(I)$ of I-to-one functions and the set $F(J)$ of J-to-one functions. The inequalities between the cardinals are expressed by morphisms between the relations. The bounding number of $(F(I), \leq_J)$ is a natural lower estimation of the cardinals $(*)$ provided that $I \subseteq J$ and we show that the cardinals $(*)$ have upper bounds in the form of dominating numbers of the partially ordered sets $(F(I), \leq_J)$ and $({}^\omega I, \leq^1_{\text{Fin}})$, where $f \leq^1_{\text{Fin}} g$ means that $f(n) \subseteq g(n)$ for all but finitely many $n \in \omega$ (see Theorem 2.12).

Kwela [10] proved that non(w(I, JQN)-space) $\leq \mathfrak{b}$ for every ideal J on ω that is contained in an F_σ-ideal. His proof is based on the fact that every F_σ-ideal is determined by a lower semi-continuous submeasure on ω. We were successful to find a similar proof for analytic P-ideals. The question whether it is possible to unify these two results in a single one led us to the notion of a capacitous ideal in Section 3. Main results of this section state that every F_σ-ideal and every analytic P-ideal is capacitous, the property "to be a capacitous ideal" is hereditary with respect to Katětov partial quasi-ordering $\leq_{\text{K}}$, and non(w(I, JQN)-space) $\leq \mathfrak{b}$ if J is capacitous (Theorem 3.5). As a consequence we prove that if J is a capacitous P-ideal, then non((I, JQN)-space) = non(($I, \leq_{\text{K}} J$QN)-space) = non(w(I, JQN)-space) = $\mathfrak{b}$ for every ideal $I \subseteq J$ and non(w(I, JQN)-space) = $\mathfrak{b}$ for every ideal $I \leq_{\text{K}} J$.

In Section 4, we present lower and upper estimations of invariants $(*)$ by bounding and dominating numbers of partial orders as general as possible. The lower estimations are uncountable cardinals.

1 Notation and terminology

We use the same notation as in [13]. By an ideal on a set S we mean any collection $I \subseteq \mathcal{P}(S)$ such that I contains all finite subsets of S, $S \notin I$, and I is closed under finite unions and subsets of its elements. If I is an ideal on S, then $I^+ = \mathcal{P}(S) \setminus I$ is a family of all I-positive sets and $I^* = \{S \setminus a : a \in I\}$ is the dual filter to the ideal I. If I and J are ideals on S and $a \subseteq S$, then by $I \vee J$ and $I \vee \langle a \rangle$ we denote the smallest ideals on S containing $I \cup J$ and $I \cup \{a\}$, respectively, if such ideals exist. Let $\mathrm{Fin} = [\omega]^{<\omega}$.

For an ideal I on ω and $\varphi \in {}^{\omega}\omega$ we define

$$\varphi^{\rightarrow}(I) = \{a \subseteq \omega : \varphi^{-1}(a) \in I\},$$
$$\varphi^{\leftarrow}(I) = \{a \subseteq \omega : \varphi(a) \in I\},$$
$$F(I) = \{\alpha \in {}^{\omega}\omega : (\forall n \in \omega)\ \alpha^{-1}(\{n\}) \in I\}.$$

Clearly, $\varphi^{\rightarrow}(I)$ is an ideal on ω if and only if $\mathrm{Fin} \subseteq \varphi^{\rightarrow}(I)$ if and only if $\varphi \in F(I)$; and $\varphi^{\leftarrow}(I)$ is an ideal on ω if and only if $\omega \notin \varphi^{\leftarrow}(I)$ if and only if $\mathrm{rng}(\varphi) \in I^{+}$. The meaning of $\alpha \in F(I)$ is $\alpha \xrightarrow{I} \infty$.

Let us recall Rudin-Keisler ($\leq_{\mathrm{RK}}$), Rudin-Blass ($\leq_{\mathrm{RB}}$), Katětov ($\leq_{\mathrm{K}}$), and Katětov-Blass ($\leq_{\mathrm{KB}}$) partial quasi-orderings of ideals I and J on ω:

$$I \leq_{\mathrm{RK}} J \Leftrightarrow (\exists \varphi \in F(J))\ I = \varphi^{\rightarrow}(J),$$
$$I \leq_{\mathrm{RB}} J \Leftrightarrow (\exists \varphi \in F(\mathrm{Fin}))\ I = \varphi^{\rightarrow}(J),$$
$$I \leq_{\mathrm{K}} J \Leftrightarrow (\exists \varphi \in F(J))\ I \subseteq \varphi^{\rightarrow}(J),$$
$$I \leq_{\mathrm{KB}} J \Leftrightarrow (\exists \varphi \in F(\mathrm{Fin}))\ I \subseteq \varphi^{\rightarrow}(J).$$

For $\alpha, \varphi \in {}^{\omega}\omega$, like in [13], the composition $\varphi \circ \alpha \in {}^{\omega}\omega$ is defined by

$$(\varphi \circ \alpha)(k) = \alpha(\varphi(k)), \quad k \in \omega.$$

Then $(\varphi \circ \alpha)^{-1}(a) = \varphi^{-1}(\alpha^{-1}(a))$ for $a \subseteq \omega$.

If ψ is a formula with parameters $\alpha, \beta, \dots \in {}^{\omega}\omega$, then we denote

$$\|\psi\| = \{k \in \omega : \psi(\alpha(k), \beta(k), \dots)\}.$$

In particular for $\alpha, \beta, \varphi \in {}^{\omega}\omega$ and $n \in \omega$, $\|\beta < \alpha\| = \{k \in \omega : \beta(k) < \alpha(k)\}$, $\|\alpha = n\| = \alpha^{-1}(\{n\})$, $\|\varphi \circ \beta < \varphi \circ \alpha\| = \{k \in \omega : \beta(\varphi(k)) < \alpha(\varphi(k))\} = \varphi^{-1}(\|\beta < \alpha\|)$. Consequently, $\varphi \circ \alpha \in F(I) \Leftrightarrow \alpha \in F(\varphi^{\rightarrow}(I))$ and $\|\varphi \circ \beta < \varphi \circ \alpha\| \in I \Leftrightarrow \|\beta < \alpha\| \in \varphi^{\rightarrow}(I)$.

Lemma 1.1. *If $\alpha \in F(I)$, $\varphi, \beta \in {}^{\omega}\omega$, and $\|\varphi \circ \beta < \alpha\| \in I$, then $\varphi \in F(I)$.*

PROOF. Let $\psi = \varphi \circ \beta$. Then $\psi^{-1}(\{n\}) = \|\psi = n\| \cap (\|\alpha \leq n\| \cup \|n < \alpha\|) \subseteq \|\alpha \leq n\| \cup \|\psi < \alpha\| \in I$. Hence $\psi \in F(I)$ and $\psi \in F(I)$ implies $\varphi \in F(I)$. □

An ideal J is said to be a $P(I)$-ideal for an ideal I, if for every partition $\{a_n : n \in \omega\} \subseteq J$ of ω there exists $c \in J^{*}$ such that $a_n \cap c \in I$ for all $n \in \omega$.

A partitions $\{a_n : n \in \omega\}$ of ω can be expressed by the function $\alpha \in {}^{\omega}\omega$ defined by $\alpha(k) = n$ for $k \in a_n$ and $n \in \omega$. Conversely, every function $\alpha \in {}^{\omega}\omega$

determines a partition $\{a_n : n \in \omega\}$ of ω by $a_n = \|\alpha = n\|$ for $n \in \omega$. Moreover, $\{a_n : n \in \omega\} \subseteq J$ if and only if $\alpha \in F(J)$.

We prefer using functions instead of partitions. In particular, an ideal J is a $P(I)$-ideal if and only if $(\forall \alpha \in F(J))(\exists c \in J^*)(\forall n \in \omega)\ \|\alpha = n\| \cap c \in I$.

Definition 1.2. Let I, J, K be ideals on ω.

J is a weak P-ideal, if $(\forall \alpha \in F(J))(\exists c \in J^+)(\forall n \in \omega)\ \|\alpha = n\| \cap c \in \mathrm{Fin}$.

J is a weak $P(I)$-ideal, if $(\forall \alpha \in F(J))(\exists c \in J^+)(\forall n \in \omega)\ \|\alpha = n\| \cap c \in I$.

J is a $W(I)$-ideal, if $(\forall \alpha \in F(J))(\forall \varphi \in F(J))(\exists c \in J^+)(\forall n \in \omega)$
$\varphi(\|\alpha = n\| \cap c) \in I$.

K is a weak $P(I, J)$-ideal if $(\forall \alpha \in F(K))(\exists c \in J^+)(\forall n \in \omega)\ \|\alpha = n\| \cap c \in I$.

K is a $W(I, J)$-ideal if $(\forall \alpha \in F(K))(\forall \varphi \in F(J))(\exists c \in J^+)(\forall n \in \omega)$
$\varphi(\|\alpha = n\| \cap c) \in I$.

Hence, J is a weak P-ideal $\Leftrightarrow$ J is a weak $P(\mathrm{Fin})$-ideal; J is a weak $P(I)$-ideal $\Leftrightarrow$ J is a weak $P(I, J)$-ideal; J is a $W(I)$-ideal $\Leftrightarrow$ J is a $W(I, J)$-ideal.

The notion of a $P(I)$-ideal was introduced in [9], the notion of a weak $P(I)$-ideal was introduced in [16] as a generalization of the dual property to a weak P-filter from [11]. The notions of a $W(I)$-ideal, a weak $P(I, J)$-ideal, a $W(I, J)$-ideal were introduced in [13]. The notion of a $W(I, J)$-ideal is related to the property $W(I, J, K)$ in [15] (if $K \subseteq J$, then $W(J, K, I) \Leftrightarrow K$ is a $P(I, J)$-ideal, see [13]).

Lemma 1.3. *An ideal J is a $W(\mathrm{Fin})$-ideal if and only if J is a weak P-ideal.*

Proof. Every $W(I)$-ideal is a weak $P(I)$-ideal. Conversely, a weak P-ideal is a $W(\mathrm{Fin})$-ideal: If $\|\alpha = n\| \cap c \in \mathrm{Fin}$, then also $\varphi(\|\alpha = n\| \cap c) \in \mathrm{Fin}$. □

Recall that $\mathfrak{p}$ is the pseudo-intersection number, $\mathfrak{b}$ is the bounding number, $\mathfrak{d}$ is the dominating number (see [7]). If P is a property of topological spaces, then $\mathrm{non}(P)$ is the minimal cardinality of a space not having the property P.

By id_X we denote the identity on X and by $\mathrm{id}_{X,Y}$ we denote the identity function from X into Y provided that $X \subseteq Y$. We use the symbol $\mathrm{id}_{X,Y}$ as an evidence about the inclusion $X \subseteq Y$ to improve the readability of formulas.

2 Bounding and dominating cardinal numbers

We say that a class of spaces is reasonable, if it contains all separable metric spaces and it is closed under homeomorphisms.

Theorem 2.1 ([13, Theorem 3.6 and Theorem 3.11]). *For all ideals I, J, and $K \subseteq J$ on ω and for all reasonable classes of spaces the following holds:*

(a) *If K is a weak $P(I,J)$-ideal, then*

$$\operatorname{non}((I, JK\mathrm{QN})\text{-}\mathit{space}) = \min\{|X| : X \subseteq F(I) \text{ and } (\forall \alpha \in F(K))(\exists \beta \in X)\ \|\beta < \alpha\| \notin J\} \leq \mathfrak{c},$$

otherwise, every space is an $(I, JK\mathrm{QN})$-space.

(b) *If K is a $W(I,J)$-ideal, then*

$$\operatorname{non}(\mathrm{w}(I, JK\mathrm{QN})\text{-}\mathit{space}) = \min\{|X| : X \subseteq F(I) \text{ and } (\forall \alpha \in F(K))(\forall \varphi \in F(J))(\exists \beta \in X)\ \|\varphi \circ \beta < \alpha\| \notin J\} \leq \mathfrak{c},$$

otherwise, every space is a $\mathrm{w}(I, JK\mathrm{QN})$-space.

(c) *If every ideal $\leq_{\mathrm{K}}$-below J is a weak $P(I)$-ideal, then*

$$\operatorname{non}((I, \leq_{\mathrm{K}} J\mathrm{QN})\text{-}\mathit{space}) = \min\{|X| : X \subseteq F(I) \text{ and } (\forall J' \leq_{\mathrm{RK}} J \text{ on } \omega) (\forall \alpha \in F(J'))(\exists \beta \in X)\ \|\beta < \alpha\| \notin J'\} \leq \mathfrak{c},$$

otherwise, every space is an $(I, \leq_{\mathrm{K}} JK\mathrm{QN})$-space.

(d) $\operatorname{non}(\mathrm{w}(I,J)\text{-}\mathit{space}) = \mathfrak{k}_{I,J}$. □

An inequality between cardinal invariants of two binary relations is sometimes a side effect of the existence of a morphism between the relations. The advantage may be the existence of the dual morphism and an inequality for the dual invariants.

We select suitable binary relations from the equalities in Theorem 2.1 and find morphisms to express known as well as some new results for the cardinals

$$\operatorname{non}((I, J\mathrm{QN})\text{-space}),\ \operatorname{non}((I, \leq_{\mathrm{K}} J\mathrm{QN})\text{-space}),\ \operatorname{non}(\mathrm{w}(I, J\mathrm{QN})\text{-space}).$$

Following [2], a binary relation is a triple (R_-, R_+, R) with $R \subseteq R_- \times R_+$. If there is no doubt about the sets R_- and R_+ we simply write R instead of the triple (R_-, R_+, R). The dual relation is $(R_+, R_-, R^\perp)$ where $R^\perp = \neg(R^{-1})$; i.e., $x\ R^\perp\ y$ if and only if $\neg(y\ R\ x)$. A morphism $(\Phi, \Psi) : (R_-, R_+, R) \to (S_-, S_+, S)$ is a pair of functions $\Phi : S_- \to R_-$ and $\Psi : R_+ \to S_+$ such that $\Phi(x)\ R\ y \Rightarrow x\ S\ \Psi(y)$; then $(\Psi, \Phi) : (S_-, S_+, S)^\perp \to (R_-, R_+, R)^\perp$ is a (dual) morphism because $\Psi(y)\ S^\perp\ x \Rightarrow y\ R^\perp\ \Phi(x)$. The composition $(\Phi_1, \Psi_1) \circ (\Phi_2, \Psi_2)$ of morphisms $(\Phi_1, \Psi_1) : (R_-, R_+, R) \to (S_-, S_+, S)$ and $(\Phi_2, \Psi_2) : (S_-, S_+, S) \to (T_-, T_+, T)$ is the morphism $(\Phi_2 \circ \Phi_1, \Psi_1 \circ \Psi_2) : (R_-, R_+, R) \to (T_-, T_+, T)$ (recall that, e.g., $\Phi_2 \circ \Phi_1(x) = \Phi_1(\Phi_2(x))$).

We write $R \preccurlyeq S$, if there is a morphism from R to S and we write $R \approx S$, if $R \preccurlyeq S$ and $S \preccurlyeq R$. We define

$$\mathfrak{d}(R) = \min\{|D| : D \subseteq R_+ \text{ and } (\forall x \in R_-)(\exists y \in D)\ x\ R\ y\},$$

if $\mathrm{dom}(R) = R_-$, and let $\mathfrak{d}(R) = \infty$, otherwise (in [2], this invariant is denoted by $\|R\|$). Obviously $\mathfrak{d}(R) < \infty \Leftrightarrow \mathfrak{d}(R) \leq |R_+|$. Dually, let $\mathfrak{b}(R) = \mathfrak{d}(R^\perp)$. If $\mathfrak{b}(R) < \infty$, then

$$\mathfrak{b}(R) = \min\{|B| : B \subseteq R_- \text{ and } (\forall x \in R_+)(\exists y \in B)\ \neg(y\ R\ x)\}.$$

A set $D \subseteq R_+$ is R-dominating, if $(\forall x \in R_-)(\exists y \in D)\ x\ R\ y$. A set $B \subseteq R_-$ is R-unbounded, if it is $R^\perp$-dominating; i.e., $(\forall x \in R_+)(\exists y \in B)\ \neg(y\ R\ x)$.

If $R \preccurlyeq S$, then $\mathfrak{d}(S) \leq \mathfrak{d}(R)$ because, if $D \subseteq R_+$ is R-dominating, then $\{\Psi(y) : y \in D\}$ is S-dominating. In particular, if there is no S-dominating set, then there is no R-dominating set; i.e., $\mathfrak{d}(S) = \infty \Rightarrow \mathfrak{d}(R) = \infty$. Dually, since $S^\perp \preccurlyeq R^\perp$, if $B \subseteq S_-$ is S-unbounded, then $\{\Phi(y) : y \in B\}$ is R-unbounded, hence $\mathfrak{b}(R) \leq \mathfrak{b}(S)$, and in particular, $\mathfrak{b}(R) = \infty \Rightarrow \mathfrak{b}(S) = \infty$.

If R is a partial ordering $\leq$ on a set P without maximal elements, then $R \preccurlyeq R^\perp$ and therefore $\mathfrak{b}(R) \leq \mathfrak{d}(R)$ (for $x \in P$ let $\Phi(x) > x$; then $(\Phi, \mathrm{id}_P) : R \to R^\perp$ is a morphism because $\Phi(x) \leq y \Rightarrow x \ngeq y$). If Q is a cofinal subset of P, then $R \approx R{\restriction}Q$ (for $x \in P$ let $x \leq \Phi(x) \in Q$; then $(\Phi, \mathrm{id}_{Q,P}) : R{\restriction}Q \to R$ and $(\mathrm{id}_{Q,P}, \Phi) : R \to R{\restriction}Q$ are morphisms). If R is a total ordering, then $R^\perp \approx R$ and therefore $\mathfrak{b}(R) = \mathfrak{d}(R)$ ($x \ngeq y \Rightarrow x \leq y$; i.e., $(\mathrm{id}_P, \mathrm{id}_P) : R^\perp \to R$ is a morphism).

Let I, J, K be ideals on ω. Denote

$$\begin{aligned} F_2(J) &= \{(\varphi, \alpha) \in {}^\omega\omega \times {}^\omega\omega : \varphi \circ \alpha \in F(J)\} \\ &= \{(\varphi, \alpha) \in F(J) \times {}^\omega\omega : \alpha \in F(\varphi^\rightarrow(J))\} \end{aligned}$$

and consider the following binary relations:

$$\begin{aligned}
&\mathrm{Z}^K_{I,J} \subseteq F(I) \times F(K), && \beta\ \mathrm{Z}^K_{I,J}\ \alpha \Leftrightarrow \|\beta < \alpha\| \in J;\\
&\mathrm{Y}^K_{I,J} \subseteq F(I) \times (F(J) \times F(K)), && \beta\ \mathrm{Y}^K_{I,J}\ (\varphi, \alpha) \Leftrightarrow \|\varphi \circ \beta < \alpha\| \in J;\\
&\mathrm{A}_{I,J} \subseteq F(I) \times F(I), && \beta\ \mathrm{A}_{I,J}\ \alpha \Leftrightarrow \|\beta < \alpha\| \in J;\\
&\mathrm{B}_{I,J} \subseteq F(I) \times F(J), && \beta\ \mathrm{B}_{I,J}\ \alpha \Leftrightarrow \|\beta < \alpha\| \in J;\\
&\mathrm{C}_{I,J} \subseteq F(I) \times F_2(J), && \beta\ \mathrm{C}_{I,J}\ (\varphi, \alpha) \Leftrightarrow \varphi^{-1}(\|\beta < \alpha\|) \in J;\\
&\mathrm{D}_{I,J} \subseteq F(I) \times (F(J) \times F(J)), && \beta\ \mathrm{D}_{I,J}\ (\varphi, \alpha) \Leftrightarrow \|\varphi \circ \beta < \alpha\| \in J;\\
&\mathrm{E}_{I,J} \subseteq I \times {}^\omega\omega, && a\ \mathrm{E}_{I,J}\ \varphi \Leftrightarrow \varphi^{-1}(a) \in J.
\end{aligned}$$

Note that $\mathrm{A}_{I,J} = \mathrm{Z}^I_{I,J}$, $\mathrm{B}_{I,J} = \mathrm{Z}^J_{I,J}$, $\mathrm{A}_{J,J} = \mathrm{B}_{J,J} = \mathrm{Z}^J_{J,J}$, and $\mathrm{D}_{I,J} = \mathrm{Y}^J_{I,J}$. Note that $\mathfrak{k}_{I,J} = \mathfrak{b}(I, {}^\omega\omega, \mathrm{E}_{I,J})$; i.e.,

$$\mathfrak{k}_{I,J} = \min\{|X| : X \subseteq I \text{ and } (\forall \varphi \in {}^\omega\omega)\ X \setminus \varphi^{\rightarrow}(J) \neq \emptyset\}, \quad \text{if } I \not\leq_{\mathrm{K}} J,$$

and $\mathfrak{k}_{I,J} = \infty$, if $I \leq_{\mathrm{K}} J$. By [13, Lemma 3.10 (b)], $\mathfrak{b}(I, {}^\omega\omega, \mathrm{E}_{I,J}) = \mathfrak{b}(I, F(J), \mathrm{E}_{I,J})$. We prefer $(I, {}^\omega\omega, \mathrm{E}_{I,J})$ to $(I, F(J), \mathrm{E}_{I,J})$ because it is fully monotone with respect to the Katětov partial ordering of ideals (see Lemma 2.7 (d)).

The following theorem is a reformulation of Theorem 2.1 and translates the cardinal invariants of the form non(...) into cardinal invariants of binary relations $\mathrm{A}_{I,J}$, $\mathrm{B}_{I,J}$, ..., $\mathrm{Z}_{I,J}$ and back. We will use it (often without any reference) throughout the whole paper whenever there will be such connection.

Theorem 2.2. *Let I, J, K be ideals on ω and let $K \subseteq J$ (the following inequalities $\leq \mathfrak{c}$ have the same meaning as $< \infty$).*

(a) non($(I, JK\mathrm{QN})$-*space*) $= \mathfrak{b}(\mathrm{Z}^K_{I,J})$ *and* $\mathfrak{b}(\mathrm{Z}^K_{I,J}) \leq \mathfrak{c}$ *if and only if K is a weak $P(I,J)$-ideal. In particular,*

- non($(I, J\mathrm{QN})$-*space*) $= \mathfrak{b}(\mathrm{Z}^J_{I,J}) = \mathfrak{b}(\mathrm{B}_{I,J})$ *and* $\mathfrak{b}(\mathrm{B}_{I,J}) \leq \mathfrak{c}$ *if and only if J is a weak $P(I)$-ideal;*
- non($(I, JI\mathrm{QN})$-*space*) $= \mathfrak{b}(\mathrm{Z}^I_{I,J}) = \mathfrak{b}(\mathrm{A}_{I,J}) \leq \mathfrak{c}$, *if $I \subseteq J$.*

(b) non(w$(I, JK\mathrm{QN})$-*space*) $= \mathfrak{b}(\mathrm{Y}^K_{I,J})$ *and* $\mathfrak{b}(\mathrm{Y}^K_{I,J}) \leq \mathfrak{c}$ *if and only if K is a $W(I,J)$-ideal. In particular,*

- non(w$(I, J\mathrm{QN})$-*space*) $= \mathfrak{b}(\mathrm{Y}^J_{I,J}) = \mathfrak{b}(\mathrm{D}_{I,J})$ *and* $\mathfrak{b}(\mathrm{D}_{I,J}) \leq \mathfrak{c}$ *if and only if J is a $W(I)$-ideal.*

(c) non($(I, \leq_{\mathrm{K}} J\mathrm{QN})$-*space*) $= \mathfrak{b}(\mathrm{C}_{I,J})$ *and* $\mathfrak{b}(\mathrm{C}_{I,J}) \leq \mathfrak{c}$ *if and only if every ideal $\leq_{\mathrm{K}}$-below J is a weak $P(I)$-ideal.*

(d) non(w(I, J)-*space*) $= \mathfrak{b}(\mathrm{E}_{I,J}) = \mathfrak{k}_{I,J}$ *and* $\mathfrak{k}_{I,J} \leq \mathfrak{c}$ *if and only if $I \not\leq_{\mathrm{K}} J$.*

PROOF. All assertions are transcriptions of Theorem 2.1. Case (c) needs some explanation. Theorem 2.1 (c) says: If every ideal $\leq_{\mathrm{K}}$-below J is a weak $P(I)$-ideal, then

$$\begin{aligned}\mathrm{non}((I, \leq_{\mathrm{K}} J\mathrm{QN})\text{-space}) = \min\{|X| : X \subseteq F(I) \text{ and } (\forall J' \leq_{\mathrm{RK}} J \text{ on } \omega)&\\ (\forall \alpha \in F(J'))(\exists \beta \in X)\ \|\beta < \alpha\| \notin J'\} \leq \mathfrak{c},&\end{aligned}$$

otherwise, non($(I, \leq_{\mathrm{K}} J\mathrm{QN})$-space) $= \infty$. This is equal to $\mathfrak{b}(\mathrm{C}_{I,J})$ because

$$\begin{aligned}&(\exists J' \leq_{\mathrm{RK}} J)(\exists \alpha \in F(J'))(\forall \beta \in X)\ \|\beta < \alpha\| \in J'\\ &\quad \Leftrightarrow (\exists \varphi \in F(J))(\exists \alpha \in F(\varphi^{\rightarrow}(J)))(\forall \beta \in X)\ \varphi^{-1}(\|\beta < \alpha\|) \in J\\ &\quad\quad \Leftrightarrow (\exists (\varphi, \alpha) \in F_2(J))(\forall \beta \in X)\ \beta\ \mathrm{C}_{I,J}\ (\varphi, \alpha). \qquad \square\end{aligned}$$

Lemma 2.3. *Consider the functions*

$$\begin{aligned}
&\Psi_1 : F(J) \to F_2(J), && \Psi_1(\alpha) = (\mathrm{id}_\omega, \alpha),\\
&\Psi_2 : F_2(J) \to F(J) \times F(J), && \Psi_2(\varphi, \alpha) = (\varphi, \varphi \circ \alpha),\\
&\Phi_3 : I \to F(I),\ \Phi_3(a)(k) = 0, \text{ if } k \in a, \text{ and } \Phi_3(a)(k) = k, \text{ if } k \in \omega \setminus a, &&\\
&\Psi_3 : F(J) \times F(J) \to {}^\omega\omega, && \Psi_3(\varphi, \alpha) = \varphi.
\end{aligned}$$

For any ideals I and J on ω the following pairs of functions are morphisms:

$$\mathrm{B}_{I,J} \xrightarrow{(\mathrm{id}_{F(I)},\Psi_1)} \mathrm{C}_{I,J} \xrightarrow{(\mathrm{id}_{F(I)},\Psi_2)} \mathrm{D}_{I,J} \xrightarrow{(\Phi_3,\Psi_3)} \mathrm{E}_{I,J}.$$

Consequently, $\mathfrak{b}(\mathrm{B}_{I,J}) \le \mathfrak{b}(\mathrm{C}_{I,J}) \le \mathfrak{b}(\mathrm{D}_{I,J}) \le \mathfrak{b}(\mathrm{E}_{I,J}) = \mathfrak{k}_{I,J}$.

Proof. For example, the implication $\Phi_3(a)\ \mathrm{D}_{I,J}\ (\varphi, \alpha) \Rightarrow a\ \mathrm{E}_{I,J}\ \Psi_3(\varphi, \alpha)$ holds because $\|\varphi \circ \Phi_3(a) < \alpha\| \in J$ implies $\varphi^{-1}(a) \subseteq \|\varphi \circ \Phi_3(a) < \alpha\| \cup \|\alpha = 0\| \in J$. □

By [13, Lemma 3.10 (a)], $\mathfrak{p} \le \mathfrak{k}_{I,J}$. The value $\mathfrak{k}_{I,J}$ gives no reasonable restriction on the value of $\mathfrak{b}(\mathrm{B}_{I,J})$: If $I \nsubseteq J$, then $\mathfrak{b}(\mathrm{B}_{I,J}) = \mathrm{non}((I, J\mathrm{QN})$-space$) = 1$ and, if $I \subseteq J$, then $\mathfrak{k}_{I,J} = \infty$.

Recall that the symbols id_X and $\mathrm{id}_{X,Y}$ denote the identity function on X. The symbol $\mathrm{id}_{X,Y}$ serves also as an evidence about the inclusion $X \subseteq Y$.

Lemma 2.4. *The inclusions $I \subseteq K$, $K \subseteq J$, and $I \subseteq J$ (separately) imply that the following pairs of identities are morphisms:*

$$\begin{array}{ccccc}
\mathrm{A}_{I,J} & \xrightarrow[I\subseteq K]{(\mathrm{id}_{F(I)},\mathrm{id}_{F(I),F(K)})} & \mathrm{Z}^K_{I,J} & \xrightarrow[K\subseteq J]{(\mathrm{id}_{F(I)},\mathrm{id}_{F(K),F(J)})} & \mathrm{B}_{I,J}\\
{\scriptstyle I\subseteq J}\uparrow{\scriptstyle (\mathrm{id}_{F(I)},\mathrm{id}_{F(I),F(J)})} & & & & {\scriptstyle I\subseteq J}\uparrow{\scriptstyle (\mathrm{id}_{F(I),F(J)},\mathrm{id}_{F(J)})}\\
\mathrm{A}_{I,I} & & & & \mathrm{A}_{J,J}
\end{array}$$ □

Let $\le_J$ denote the partial quasi-ordering of ${}^\omega\omega$ defined by

$$\alpha \le_J \beta \Leftrightarrow \|\alpha < \beta\| \in J, \qquad \alpha, \beta \in {}^\omega\omega.$$

If J is a maximal ideal, then $\le_J$ is a total quasi-ordering.

Lemma 2.5. *For any ideals I and J on ω, $\mathrm{A}_{J,J} = (F(J), \le_J)$ and $\mathrm{A}_{I,J} = (F(I), \le_J)$ are directed partially quasi-ordered sets. $\mathrm{A}_{I,J}$ has a largest element if and only if $I \cap J^* \ne \emptyset$.*

Proof. Let $\gamma_{\alpha,\beta}$ be the pointwise minimum of $\alpha, \beta \in F(I)$. Then $\gamma_{\alpha,\beta} \in F(I)$ because $\|\gamma_{\alpha,\beta} = n\| \subseteq \|\alpha = n\| \cup \|\beta = n\| \in I$ for all $n \in \omega$ and $\gamma_{\alpha,\beta}$ is an upper bound of α and β in $\mathrm{A}_{I,J}$ because $\|\alpha < \gamma_{\alpha,\beta}\| = \|\beta < \gamma_{\alpha,\beta}\| = \emptyset \in J$. Any $\beta \in F(I)$ is a largest element in $\mathrm{A}_{I,J}$ if and only if $\|\beta = 0\| \in I \cap J^*$ because for every $\alpha \in F(I)$, $\|\alpha < \beta\| \subseteq \omega \setminus \|\beta = 0\|$. □

Assuming $I \cap J^* = \emptyset$ we denote (in most cases $I \subseteq J$):

$$\mathfrak{b}_{I,J} = \mathfrak{b}(\mathrm{A}_{I,J}), \quad \mathfrak{d}_{I,J} = \mathfrak{d}(\mathrm{A}_{I,J}), \quad \mathfrak{b}_J = \mathfrak{b}(\mathrm{A}_{J,J}), \quad \mathfrak{d}_J = \mathfrak{d}(\mathrm{A}_{J,J}).$$

Since $\mathrm{A}_{I,J}$ is a partially quasi-ordered set on a subset of ${}^{\omega}\omega$, $\mathfrak{b}_{I,J} \leq \mathfrak{d}_{I,J} \leq \mathfrak{c}$ and $\mathfrak{b}_J \leq \mathfrak{d}_J \leq \mathfrak{c}$.

Corollary 2.6. *If $I \subseteq J$ are ideals on ω, then $\omega_1 \leq \mathfrak{b}_J$ and $\omega_1 \leq \mathfrak{b}_I \leq \mathfrak{b}_{I,J}$.*

Proof. By [13, Theorem 2.12 (a)], non($(I, JK$QN)-space) $\geq \omega_1$, if $I \subseteq K \cap J$. Therefore by Theorem 2.2 (a), $\mathfrak{b}_I =$ non($(I, II$QN)-space) $\geq \omega_1$. Then by Lemma 2.4, $\omega_1 \leq \mathfrak{b}_I \leq \mathfrak{b}_{I,J}$ and, in particular for $I = J$, $\omega_1 \leq \mathfrak{b}_{J,J} = \mathfrak{b}_J$. □

We consider also restrictions of two partial quasi-orderings of ${}^{\omega}\mathcal{P}(\omega)$:

$$f \leq^0_J g \Leftrightarrow (\exists x \in J)(\forall n \in \omega)\ f(n) \subseteq g(n) \cup x,$$
$$f \leq^1_J g \Leftrightarrow \{n \in \omega : f(n) \nsubseteq g(n)\} \in J.$$

The relations $\leq^0_J$ and $\leq^1_J$ are natural generalizations of the eventual partial quasi-ordering $\leq^*$ on ${}^{\omega}\omega$ and $({}^{\omega}\mathrm{Fin}, \leq^0_{\mathrm{Fin}}) \approx ({}^{\omega}\mathrm{Fin}, \leq^1_{\mathrm{Fin}}) \approx ({}^{\omega}\omega, \leq^*)$ because ${}^{\omega}\omega$ is cofinal in ${}^{\omega}\mathrm{Fin}$. Note that for $\alpha, \beta \in {}^{\omega}\omega$, $\alpha \leq_J \beta \Leftrightarrow \beta \leq^1_J \alpha$; i.e., $\leq_J$ is the inverse of the relation $\leq^1_J$.

Be aware that we use $\mathfrak{b}_J$ and $\mathfrak{d}_J$ differently from other authors (e.g., [8, 10]) and by definition they are different from $\mathfrak{b}({}^{\omega}\omega, \leq^1_J)$ and $\mathfrak{d}({}^{\omega}\omega, \leq^1_J)$.

By the next two lemmas, $({}^{\omega}I, \leq^1_{\mathrm{Fin}}) \preccurlyeq \mathrm{A}_{I,I} \preccurlyeq \mathrm{A}_{I,J} \approx ({}^{\omega}I, \leq^0_J)$ and hence Corollary 2.6 can be proved also by the diagonal method.

Lemma 2.7. *Let I, J, K, I', J', J'', K' be ideals on ω.*

(a) $\mathrm{Z}^K_{I,J} \approx ({}^{\omega}I, {}^{\omega}K, \leq^0_J)$, $\mathrm{A}_{I,J} \approx ({}^{\omega}I, \leq^0_J)$, *and* $\mathrm{B}_{I,J} \approx ({}^{\omega}I, {}^{\omega}J, \leq^0_J)$.

(b) *If* $I' \subseteq I$, $J \subseteq J'$, *and* $K \subseteq K'$, *then* $\mathrm{Z}^K_{I,J} \preccurlyeq \mathrm{Z}^{K'}_{I',J'}$, $\mathrm{Y}^K_{I,J} \preccurlyeq \mathrm{Y}^{K'}_{I',J'}$, $\mathrm{A}_{I,J} \preccurlyeq \mathrm{A}_{I,J'}$, $\mathrm{B}_{I,J} \preccurlyeq \mathrm{B}_{I',J'}$, $\mathrm{C}_{I,J} \preccurlyeq \mathrm{C}_{I',J'}$, $\mathrm{D}_{I,J} \preccurlyeq \mathrm{D}_{I',J'}$, $\mathrm{E}_{I,J} \preccurlyeq \mathrm{E}_{I',J'}$ (*omitting* $\mathrm{A}_{I,J} \preccurlyeq \mathrm{A}_{I',J}$).

(c) *If* $J \leq_{\mathrm{KB}} J'$, *then* $\mathrm{A}_{\mathrm{Fin},J} \preccurlyeq \mathrm{A}_{\mathrm{Fin},J'}$.

(d) *If* $I' \leq_{\mathrm{K}} I$ *and* $J \leq_{\mathrm{K}} J'$, *then* $\mathrm{B}_{\mathrm{Fin},J} \preccurlyeq \mathrm{B}_{\mathrm{Fin},J'}$, $\mathrm{C}_{I,J} \preccurlyeq \mathrm{C}_{I,J'}$, $\mathrm{D}_{I,J} \preccurlyeq \mathrm{D}_{I',J'}$, $\mathrm{E}_{I,J} \preccurlyeq \mathrm{E}_{I',J'}$.

Proof. (a) Define $\Phi_I : F(I) \to {}^\omega I$ and $\Psi_I : {}^\omega I \to F(I)$ by $\Phi_I(\alpha)(n) = \|\alpha \le n\|$ and $\Psi_I(f)(n) = \min\{m \in \omega : n \in m \cup f(m)\}$ for $\alpha \in F(I)$, $f \in {}^\omega I$, and $n \in \omega$. We show that the following pairs of functions are morphisms:

$$(\Phi_I, \Psi_K) : ({}^\omega I, {}^\omega K, \le^0_J) \to (F(I), F(K), \le_J),$$
$$(\Psi_I, \Phi_K) : (F(I), F(K), \le_J) \to ({}^\omega I, {}^\omega K, \le^0_J).$$

Assume that $\alpha \in F(I)$, $f \in {}^\omega K$, and $\Phi_I(\alpha) \le^0_J f$. There is an $x \in J$ such that $\|\alpha = n\| \subseteq f(n) \cup x$ for all $n \in \omega$ and then, $\alpha(k) = n$ and $k \notin x$ implies $\Psi_K(f)(k) \le n = \alpha(k)$. Therefore $\|\alpha < \Psi_K(f)\| \subseteq x \in J$; i.e., $\alpha \le_J \Psi_K(f)$.

Assume that $\alpha \in F(K)$, $f \in {}^\omega I$, and $\Psi_I(f) \le_J \alpha$, i.e., $\|\Psi_I(f) < \alpha\| \in J$. Then for every $n \in \omega$, $f(n) \subseteq \|\Psi_I(f) \le n\| \subseteq \|\alpha \le n\| \cup \|\Psi_I(f) < \alpha\| = \Phi_K(\alpha)(n) \cup \|\Psi_I(f) < \alpha\|$. Therefore $f \le^0_J \Phi_K(\alpha)$.

(b) These morphisms consists of identity functions (similar to Lemma 2.4).

(c)–(d) Let $\nu \in F(I)$ and $\eta \in F(J')$ be such that $I' \subseteq \nu^\to(I) = \{a \subseteq \omega : \nu^{-1}(a) \in I\}$ and $J \subseteq \eta^\to(J') = \{a \subseteq \omega : \eta^{-1}(a) \in J'\}$.

By (a), to prove $\mathrm{B}_{\mathrm{Fin},J} \preccurlyeq \mathrm{B}_{\mathrm{Fin},J'}$ and $\mathrm{A}_{\mathrm{Fin},J} \preccurlyeq \mathrm{A}_{\mathrm{Fin},J'}$ we find morphisms $(\Phi, \Psi) : ({}^\omega\mathrm{Fin}, {}^\omega J, \le^0_J) \to ({}^\omega\mathrm{Fin}, {}^\omega J', \le^0_{J'})$ and $(\Phi, \Psi') : ({}^\omega\mathrm{Fin}, \le^0_J) \to ({}^\omega\mathrm{Fin}, \le^0_{J'})$. Define $\Phi : {}^\omega\mathrm{Fin} \to {}^\omega\mathrm{Fin}$ and $\Psi : {}^\omega J \to {}^\omega J'$ by $\Phi(f)(n) = \eta[f(n)]$ and $\Psi(g)(n) = \eta^{-1}(g(n))$; if η is finite-to-one, then let $\Psi' = \Psi\restriction({}^\omega\mathrm{Fin})$. If $\Phi(f) \le^0_J g$ and $x \in J$ is such that $\Phi(f)(n) \subseteq g(n) \cup x$ for all $n \in \omega$, then $f(n) \subseteq \eta^{-1}(g(n) \cup x) = \Psi(g)(n) \cup \eta^{-1}(x)$ for all $n \in \omega$ and hence, $f \le^0_{J'} \Psi(g)$ because $\eta^{-1}(x) \in J'$.

A morphism $(\mathrm{id}_{F(I)}, \Psi) : \mathrm{C}_{I,J} \to \mathrm{C}_{I,J'}$. Define $\Psi : F_2(J) \to F_2(J')$ by $\Psi(\varphi, \alpha) = (\eta \circ \varphi, \alpha)$; if $\varphi \circ \alpha \in F(J)$, then $\Psi(\varphi, \alpha) \in F(J')$ because $(\eta \circ \varphi \circ \alpha)^{-1}(\{n\}) = \eta^{-1}[(\varphi \circ \alpha)^{-1}(\{n\})]$. Now, $\beta\, \mathrm{C}_{I,J}\, (\varphi, \alpha) \Rightarrow \varphi^{-1}(\|\beta < \alpha\|) \in J \subseteq \eta^\to(J') \Rightarrow \eta^{-1}(\varphi^{-1}(\|\beta < \alpha\|)) \in J' \Rightarrow \beta\, \mathrm{C}_{I,J'}\, (\eta \circ \varphi, \alpha) \Rightarrow \beta\, \mathrm{C}_{I,J'}\, \Psi(\varphi, \alpha)$.

A morphism $(\Phi, \Psi) : \mathrm{D}_{I,J} \to \mathrm{D}_{I',J'}$. We define $\Phi : F(I') \to F(I)$ and $\Psi : F(J) \times F(J) \to F(J') \times F(J')$. For $\beta \in F(I')$ let $\Phi(\beta) = \nu \circ \beta$ and for $\varphi, \alpha \in F(J)$ let $\Psi(\varphi, \alpha) = (\eta \circ \varphi \circ \nu, \eta \circ \alpha)$, if $\eta \circ \varphi \circ \nu \in F(J')$, and $\Psi(\varphi, \alpha) \in F(J')$ be arbitrary, otherwise. Then $\Phi(\beta) \in F(I)$ because $\|\nu \circ \beta = n\| = \nu^{-1}(\|\beta = n\|)$; similarly, $\eta \circ \alpha \in F(J')$ and $\Psi(\varphi, \alpha) \in F(J') \times F(J')$. Now, $\Phi(\beta)\, \mathrm{D}_{I,J}\, (\varphi, \alpha) \Rightarrow \|\varphi \circ \Phi(\beta) < \alpha\| \in J \subseteq \eta^\to(J') \Rightarrow \eta^{-1}(\|\varphi \circ \Phi(\beta) < \alpha\|) \in J' \Rightarrow \|\eta \circ \varphi \circ \nu \circ \beta < \eta \circ \alpha\| \in J'$. Hence by Lemma 1.1, if $\Phi(\beta)\, \mathrm{D}_{I,J}\, (\varphi, \alpha)$, then $\eta \circ \varphi \circ \nu \in F(J')$ and consequently, $\Psi(\varphi, \alpha) = (\eta \circ \varphi \circ \nu, \eta \circ \alpha)$ and $\beta\, \mathrm{D}_{I,J'}\, \Psi(\varphi, \alpha)$.

A morphism $(\Phi, \Psi) : \mathrm{E}_{I,J} \to \mathrm{E}_{I',J'}$. Let $\Phi : I' \to I$ and $\Psi : {}^\omega\omega \to {}^\omega\omega$ be defined by $\Phi(a) = \nu^{-1}(a)$ and $\Psi(\varphi) = \eta \circ \varphi \circ \nu$.[1] If $\Phi(a)\, \mathrm{E}_{I,J}\, \varphi$, i.e.,

[1] The same argument in proving monotonicity of the relation $(I, F(J), \mathrm{E}_{I,J})$ as opposed to $(I, {}^\omega\omega, \mathrm{E}_{I,J})$ requires ν finite-to-one: If $\varphi \in F(J)$ and $\nu \in F(\mathrm{Fin})$, then $\Psi(\varphi) \in F(J')$.

$\varphi^{-1}(\nu^{-1}(a)) \in J \subseteq \eta^{\rightarrow}(J')$, then $\eta^{-1}(\varphi^{-1}(\nu^{-1}(a))) = (\eta \circ \varphi \circ \nu)^{-1}(a) \in J'$; i.e., $a \mathrm{E}_{I,J} \Psi(\varphi)$. $\square$

It seems that the monotonicity of the relations $\mathrm{A}_{I,J}$, $\mathrm{B}_{I,J}$, $\mathrm{C}_{I,J}$ with respect to the ideals I and J in Lemma 2.7 (c)–(d) differs from the monotonicity of $\mathrm{D}_{I,J}$ and $\mathrm{E}_{I,J}$. Some additional monotonicity properties of $\mathrm{A}_{I,J}$ and $\mathrm{B}_{I,J}$ we prove in Proposition 2.11 below.

Lemma 2.8. *Let I, J, K, L denote ideals on ω.*

(a) *There are morphisms according to the diagram*

$$\begin{array}{ccccccc} & & \mathrm{A}_{I,I} & & & & \\ & & \uparrow & & & & \\ \mathrm{A}_{I,\mathrm{Fin}} & \rightarrow & ({}^{\omega}I, \leq^1_{\mathrm{Fin}}) & \rightarrow & ({}^{\omega}\omega, \leq^*) & \rightarrow & ({}^{\omega}\omega, \leq^0_J) \approx \mathrm{A}_{\mathrm{Fin},J} \\ & & \downarrow & & \downarrow & & \\ & & ({}^{\omega}I, \leq^1_J) & \rightarrow & ({}^{\omega}\omega, \leq^1_J) & & \end{array}$$

(b) $\mathfrak{b}_{\mathrm{Fin}} = \mathfrak{b}$ *and* $\mathfrak{d}_{\mathrm{Fin}} = \mathfrak{d}$.

(c) *If $I \subseteq L \subseteq K \subseteq J$ and K is a $P(I)$-ideal, then* $\mathrm{A}_{L,L} \preccurlyeq \mathrm{A}_{I,K} \approx \mathrm{A}_{L,K} \approx \mathrm{A}_{K,K} \approx \mathrm{B}_{I,K} \approx \mathrm{B}_{L,K} \approx \mathrm{B}_{K,K} \preccurlyeq \mathrm{A}_{I,J} \approx \mathrm{A}_{L,J} \approx \mathrm{A}_{K,J} \preccurlyeq \mathrm{B}_{I,J} \approx \mathrm{B}_{L,J} \approx \mathrm{B}_{K,J}$.

(d) *If J is a P-ideal, then $\mathfrak{b} \leq \mathfrak{b}_J \leq \mathfrak{d}_J \leq \mathfrak{d}$ and for every ideal $I \subseteq J$ on ω, $\mathfrak{b}_I \leq \mathfrak{b}_J = \mathfrak{b}_{I,J} \leq \mathfrak{d}_{I,J} = \mathfrak{d}_J \leq \mathfrak{d}_I$.*

Proof. (a) The morphisms $({}^{\omega}I, \leq^1_{\mathrm{Fin}}) \to ({}^{\omega}I, \leq^0_I)$, $({}^{\omega}I, \leq^1_{\mathrm{Fin}}) \to ({}^{\omega}I, \leq^1_J)$, $({}^{\omega}\omega, \leq^*) \to ({}^{\omega}\omega, \leq^0_J)$, $({}^{\omega}\omega, \leq^*) \to ({}^{\omega}\omega, \leq^1_J)$ are created by pairs of identities and by Lemma 2.7 (a), $\mathrm{A}_{I,J} \approx ({}^{\omega}I, \leq^0_J)$. Define $\Psi_1 : {}^{\omega}I \to {}^{\omega}I$ and $\Psi_2 : {}^{\omega}I \to {}^{\omega}\omega$ by $\Psi_1(f)(n) = f(n) \cup n$ and $\Psi_2(f)(k) = \max\{m \in \omega : m \subseteq f(k)\}$ for $f \in {}^{\omega}I$. Then $(\mathrm{id}_{{}^{\omega}I}, \Psi_1) : ({}^{\omega}I, \leq^0_{\mathrm{Fin}}) \to ({}^{\omega}I, \leq^1_{\mathrm{Fin}})$ and $(\mathrm{id}_{{}^{\omega}I, {}^{\omega}\omega}, \Psi_2) : ({}^{\omega}I, \leq^1_J) \to ({}^{\omega}\omega, \leq^1_J)$ are morphisms. Taking $J = \mathrm{Fin}$ we get also a morphism $({}^{\omega}I, \leq^1_{\mathrm{Fin}}) \to ({}^{\omega}\omega, \leq^*)$.

(b) By (a), $\mathrm{A}_{\mathrm{Fin},\mathrm{Fin}} \approx ({}^{\omega}\omega, \leq^*)$.

(c) Let L and L' be arbitrary ideals on ω such that $I \subseteq L \subseteq K$ and $I \subseteq L' \subseteq K$. By Lemma 2.4 and Lemma 2.7 (b), $\mathrm{A}_{L,L} \preccurlyeq \mathrm{A}_{L,K} \preccurlyeq \mathrm{B}_{L,K}$ and $\mathrm{A}_{L,K} \preccurlyeq \mathrm{A}_{L,J} \preccurlyeq \mathrm{B}_{L,J}$. We prove $\mathrm{A}_{L,J} \preccurlyeq \mathrm{A}_{L',J}$, $\mathrm{B}_{L,J} \preccurlyeq \mathrm{B}_{L',J}$, and $\mathrm{B}_{L,K} \preccurlyeq \mathrm{A}_{L',K}$. By replacing L and L' we get the (first two) inverse morphisms and we get $\mathrm{A}_{L,L} \preccurlyeq \mathrm{A}_{L',K} \approx \mathrm{A}_{L,K} \approx \mathrm{B}_{L',K} \approx \mathrm{B}_{L,K} \preccurlyeq \mathrm{A}_{L',J} \approx \mathrm{A}_{L,J} \preccurlyeq \mathrm{B}_{L',J} \approx \mathrm{B}_{L,J}$. For $L' = I$ and $L' = K$ we get the remaining equalities.

Let $\Phi : {}^{\omega}K \to {}^{\omega}I$ be defined as follows: Since K is a $P(I)$-ideal for every $g \in {}^{\omega}K$ fix a set $x_g \in K$ such that $g(n) \setminus x_g \in I$ for all $n \in \omega$ and let

$\Phi(g)(n) = g(n) \setminus x_g$. Then $g \leq^0_K \Phi(g)$. By Lemma 2.7 (a) it is enough to show that the following pairs of functions are morphisms:

$$(\Phi\restriction({}^\omega L'), \Phi\restriction({}^\omega L)) : ({}^\omega L, \leq^0_J) \to ({}^\omega L', \leq^0_J), \tag{2.1}$$

$$(\Phi\restriction({}^\omega L'), \mathrm{id}_{{}^\omega J}) : ({}^\omega L, {}^\omega J, \leq^0_J) \to ({}^\omega L', {}^\omega J, \leq^0_J), \tag{2.2}$$

$$(\Phi\restriction({}^\omega L'), \Phi) : ({}^\omega L, {}^\omega K, \leq^0_K) \to ({}^\omega L', \leq^0_K). \tag{2.3}$$

These functions are well defined because ${}^\omega I \subseteq {}^\omega L \subseteq {}^\omega K$ and ${}^\omega I \subseteq {}^\omega L' \subseteq {}^\omega K$.

Assume that $\Phi(g) \leq^0_J f$ for some $g \in {}^\omega L'$ and $f \in {}^\omega J$, i.e., there is $x \in J$ such that $\Phi(g)(n) \subseteq f(n) \cup x$ for all $n \in \omega$. Then $g \leq^0_J f$ because $g(n) \subseteq f(n) \cup (x \cup x_g)$ for all $n \in \omega$ and $x \cup x_g \in J$. This shows that (2.2) is a morphism. If $f \in {}^\omega L$, then $g \leq^0_J \Phi(f)$ because $g \leq^0_J f \leq^0_K \Phi(f)$ and $K \subseteq J$. Therefore (2.1) is a morphism. Taking $J = K$ and $f \in {}^\omega K$ in the same way we get $g \leq^0_K \Phi(f)$ because $g \leq^0_K f \leq^0_K \Phi(f)$. Therefore (2.3) is a morphism.

(d) $\mathrm{A}_{\mathrm{Fin},\mathrm{Fin}} \preccurlyeq \mathrm{A}_{J,J}$ and $\mathrm{A}_{I,I} \preccurlyeq \mathrm{A}_{I,J} \approx \mathrm{A}_{J,J}$ holds for all $I \subseteq J$: $\mathrm{A}_{I,I} \preccurlyeq \mathrm{A}_{I,J}$ holds by Lemma 2.4 and $\mathrm{A}_{I,J} \approx \mathrm{A}_{J,J}$ holds by (c) because "J is a $P(\mathrm{Fin})$-ideal and $\mathrm{A}_{I,J} \approx \mathrm{A}_{J,J}$" is an instance of "$K$ is a $P(I)$-ideal and $\mathrm{A}_{L,K} \approx \mathrm{A}_{K,K}$" in (c) by the substitution $(K/J, I/\mathrm{Fin}, L/I)$. Therefore $\mathfrak{b}_I \leq \mathfrak{b}_{I,J} = \mathfrak{b}_J \leq \mathfrak{d}_J = \mathfrak{d}_{I,J} \leq \mathfrak{d}_I$ and $\mathfrak{b} = \mathfrak{b}_{\mathrm{Fin}} \leq \mathfrak{b}_J \leq \mathfrak{d}_J \leq \mathfrak{d}_{\mathrm{Fin}} = \mathfrak{d}$ ($\mathfrak{b}_{\mathrm{Fin}} = \mathfrak{b}$ and $\mathfrak{d}_{\mathrm{Fin}} = \mathfrak{d}$ by (b); $\mathfrak{b}_J \leq \mathfrak{d}_J$ because $\mathrm{A}_{J,J}$ is a partially quasi-ordered set). □

By lemmas 2.4, 2.7 (b), 2.8 (c), if $I \subseteq K \subseteq J$ and K is a $P(I)$-ideal, then

$$\begin{aligned}\mathfrak{b}_I \leq \mathfrak{b}_{I,K} = \mathfrak{b}_K &= \mathrm{non}((I, K\mathrm{QN})\text{-space}) = \mathrm{non}((K, K\mathrm{QN})\text{-space}) \\ &\leq \mathfrak{b}_{I,J} = \mathfrak{b}_{K,J} \leq \mathrm{non}((I, J\mathrm{QN})\text{-space}) = \mathrm{non}((K, J\mathrm{QN})\text{-space}).\end{aligned}$$

The direct product of relations (R_-, R_+, R) and (S_-, S_+, S) is the relation $(R_- \times S_-, R_+ \times S_+, R \otimes S)$ where $(x, u)\ R \otimes S\ (y, v) \Leftrightarrow (x\ R\ y$ and $u\ R\ v)$. Obviously, $R \otimes S \approx S \otimes R \preccurlyeq R$. Note that $\mathfrak{b}(R \otimes S) = \min\{\mathfrak{b}(R), \mathfrak{b}(S)\}$ and, if $\mathfrak{d}(R \otimes S)$ is infinite, then $\mathfrak{d}(R \otimes S) = \max\{\mathfrak{d}(R), \mathfrak{d}(S)\}$.

An ideal J is a weak $P(I_1 \vee I_2)$-ideal if and only if J is a weak $P(I_1)$-ideal or J is a weak $P(I_2)$-ideal (see Proposition 2.9 (d) and Theorem 2.2 (a)).

Proposition 2.9. *Let $I = I_1 \vee I_2$, $K = K_1 \vee K_2$, and $J = J_1 \vee J_2$ be ideals on ω. Then $\mathrm{Z}^{K_1}_{I_1,J_1} \otimes \mathrm{Z}^{K_2}_{I_2,J_2} \preccurlyeq \mathrm{Z}^K_{I,J} \approx \mathrm{Z}^K_{I_1,J} \otimes \mathrm{Z}^K_{I_2,J}$. Consequently:*

(a) $\min\{\mathfrak{b}_{I_1,J_1}, \mathfrak{b}_{I_2,J_2}\} \leq \mathfrak{b}_{I_1 \vee I_2, J_1 \vee J_2}$ *and* $\mathfrak{d}_{I_1 \vee I_2, J_1 \vee J_2} \leq \max\{\mathfrak{d}_{I_1,J_1}, \mathfrak{d}_{I_2,J_2}\}$.

(b) $\min\{\mathfrak{b}_{I_1}, \mathfrak{b}_{I_2}\} \leq \mathfrak{b}_{I_1 \vee I_2}$ *and* $\mathfrak{d}_{I_1 \vee I_2} \leq \max\{\mathfrak{d}_{I_1}, \mathfrak{d}_{I_2}\}$.

(c) *If $K \subseteq J$, then* $\mathrm{non}((I_1 \vee I_2, JK\mathrm{QN})$*-space*$) = \min\{\mathrm{non}((I_1, JK\mathrm{QN})$*-space*$), \mathrm{non}((I_2, JK\mathrm{QN})$*-space*$)\}$.

(d) $\operatorname{non}((I_1 \vee I_2, J\mathrm{QN})\text{-}space) = \min\{\operatorname{non}((I_1, J\mathrm{QN})\text{-}space), \operatorname{non}((I_2, J\mathrm{QN})\text{-}space)\}$.

PROOF. By Lemma 2.7 (a) we get $\mathrm{Z}^{K_1}_{I_1,J_1} \otimes \mathrm{Z}^{K_2}_{I_2,J_2} \preccurlyeq \mathrm{Z}^{K}_{I,J}$, if we find a morphism

$$(\Phi, \Psi) : ({}^\omega I_1 \times {}^\omega I_2, {}^\omega K_1 \times {}^\omega K_2, \leq^0_{J_1} \otimes \leq^0_{J_2}) \to ({}^\omega I, {}^\omega K, \leq^0_J).$$

Let $\Phi = (\Phi_1, \Phi_2) : {}^\omega I \to {}^\omega I_1 \times {}^\omega I_2$ be such that $\Phi_1(f)(n) \cup \Phi_2(f)(n) = f(n)$ for all $f \in {}^\omega(I_1 \vee I_2)$ and $n \in \omega$ and let $\Psi : {}^\omega K_1 \times {}^\omega K_2 \to {}^\omega K$ be defined by $\Psi(g_1, g_2)(n) = g_1(n) \cup g_2(n)$ for $(g_1, g_2) \in {}^\omega K_1 \times {}^\omega K_2$ and $n \in \omega$. Assume that $\Phi(f) \leq^0_{J_1} \otimes \leq^0_{J_2} (g_1, g_2)$, i.e., $\Phi_1(f) \leq^0_{J_1} g_1$ and $\Phi_2(f) \leq^0_{J_2} g_2$ which means that there are $x_i \in J_i$, $i = 1, 2$, such that $\Phi_1(f)(n) \subseteq g_1(n) \cup x_1$ and $\Phi_2(f)(n) \subseteq g_2(n) \cup x_2$ for all $n \in \omega$. Then $x_1 \cup x_2 \in J$ and for every $n \in \omega$, $f(n) = \Phi_1(f)(n) \cup \Phi_2(f)(n) \subseteq g_1(n) \cup g_2(n) \cup x_1 \cup x_2 = \Psi(g_1, g_2)(n) \cup (x_1 \cup x_2)$. Therefore $f \leq^0_J \Psi(g_1, g_2)$. This proves $\mathrm{Z}^{K_1}_{I_1,J_1} \otimes \mathrm{Z}^{K_2}_{I_2,J_2} \preccurlyeq \mathrm{Z}^{K}_{I,J}$. By substitution $K_1 = K_2 = K$ and $J_1 = J_2 = J$ we get $\mathrm{Z}^{K}_{I_1,J} \otimes \mathrm{Z}^{K}_{I_2,J} \preccurlyeq \mathrm{Z}^{K}_{I,J}$.

To get $\mathrm{Z}^{K}_{I,J} \preccurlyeq \mathrm{Z}^{K}_{I_1,J} \otimes \mathrm{Z}^{K}_{I_2,J}$ we find a morphism

$$(\Phi', \Psi') : ({}^\omega I, {}^\omega K, \leq^0_J) \to ({}^\omega I_1 \times {}^\omega I_2, {}^\omega K \times {}^\omega K, \leq^0_J \otimes \leq^0_J).$$

Define $\Phi' : {}^\omega I_1 \times {}^\omega I_2 \to {}^\omega I$ and $\Psi' : {}^\omega K \to {}^\omega K \times {}^\omega K$ by $\Phi'(f_1, f_2)(n) = f_1(n) \cup f_2(n)$ and $\Psi'(g) = (g, g)$. Assume that $\Phi'(f_1, f_2) \leq^0_J g$ and let $x \in J$ be such that $f_1(n) \cup f_2(n) \subseteq g(n) \cup x$ for all $n \in \omega$. Then $f_1 \leq^0_J g$ and $f_2 \leq^0_J g$, i.e., $(f_1, f_2) \leq^0_J \otimes \leq^0_J \Psi'(g)$.

For (a) take $K_1 = I_1$, $K_2 = I_2$; for (b) take $K_1 = J_1 = I_1$, $K_2 = J_2 = I_2$; for (c) take $K_1 = K_2 = K$; for (d) take $J_1 = J_2 = K_1 = K_2 = J$. □

The J-sum of ideals L_n, $n \in \omega$, on ω is the ideal $\sum^J_{n\in\omega} L_n = \{a \subseteq \omega \times \omega : \{n \in \omega : a_{(n)} \notin L_n\} \in J\}$ where $a_{(n)} = \{k \in \omega : (n, k) \in a\}$ for $a \subseteq \omega \times \omega$. If $J' = \sum^J_{n\in\omega} L_n$, then $J \leq_{\mathrm{RK}} J'$ and by the next proposition, in this special case, $\mathrm{A}_{J',J'} \preccurlyeq \mathrm{A}_{J,J}$.

Proposition 2.10. *Let I, J, K, L_n, $n \in \omega$, be ideals on ω and let $I' = \sum^I_{n\in\omega} L_n$, $J' = \sum^J_{n\in\omega} L_n$, and $K' = \sum^K_{n\in\omega} L_n$. Then $\mathrm{Z}^{K'}_{I',J'} \preccurlyeq \mathrm{Z}^{K}_{I,J}$. In particular, $\mathfrak{b}_{I',J'} \leq \mathfrak{b}_{I,J}$ and $\operatorname{non}((I', J'\mathrm{QN})\text{-}space) \leq \operatorname{non}((I, J\mathrm{QN})\text{-}space)$.*

PROOF. Define $\Phi : {}^\omega I \to {}^\omega I'$ and $\Psi : {}^\omega K' \to {}^\omega K$ by $\Phi(f)(k) = f(k) \times \omega$ for $f \in {}^\omega I$ and $\Psi(g)(k) = \{n \in \omega : g(k)_{(n)} \notin L_n\}$ for $g \in {}^\omega K'$. Let $\Phi(f) \leq^0_{J'} g$ (where $f \in {}^\omega I$ and $g \in {}^\omega K'$) and let $x \in J'$ be such that $f(k) \times \omega \subseteq g(k) \cup x$ for all $k \in \omega$. Denote $y = \{n \in \omega : x_{(n)} \notin L_n\}$. Then $y \in J$ and for every $k \in \omega$, $f(k) \subseteq \{n \in \omega : (g(k) \cup x)_{(n)} \notin L_n\} = \Psi(g)(k) \cup y$. Therefore $f \leq^0_J \Psi(g)$. □

Proposition 2.11 ([13, Lemma 4.1]). *Let I, J, K be ideals on ω, $\eta \in {}^{\omega}\omega$, and $a = \omega \setminus \mathrm{rng}(\eta)$.*

(a) *If $\eta \in F(J)$ and $\mathrm{rng}(\eta) \in I^+$, then $\mathrm{A}_{I,\eta^{\to}(J)} \preccurlyeq \mathrm{A}_{\eta^{\leftarrow}(I),J}$ and $\mathrm{B}_{I,\eta^{\to}(J)} \preccurlyeq \mathrm{B}_{\eta^{\leftarrow}(I),J}$. More generally:*

 (1) *If $\eta \in F(J)$ and $\mathrm{rng}(\eta) \in I^+ \cap K^+$, then $\mathrm{Z}^K_{I,\eta^{\to}(J)} \preccurlyeq \mathrm{Z}^{\eta^{\leftarrow}(K)}_{\eta^{\leftarrow}(I),J}$.*

 (2) *If $\eta \in F(J) \cap F(K)$ and $\mathrm{rng}(\eta) \in I^+$, then $\mathrm{Z}^{\eta^{\to}(K)}_{I,\eta^{\to}(J)} \preccurlyeq \mathrm{Z}^K_{\eta^{\leftarrow}(I),J}$.*

(b) $\mathrm{Z}^K_{I,J} \preccurlyeq \mathrm{Z}^K_{I,J\vee\langle a\rangle} \approx \mathrm{Z}^{K\vee\langle a\rangle}_{I,J\vee\langle a\rangle} \approx \mathrm{Z}^{\eta^{\leftarrow}(K)}_{\eta^{\leftarrow}(I),\eta^{\leftarrow}(J)}$, *if* $\mathrm{rng}(\eta) \in I^+ \cap J^+ \cap K^+$. *Consequently,* $\mathrm{A}_{I,J} \preccurlyeq \mathrm{A}_{I,J\vee\langle a\rangle} \approx \mathrm{A}_{\eta^{\leftarrow}(I),\eta^{\leftarrow}(J)}$ *and* $\mathrm{B}_{I,J} \preccurlyeq \mathrm{B}_{I,J\vee\langle a\rangle} \approx \mathrm{B}_{\eta^{\leftarrow}(I),\eta^{\leftarrow}(J)}$.

(c) *If η is one-to-one, then $\mathrm{Z}^K_{I,J} \approx \mathrm{Z}^{\eta^{\to}(K)}_{\eta^{\to}(I),\eta^{\to}(J)}$ and consequently, $\mathrm{A}_{I,J} \approx \mathrm{A}_{\eta^{\to}(I),\eta^{\to}(J)}$ and $\mathrm{B}_{I,J} \approx \mathrm{B}_{\eta^{\to}(I),\eta^{\to}(J)}$.*

(d) *If $\eta \in F(I)$ and $\mathrm{rng}(\eta) \in J^+$, then $\mathrm{B}_{I,\eta^{\leftarrow}(J)} \preccurlyeq \mathrm{B}_{\eta^{\to}(I),J\vee\langle a\rangle}$.*

PROOF. (a) Define $\Phi : {}^{\omega}(\eta^{\leftarrow}(I)) \to {}^{\omega}I$ and $\Psi : {}^{\omega}K \to {}^{\omega}(\eta^{\leftarrow}(K))$ (respectively, $\Psi : {}^{\omega}(\eta^{\to}(K)) \to {}^{\omega}K$) by $\Phi(f)(n) = \eta[f(n)]$ and $\Psi(g)(n) = \eta^{-1}(g(n))$. If $\Phi(f) \leq^0_{\eta^{\to}(J)} g$, then there is $x \subseteq \omega$ such that $\eta^{-1}(x) \in J$ and for every n, $\eta[f(n)] \subseteq g(n) \cup x$ and so, $f(n) \subseteq \eta^{-1}(g(n)) \cup \eta^{-1}(x)$. Therefore $f \leq^0_J \Psi(g)$.

(b) $\mathrm{Z}^K_{I,J} \preccurlyeq \mathrm{Z}^K_{I,J\vee\langle a\rangle} \preccurlyeq \mathrm{Z}^{K\vee\langle a\rangle}_{I,J\vee\langle a\rangle}$ hold by Lemma 2.7 (b). Since $J \vee \langle a\rangle = \eta^{\to}(\eta^{\leftarrow}(J))$, $K \vee \langle a\rangle = \eta^{\to}(\eta^{\leftarrow}(K))$, and $\eta \in F(\eta^{\leftarrow}(J)) \cap F(\eta^{\leftarrow}(K))$, by (a2), $\mathrm{Z}^{K\vee\langle a\rangle}_{I,J\vee\langle a\rangle} = \mathrm{Z}^{\eta^{\to}(\eta^{\leftarrow}(K))}_{I,\eta^{\to}(\eta^{\leftarrow}(J))} \preccurlyeq \mathrm{Z}^{\eta^{\leftarrow}(K)}_{\eta^{\leftarrow}(I),\eta^{\leftarrow}(J)}$ We prove $\mathrm{Z}^{\eta^{\leftarrow}(K)}_{\eta^{\leftarrow}(I),\eta^{\leftarrow}(J)} \preccurlyeq \mathrm{Z}^K_{I,J\vee\langle a\rangle}$. Define $\Phi : {}^{\omega}I \to {}^{\omega}(\eta^{\leftarrow}(I))$ and $\Psi : {}^{\omega}(\eta^{\leftarrow}(K)) \to {}^{\omega}K$ by $\Phi(f)(n) = \eta^{-1}(f(n))$ and $\Psi(g)(n) = \eta[g(n)]$. If $\Phi(f) \leq^0_{\eta^{\leftarrow}(J)} g$, then there is $x \subseteq \omega$ such that $\eta[x] \in J$ and for every $n \in \omega$, $\eta^{-1}(f(n)) \subseteq g(n) \cup x$ and then $f(n) \subseteq \eta[g(n)] \cup \eta[x] \cup a$. Therefore $f \leq^0_{J\vee\langle a\rangle} \Psi(g)$.

(c) For every ideal L on ω, if $\eta \in F(L)$, then $\mathrm{rng}(\eta) \in \eta^{\to}(L)^*$ and $\eta^{\to}(L) = \eta^{\to}(L) \vee \langle a\rangle$; if η is injective, then $\eta^{\leftarrow}(\eta^{\to}(L)) = L$. Consequently by (b), $\mathrm{Z}^{\eta^{\to}(K)}_{\eta^{\to}(I),\eta^{\to}(J)} = \mathrm{Z}^{\eta^{\to}(K)}_{\eta^{\to}(I),\eta^{\to}(J)\vee\langle a\rangle} \approx \mathrm{Z}^{\eta^{\leftarrow}(\eta^{\to}(K))}_{\eta^{\leftarrow}(\eta^{\to}(I)),\eta^{\leftarrow}(\eta^{\to}(J))} = \mathrm{Z}^K_{I,J}$.

(d) $\mathrm{B}_{I,\eta^{\leftarrow}(J)} \preccurlyeq \mathrm{B}_{\eta^{\leftarrow}(\eta^{\to}(I)),\eta^{\leftarrow}(J)}$ because $\eta^{\leftarrow}(\eta^{\to}(I)) \subseteq I$. Since $\mathrm{rng}(\eta) \in \eta^{\to}(I)^+ \cap J^+$, by (b), $\mathrm{B}_{\eta^{\to}(I),J\vee\langle a\rangle} \approx \mathrm{B}_{\eta^{\leftarrow}(\eta^{\to}(I)),\eta^{\leftarrow}(J)}$. □

Theorem 2.12. *Let I and J be ideals on ω.*

(a) *If J is a weak $P(I)$-ideal, then $\mathrm{B}_{I,J} \preccurlyeq \mathrm{A}^{\perp}_{I,J}$ and hence,* $\mathrm{non}((I, J\mathrm{QN})$*-space*$) \leq \mathfrak{d}_{I,J}$.

(b) *If every ideal $\leq_{\mathrm{K}}$-below J is a weak $P(I)$-ideal, then* $\mathrm{C}_{I,J} \preccurlyeq ({}^\omega I, \leq^1_{\mathrm{Fin}})^\perp$ *and hence,* $\mathrm{non}((I, \leq_{\mathrm{K}} J\mathrm{QN})\text{-}space) \leq \mathfrak{d}({}^\omega I, \leq^1_{\mathrm{Fin}})$.

(c) *If J is a $W(I)$-ideal, then* $\mathrm{D}_{I,J} \preccurlyeq ({}^\omega I, \leq^1_{\mathrm{Fin}})^\perp$ *and hence,* $\mathrm{non}(\mathrm{w}(I, J\mathrm{QN})\text{-}space) \leq \mathfrak{d}({}^\omega I, \leq^1_{\mathrm{Fin}})$.

Proof. (a) J is a weak $P(I)$-ideal if and only if $(\forall \alpha \in F(J))(\exists \beta \in F(I))$ $\|\beta < \alpha\| \notin J$ ([13, Theorem 3.4 (a)]). For every $\alpha \in F(J)$ choose $\Psi(\alpha) \in F(I)$ such that $\Psi(\alpha) \not\leq_J \alpha$. We show that $(\mathrm{id}_{F(I)}, \Psi) : \mathrm{B}_{I,J} \to \mathrm{A}^\perp_{I,J}$ is a morphism, i.e., $\beta\ \mathrm{B}_{I,J}\ \alpha \Rightarrow \beta\ \mathrm{A}^\perp_{I,J}\ \Psi(\alpha)$ for $\beta \in F(I)$ and $\alpha \in F(J)$. Let $\beta\ \mathrm{B}_{I,J}\ \alpha$, i.e., $\beta \leq_J \alpha$. Then $\Psi(\alpha) \not\leq_J \beta$ by transitivity of $\leq_J$, i.e., $\beta\ \mathrm{A}^\perp_{I,J}\ \Psi(\alpha)$.

(b)–(c) Under the hypotheses we define $\Phi : {}^\omega I \to F(I)$, $\Psi_1 : F_2(J) \to {}^\omega I$, and $\Psi_2 : F(J) \times F(J) \to {}^\omega I$ such that

$$\Phi(f)\ \mathrm{C}_{I,J}\ (\varphi, \alpha) \Rightarrow f \not\geq^1_{\mathrm{Fin}} \Psi_1(\varphi, \alpha), \qquad f \in {}^\omega I,\ (\varphi, \alpha) \in F_2(J),$$
$$\Phi(f)\ \mathrm{D}_{I,J}\ (\varphi, \alpha) \Rightarrow f \not\geq^1_{\mathrm{Fin}} \Psi_2(\varphi, \alpha), \qquad f \in {}^\omega I,\ (\varphi, \alpha) \in F(J) \times F(J).$$

Let $\Phi(f)(k) = \min\{m \in \omega : k \in m \cup f(m)\}$ for $f \in {}^\omega I$ and $n \in \omega$. Then $\|\Phi(f) \leq n\| = n \cup \bigcup_{m \leq n} f(m) \in I$.

If all ideals $\leq_{\mathrm{K}}$-below J are weak $P(I)$-ideals, then for every $(\varphi, \alpha) \in F_2(J)$ there is a set $c_{\varphi,\alpha} \in \varphi^{\to}(J)^+$ such that $\|\alpha = n\| \cap c_{\varphi,\alpha} \in I$ for all $n \in \omega$. Let $\Psi_1(\varphi, \alpha)(n) = \|\alpha \leq n+1\| \cap c_{\varphi,\alpha}$ for all $n \in \omega$.

Assume that $\Psi_1(\varphi, \alpha) \leq^1_{\mathrm{Fin}} f$. There is $n_0 \in \omega$ such that for every $n \geq n_0$, $\|\alpha = n+1\| \cap c_{\varphi,\alpha} \subseteq f(n) \subseteq \|\Phi(f) \leq n\|$ and hence $\|\alpha = n+1\| \cap c_{\varphi,\alpha} \subseteq \|\Phi(f) < \alpha\|$. Then $\|\Phi(f) < \alpha\| \supseteq c_{\varphi,\alpha} \setminus \|\alpha \leq n_0\| \in \varphi^{\to}(J)^+$ because $\|\alpha \leq n_0\| \in \varphi^{\to}(J)$. Therefore $\neg(\Phi(f)\ \mathrm{C}_{I,J}\ (\varphi, \alpha))$.

If J is a $W(I)$-ideal, then for every $\varphi, \alpha \in F(J)$ there is a set $d_{\varphi,\alpha} \in J^+$ such that $\varphi(\|\alpha = n\| \cap d_{\varphi,\alpha}) \in I$ for all $n \in \omega$. Let $\Psi_2(\varphi, \alpha)(n) = \varphi(\|\alpha \leq n+1\| \cap d_{\varphi,\alpha})$ for all $n \in \omega$.

Assume that $\Psi_2(\varphi, \alpha) \leq^1_{\mathrm{Fin}} f$. Let $n_0 \in \omega$ be such that for every $n \geq n_0$, $\varphi(\|\alpha = n+1\| \cap d_{\varphi,\alpha}) \subseteq f(n) \subseteq \|\Phi(f) \leq n\|$ and hence $\|\alpha = n+1\| \cap d_{\varphi,\alpha} \subseteq \{k \in \omega : \Phi(f)(\varphi(k)) < \alpha(k)\} = \|\varphi \circ \Phi(f) < \alpha\|$. Then $\neg(\Phi(f)\ \mathrm{D}_{I,J}\ (\varphi, \alpha))$ because $\|\varphi \circ \Phi(f) < \alpha\| \supseteq d_{\varphi,\alpha} \setminus \|\alpha \leq n_0\| \in J^+$. □

Corollary 2.13 ([10, Theorem 2.7 (a)]). *If J is a weak P-ideal, then*

$$\mathfrak{b} \leq \max\{\mathfrak{b}_{\mathrm{Fin},J}, \mathfrak{b}_J\} \leq \mathrm{non}((\mathrm{Fin}, J\mathrm{QN})\text{-}space)$$
$$\leq \mathrm{non}((\mathrm{Fin}, \leq_{\mathrm{K}} J\mathrm{QN})\text{-}space) \leq \mathrm{non}(\mathrm{w}(\mathrm{Fin}, J\mathrm{QN})\text{-}space) \leq \mathfrak{d}.$$

Proof. Let $\kappa = \max\{\mathfrak{b}_{\mathrm{Fin},J}, \mathfrak{b}_J\}$. By Lemma 2.8 (a), twice, $({}^\omega\omega, \leq^*) \approx \mathrm{A}_{\mathrm{Fin},\mathrm{Fin}} \preccurlyeq \mathrm{A}_{\mathrm{Fin},J}$ and hence, $\mathfrak{b} \leq \mathfrak{b}_{\mathrm{Fin},J} \leq \kappa$. By Lemma 2.4, $\kappa \leq \mathfrak{b}(\mathrm{B}_{\mathrm{Fin},J})$

because $\mathrm{A}_{\mathrm{Fin},J} \preccurlyeq \mathrm{B}_{\mathrm{Fin},J}$ and $\mathrm{A}_{J,J} \preccurlyeq \mathrm{B}_{\mathrm{Fin},J}$. By Lemma 2.3, $\mathrm{B}_{\mathrm{Fin},J} \preccurlyeq \mathrm{C}_{\mathrm{Fin},J} \preccurlyeq \mathrm{D}_{\mathrm{Fin},J}$. By Lemma 1.3, J is a $W(\mathrm{Fin})$-ideal and by Theorem 2.12 (c) for $I = \mathrm{Fin}$, $\mathrm{D}_{\mathrm{Fin},J} \preccurlyeq ({}^\omega\mathrm{Fin}, \leq^1_{\mathrm{Fin}})^\perp \approx ({}^\omega\omega, \leq^*)^\perp$. Therefore $\kappa \leq \mathfrak{b}(\mathrm{B}_{\mathrm{Fin},J}) \leq \mathfrak{b}(\mathrm{C}_{\mathrm{Fin},J}) \leq \mathfrak{b}(\mathrm{D}_{\mathrm{Fin},J}) \leq \mathfrak{d}$. Now apply Theorem 2.2. □

3 Ideals related to submeasures and capacities

Recall that $\mu : \mathcal{P}(\omega) \to [0,\infty]$ is a submeasure on ω if $\mu(\emptyset) = 0$, $\mu(\{n\}) < \infty$ for all $n \in \omega$, and $\mu(a) \leq \mu(a \cup b) \leq \mu(a) + \mu(b)$ for all $a, b \subseteq \omega$. Denote $\mathrm{Fin}(\mu) = \{a \subseteq \omega : \mu(a) < \infty\}$ and $\mathrm{Exh}(\mu) = \{a \subseteq \omega : \lim_{n\in\omega} \mu(a \setminus n) = 0\}$. We say that μ is unbounded on $I \subseteq \mathrm{Fin}(\mu)$, if for every $n \in \omega$ there is $a \in I$ such that $\mu(a) \geq n$; μ is unbounded, if it is unbounded on $\mathrm{Fin}(\mu)$; μ is lower semi-continuous, if $\mu(a) = \lim_{n\in\omega} \mu(a \cap n)$ for all $a \subseteq \omega$.

Clearly, $\mathrm{Exh}(\mu)$ is an ideal on ω for every submeasure μ on ω and $\mathrm{Fin}(\mu)$ is an ideal if and only if $\mu(\omega) = \infty$. For every ideal I there is a $\{0,\infty\}$-valued submeasure μ such that $I = \mathrm{Fin}(\mu)$. However, this submeasure is not unbounded. On the other hand, if μ is unbounded, then μ is unbounded on an ideal $I \subseteq \mathrm{Fin}(\mu)$ generated by a countable family.

Proposition 3.1. *Let $I \subseteq J$ be ideals on ω.*

(a) *If $I \subseteq J \subseteq \mathrm{Fin}(\mu)$ for a submeasure μ that is unbounded on I, then $\mathrm{B}_{I,J} \preccurlyeq ({}^\omega\omega, \leq^*)$ and hence,* $\mathrm{non}((I, J\mathrm{QN})$*-space*$) \leq \mathfrak{b}$.

(b) *If $J = \bigcup_{n\in\omega} J_n$ for an increasing sequence of ideals J_n and for infinitely many $n \in \omega$, $I \cap (J_{n+1} \setminus J_n) \neq \emptyset$, then $\mathrm{B}_{I,J} \preccurlyeq ({}^\omega\omega, \leq^*)$ and hence,* $\mathrm{non}((I, J\mathrm{QN})$*-space*$) \leq \mathfrak{b}$.

Proof. (a) We find a morphism $(\Phi, \Psi) : ({}^\omega I, {}^\omega J, \leq^0_J) \to ({}^\omega\omega, \leq^*)$; this is sufficient by Lemma 2.7 (a). Fix $a_n \in I$, $n \in \omega$, such that $n \leq \mu(a_n) < \infty$. Define $\Phi : {}^\omega\omega \to {}^\omega I$ and $\Psi : {}^\omega J \to {}^\omega\omega$ by $\Phi(\alpha)(n) = a_{\alpha(n)+n}$ and $\Psi(g)(n) = \lceil \mu(g(n)) \rceil$ where $\alpha \in {}^\omega\omega$, $g \in {}^\omega J$, $n \in \omega$. Let $\Phi(\alpha) \leq^0_J g$ and let $x \in J$ be such that for every $n \in \omega$, $\Phi(\alpha)(n) \subseteq g(n) \cup x$ and hence $\alpha(n) + n \leq \mu(g(n) \cup x) \leq \Psi(g(n)) + \mu(x)$. Then $\alpha \leq^* \Psi(g)$ because $\alpha(n) \leq \Psi(g(n))$ for all $n \geq \mu(x)$.

(b) Define a submeasure μ on ω by $\mu(a) = \inf\{n \in \omega : a \in J_n\}$ for $a \subseteq \omega$ where we let $\inf \emptyset = \infty$. Then $J = \mathrm{Fin}(\mu)$ and μ is unbounded on I. □

Mazur [12] proved that an ideal J on ω is an F_σ ideal if and only if $J = \mathrm{Fin}(\mu)$ for some lower semi-continuous submeasure μ on ω such that $\mu(\omega) = \infty$. Solecki [14] proved that an ideal J on ω is an analytic P-ideal if and only if $J = \mathrm{Exh}(\mu)$ for a bounded lower semi-continuous submeasure μ on ω. Kwela [10, Theorem 2.7 (b)] proved a result paraphrasing $\mathrm{D}_{\mathrm{Fin},J} \preccurlyeq ({}^\omega\omega, \leq^*)$ for all subideals J of F_σ-ideals (see also weaker Proposition 3.1 (a)). Obviously, $\mathrm{Exh}(\mu) \subseteq \mathrm{Fin}(\mu)$ but $\mathrm{Fin}(\mu)$ is a proper F_σ ideal only if $\mu(\omega) = \infty$ and it remained open whether this result includes also all analytic P-ideals. This question led us to the following notion of capacity on ω familiar to the notion of capacity in topological spaces in sense of [3].

A capacity on ω is a function $\nu : \mathcal{P}(\omega) \to [0, \infty]$ with these properties:

(i) $\nu(\emptyset) = 0$ and $a \subseteq b \subseteq \omega$ implies $\nu(a) \leq \nu(b)$.

(ii) $\nu(a) < \infty$ and $\nu(\omega \setminus a) = \infty$ for every $a \in [\omega]^{<\omega}$.

(iii) $\lim_{n\in\omega} \nu(a \cap n) = \nu(a)$ for every $a \subseteq \omega$.

We say that an ideal J on ω is capacitous, if there is a capacity ν on ω such that $J \subseteq \mathrm{Fin}^*(\nu) = \{a \subseteq \omega : \nu(a) < \infty \text{ and } \nu(\omega \setminus a) = \infty\}$. Condition (iii) is equivalent to the next condition:

(iii′) $\nu(\bigcup_{n\in\omega} a_n) = \lim_{k\in\omega} \nu(\bigcup_{n<k} a_n)$ for any $a_n \in [\omega]^{<\omega}$.

Therefore a capacity and capacitous ideals can be considered on any infinite countable set. The ceiling of a capacity is a capacity and thus it is enough to consider capacities with values in $\omega \cup \{\infty\}$.

Lemma 3.2. *Let μ be a lower semi-continuous submeasure on ω and let I and J be ideals on ω.*

(1) *If $\omega \notin \mathrm{Fin}(\mu)$, then $\mathrm{Fin}(\mu)$ is a capacitous ideal on ω.*

(2) *If $\omega \notin \mathrm{Exh}(\mu)$, then $\mathrm{Exh}(\mu)$ is a capacitous ideal on ω.*

(3) *If J is a capacitous ideal and $I \leq_{\mathrm{K}} J$, then I is a capacitous ideal.*

(4) *Every capacitous ideal on ω is a meager subset of $\mathcal{P}(\omega) \simeq {}^\omega 2$.*

PROOF. (1) If $\mu(\omega) = \infty$, then μ is a capacity and $\mathrm{Fin}(\mu) = \mathrm{Fin}^*(\mu)$.

(2) By (1) we can assume that $\mu(\omega) < \infty$ because $\mathrm{Exh}(\mu) \subseteq \mathrm{Fin}(\mu)$. Denote $\varepsilon = \inf_{n\in\omega} \mu(\omega \setminus n)$. Since $\omega \notin \mathrm{Exh}(\mu)$, $\varepsilon > 0$ and we can define $\nu(a) = \min\{n \leq \omega : \mu(a \setminus n) \leq \varepsilon/2\}$ for all $a \subseteq \omega$. Identifying the value ω with ∞ we show that ν is a capacity on ω. Obviously, ν fulfills (i) and (ii). To verify (iii) note that, if $\nu(a) \geq k + 1$, then $\mu(a \setminus k) > \varepsilon/2$ and there is $n \in \omega$ such that $\mu((a \setminus k) \cap n) > \varepsilon/2$, and then $\nu(a \cap n) \geq k + 1$.

(3) Let $\varphi \in {}^\omega\omega$ be such that $I \subseteq \varphi^{\to}(J)$ and assume that $J \subseteq \mathrm{Fin}^*(\nu)$ for a capacity ν. Let $\nu_\varphi : \mathcal{P}(\omega) \to [0, \infty]$ be the capacity on ω defined by $\nu_\varphi(a) = \nu(\varphi^{-1}(a))$ for $a \in \mathcal{P}(\omega)$. Then $I \subseteq \varphi^{\to}(\mathrm{Fin}^*(\nu)) = \mathrm{Fin}^*(\nu_\varphi)$.

(4) Let $J \subseteq \mathrm{Fin}^*(\nu)$ be an ideal on ω. By (ii) and (iii) we can inductively define a partition $\{a_n : n \in \omega\}$ of ω into finite intervals such that $\nu(a_n) > n$. Then $\bigcup_{n\in b} a_n \notin J$ for every infinite set $b \subseteq \omega$, i.e., $\mathrm{Fin} \leq_{\mathrm{RB}} J$. By Talagrand's theorem, $\mathrm{Fin} \leq_{\mathrm{RB}} J \Leftrightarrow J$ has the Baire property (see [1, Theorem 4.1.2] and [17]; for ideals, meager and the Baire property mean the same). $\square$

Theorem 3.3. *Every ideal on ω that is $\leq_{\mathrm{K}}$-below of an F_σ-ideal or an analytic P-ideal is capacitous.*

Proof. Lemma 3.2 (1)–(3). $\square$

Example 3.4. Let T be a countable set and let I_t, $t \in T$, be ideals on ω. Define $\bigoplus_{t\in T} I_t = \{a \subseteq T \times \omega : (\forall t \in T)\ a_{(t)} \in I_t \text{ and } |\{t \in T : a_{(t)} \neq \emptyset\}| < \omega\}$ and $\sum_{t\in T} I_t = \{a \subseteq T \times \omega : (\forall t \in T)\ a_{(t)} \in I_t\}$, where $a_{(t)} = \{k \in \omega : (t,k) \in a\}$. For $t_0 \in T$ let $\varphi_{t_0} : \omega \to T \times \omega$ be defined by $\varphi_{t_0}(k) = (t_0, k)$. Then $\bigoplus_{t\in T} I_t \subseteq \sum_{t\in T} I_t \subseteq \varphi_{t_0}^{\to}(I_{t_0})$ and hence $\bigoplus_{t\in T} I_t \leq_{\mathrm{K}} \sum_{t\in T} I_t \leq_{\mathrm{K}} I_{t_0}$. If $|T| = \omega$, then $\bigoplus_{t\in T} I_t \leq_{\mathrm{K}} \mathrm{Fin}$ because $\bigoplus_{t\in T} I_t$ is not tall. Therefore, by Lemma 3.2 (3),

(1) $\bigoplus_{t\in T} I_t$ is capacitous, if $|T| = \omega$ or one of the ideals I_t is capacitous;

(2) $\sum_{t\in T} I_t$ is capacitous, if one of the ideals I_t is capacitous.

The following theorem generalizes a result of Kwela [10, Theorem 2.7 (b)] with the same proof:

Theorem 3.5. *For every capacitous ideal J on ω,*

$$\mathrm{A}_{\mathrm{Fin},J} \approx \mathrm{B}_{\mathrm{Fin},J} \approx \mathrm{C}_{\mathrm{Fin},J} \approx \mathrm{D}_{\mathrm{Fin},J} \approx ({}^\omega\omega, \leq^*)$$

and hence,

$$\begin{aligned}\mathfrak{b}_{\mathrm{Fin},J} = \mathrm{non}((\mathrm{Fin}, J\mathrm{QN})\text{-}\mathit{space}) &= \mathrm{non}((\mathrm{Fin}, \leq_{\mathrm{K}} J\mathrm{QN})\text{-}\mathit{space})\\ &= \mathrm{non}(\mathrm{w}(\mathrm{Fin}, J\mathrm{QN})\text{-}\mathit{space}) = \mathfrak{b}.\end{aligned}$$

Proof. By Lemma 2.8 (a), Lemma 2.3, and Lemma 2.4, $({}^\omega\omega, \leq^*) \preccurlyeq \mathrm{A}_{\mathrm{Fin},J} \preccurlyeq \mathrm{B}_{\mathrm{Fin},J} \preccurlyeq \mathrm{C}_{\mathrm{Fin},J} \preccurlyeq \mathrm{D}_{\mathrm{Fin},J}$. We prove $\mathrm{D}_{\mathrm{Fin},J} \preccurlyeq ({}^\omega\omega, \leq^*)$. Let ν be a capacity on ω such that $J \subseteq \mathrm{Fin}^*(\nu)$. We find $\Phi : {}^\omega\omega \to F(\mathrm{Fin})$ and $\Psi : F(J) \times F(J) \to {}^\omega\omega$ such that $\Phi(f)\ \mathrm{D}_{\mathrm{Fin},J}\ (\varphi, \alpha) \Rightarrow f \leq^* \Psi(\varphi, \alpha)$.

Define $\Phi(f)(m) = \min\{n \in \omega : m < f(n) + n\}$ for $f \in {}^\omega\omega$ and $m \in \omega$.

We can define $\Psi(\varphi, \alpha)(n) = \max\{m \in \omega : \nu(\|\varphi < m\| \setminus \|\alpha \leq n\|) \leq n\}$ for $(\varphi, \alpha) \in F(J) \times F(J)$ and $n \in \omega$ because $\lim_{k\in\omega} \nu(k \setminus \|\alpha \leq n\|) = \infty$.

Assume that $\Phi(f)\ \mathrm{D}_{\mathrm{Fin},J}\ (\varphi, \alpha)$. Then $\nu(\|\varphi \circ \Phi(f) < \alpha\|) < \infty$. For every $n \geq \nu(\|\varphi \circ \Phi(f) < \alpha\|)$ then $\nu(\|\varphi < f(n) + n\| \setminus \|\alpha \leq n\|) = \nu(\{k \in \omega : \Phi(f)(\varphi(k)) \leq n < \alpha(k)\}) \leq \nu(\|\varphi \circ \Phi(f) < \alpha\|) \leq n$ and hence $f(n) + n \leq \Psi(\varphi, \alpha)(n)$. Therefore $f \leq^* \Psi(\varphi, \alpha)$. $\square$

Corollary 3.6. *Let J be a capacitous ideal on ω.*

(a) *J is a weak P-ideal.*

(b) *For every ideal I on ω, $\mathrm{B}_{I,J} \preccurlyeq \mathrm{C}_{I,J} \preccurlyeq \mathrm{D}_{I,J} \preccurlyeq ({}^{\omega}\omega, \leq^*)$ and hence,*

$$\begin{aligned}&\mathrm{non}((I, J\mathrm{QN}\text{-}space)) \leq \mathrm{non}((I, \leq_{\mathrm{K}} J\mathrm{QN}\text{-}space))\\ &\qquad\qquad\qquad\qquad \leq \mathrm{non}(\mathrm{w}(I, J\mathrm{QN}\text{-}space)) \leq \mathfrak{b} \quad and\\ &\max\{\mathfrak{b}_{I,J}, \mathfrak{b}_J\} \leq \mathrm{non}((I, J\mathrm{QN}\text{-}space)), \quad if\ I \subseteq J.\end{aligned}$$

(c) *If J is a P-ideal, then for every ideal $I \subseteq J$, $\mathrm{A}_{I,J} \approx \mathrm{B}_{I,J} \approx \mathrm{C}_{I,J} \approx \mathrm{D}_{I,J} \approx ({}^{\omega}\omega, \leq^*)$ and hence,*

$$\begin{aligned}\mathfrak{b}_{I,J} = \mathfrak{b}_J = \mathrm{non}((I, J\mathrm{QN})\text{-}space) &= \mathrm{non}((I, \leq_{\mathrm{K}} J\mathrm{QN})\text{-}space)\\ &= \mathrm{non}(\mathrm{w}(I, J\mathrm{QN})\text{-}space) = \mathfrak{b}.\end{aligned}$$

Moreover for every ideal $I \leq_{\mathrm{K}} J$, $\mathrm{D}_{I,J} \approx ({}^{\omega}\omega, \leq^*)$ and hence,

$$\mathrm{non}(\mathrm{w}(I, J\mathrm{QN})\text{-}space) = \mathfrak{b}.$$

PROOF. (a) A capacitous ideal J is a weak P-ideal by Theorem 2.2 (a) because by Theorem 3.5, $\mathfrak{b}(\mathrm{B}_{\mathrm{Fin},J}) \leq \mathfrak{c}$.

(b) By Lemma 2.3 and Lemma 2.7 (b), $\mathrm{B}_{I,J} \preccurlyeq \mathrm{C}_{I,J} \preccurlyeq \mathrm{D}_{I,J} \preccurlyeq \mathrm{D}_{\mathrm{Fin},J}$ and, by Theorem 3.5, $\mathrm{D}_{\mathrm{Fin},J} \approx ({}^{\omega}\omega, \leq^*)$. If, moreover, $I \subseteq J$, then by Lemma 2.4, $\mathrm{A}_{I,J} \preccurlyeq \mathrm{B}_{I,J}$ and $\mathrm{A}_{J,J} \preccurlyeq \mathrm{B}_{I,J}$. This finishes the proof of (b).

(c) Let J be a $P(\mathrm{Fin})$-ideal and $I \subseteq J$. By Lemma 2.8 (a) and (c), $({}^{\omega}\omega, \leq^*) \preccurlyeq \mathrm{A}_{\mathrm{Fin},J} \approx \mathrm{A}_{I,J} \approx \mathrm{B}_{I,J}$. This together with (b) gives the first part of (c). In particular, $\mathrm{D}_{J,J} \approx \mathrm{D}_{\mathrm{Fin},J} \approx ({}^{\omega}\omega, \leq^*)$. If $I \leq_{\mathrm{K}} J$, then by Lemma 2.7 (d), $\mathrm{D}_{J,J} \preccurlyeq \mathrm{D}_{I,J} \preccurlyeq \mathrm{D}_{\mathrm{Fin},J}$ and hence $\mathrm{D}_{I,J} \approx ({}^{\omega}\omega, \leq^*)$. □

Kwela [10] proved that it is consistent with ZFC to assume the existence of an ideal J for which $\mathfrak{c} \geq \mathrm{non}(\mathrm{w}(\mathrm{Fin}, J\mathrm{QN})\text{-space}) > \mathfrak{b}$. The next proposition slightly improves his main argument and allows to prove the consistency of $\mathfrak{c} \geq \mathrm{non}((\mathrm{Fin}, \leq_{\mathrm{K}} J\mathrm{QN})\text{-space}) > \mathfrak{b}$ in this way: (1) It is consistent that there exists an ideal I such that $\mathfrak{b} < \mathfrak{b}({}^{\omega}\omega, \leq^1_I)$. (2) By Proposition 3.7 for this ideal I find an ideal J such that $\mathfrak{b}({}^{\omega}\omega, \leq^1_I) \leq \mathrm{non}((\mathrm{Fin}, \leq_{\mathrm{K}} J\mathrm{QN})\text{-space}) \leq \mathfrak{d}$. Every such ideal J is a non-capacitous weak P-ideal (J is not capacitous by Theorem 3.5 because $\mathrm{non}((\mathrm{Fin}, \leq_{\mathrm{K}} J\mathrm{QN})\text{-space}) \neq \mathfrak{b}$; J is a weak P-ideal by Theorem 2.2 (a) because $\mathrm{non}((\mathrm{Fin}, J\mathrm{QN})\text{-space}) \leq \mathrm{non}((\mathrm{Fin}, \leq_{\mathrm{K}} J\mathrm{QN})\text{-space}) \leq \mathfrak{c}$).

Proposition 3.7 ([10, Lemma 2.13]). *For every ideal I on ω there is a weak P-ideal J such that $({}^{\omega}\omega, \leq^1_I) \preccurlyeq \mathrm{C}_{\mathrm{Fin},J}$ and consequently,*

$$\mathfrak{b}({}^{\omega}\omega, \leq^1_I) \leq \mathrm{non}((\mathrm{Fin}, \leq_{\mathrm{K}} J\mathrm{QN})\text{-}space) \leq \mathrm{non}(\mathrm{w}(\mathrm{Fin}, J\mathrm{QN})\text{-}space) \leq \mathfrak{d}.$$

Proof. Fix $\alpha \in {}^{\omega}\omega$ such that $\|\alpha = n\|$ is infinite for every $n \in \omega$ and denote

$$J = J_I = \{x \subseteq \omega : \alpha(x) \in I \text{ and } (\forall^\infty n \in \omega)\ \|\alpha = n\| \cap x \in \mathrm{Fin}\}.$$

(i) We show that J is a weak P-ideal. Let $\beta \in F(J)$. By induction define $m_n \in \|\alpha = n\|$ and $k_n = \beta(m_n)$ with $a_n = \{k \in \omega : \|\alpha = n\| \cap \|\beta = k\| \notin \mathrm{Fin}\}$:

$$m_n = \begin{cases} \min(\|\alpha = n\| \setminus \bigcup\{\|\beta = k_i\| : i < n\}), & \text{if } \|\alpha = n\| \not\subseteq \bigcup_{i<n} \|\beta = k_i\|, \\ \min(\|\alpha = n\| \cap \bigcup\{\|\beta = k\| : k \in a_n\}), & \text{otherwise.} \end{cases}$$

Let $c = \{m_n : n \in \omega\}$. Then $c \notin J$ because $\alpha(c) = \omega \notin I$. For every $k \in \omega$, $\|\beta = k\| \cap c \in \mathrm{Fin}$ because $\|\alpha = n\| \cap \|\beta = k\| \notin \mathrm{Fin}$ only for finitely many n.

(ii) We find a morphism $(\Phi, \Psi) : ({}^{\omega}\omega, \leq^1_I) \to \mathrm{C}_{\mathrm{Fin},J}$. For $\beta \in F(\mathrm{Fin})$ define $\Phi(\beta) \in {}^{\omega}\omega$ by $\Phi(\beta)(n) = \min\{m \in \omega : \|\beta < n\| \subseteq m\}$. For $g \in {}^{\omega}\omega$ find a one-to-one function $\varphi \in {}^{\omega}\omega$ such that for every $n \in \omega$, φ maps $\|\alpha = n\|$ into $\|\alpha = n\| \setminus g(n)$, i.e., $\varphi(k) \geq g(n)$ for $k \in \|\alpha = n\|$. Then $(\varphi, \alpha) \in F_2(J)$ because $\|\varphi \circ \alpha = n\| = \varphi^{-1}(\|\alpha = n\|) = \|\alpha = n\| \in J$. Let $\Psi(g) = (\varphi, \alpha)$.

Let $\beta \in F(\mathrm{Fin})$ and $g \in {}^{\omega}\omega$ be such that $\Phi(\beta) \leq^1_I g$, i.e., $y = \{n \in \omega : \Phi(\beta)(n) > g(n)\} \in I$. We prove $\beta\ \mathrm{C}_{\mathrm{Fin},J}\ \Psi(g)$, i.e., $x = \varphi^{-1}(\|\beta < \alpha\|) \in J$ where $\Psi(g) = (\varphi, \alpha)$. By choice of φ for every $n \in \omega$, $\|\alpha = n\| \cap x = \varphi^{-1}(\|\alpha = n\| \cap \|\beta < \alpha\|) = \|\alpha = n\| \cap \varphi^{-1}(\|\beta < n\|)$ and this set is finite because $\beta, \varphi \in F(\mathrm{Fin})$. For every $n \in \omega \setminus y$, $\|\alpha = n\| \cap \varphi^{-1}(\|\beta < n\|) = \emptyset$ because for every $k \in \|\alpha = n\|$, $\varphi(k) \geq g(n) \geq \Phi(\beta)(n)$ and hence $\beta(\varphi(k)) \geq n$. Then $x \in J$ because $\alpha(x) \subseteq y \in I$ and $\|\alpha = n\| \cap x \in \mathrm{Fin}$ for all $n \in \omega$.

(iii) By (ii), $\mathfrak{b}({}^{\omega}\omega, \leq^1_I) \leq \mathfrak{b}(\mathrm{C}_{\mathrm{Fin},J}) = \mathrm{non}((\mathrm{Fin}, \leq_{\mathrm{K}} J\mathrm{QN})\text{-space})$. By Corollary 2.13, $\mathrm{non}((\mathrm{Fin}, \leq_{\mathrm{K}} J\mathrm{QN})\text{-space}) \leq \mathrm{non}(\mathrm{w}(\mathrm{Fin}, J\mathrm{QN})\text{-space}) \leq \mathfrak{d}$. □

The ideal J in the proof of Proposition 3.7 is meager because it is contained in the Borel ideal $\{x \subseteq \omega : (\forall^\infty n \in \omega)\ \|\alpha = n\| \cap x \in \mathrm{Fin}\} \simeq \mathrm{Fin} \times \mathrm{Fin}$. By Corollary 3.6 (a), $\mathrm{Fin} \times \mathrm{Fin}$ is not capacitous because it is not a weak P-ideal.

Question 3.8. Is there in ZFC a weak P-ideal that is not a capacitous ideal?

4 Estimations by invariants of partial orders

Let I and J be ideals on ω. If $I \subseteq J$, denote

$$\nu_{I,J} = \sup\{\mathfrak{b}_{I',J} : I \subseteq I' \subseteq J\};$$

then $\mathfrak{b}_I \leq \mathfrak{b}_{I,J} \cdot \mathfrak{b}_J \leq \nu_{I,J} \leq \mathfrak{c}$. If $I \leq_{\mathrm{K}} J$, then let

$$\kappa_{I,J} = \sup\{\mathfrak{b}_{I',J'} : I \subseteq I' \subseteq J' \leq_{\mathrm{K}} J\},$$
$$\lambda_{I,J} = \sup\{\mathfrak{b}_{I',J'} : I \leq_{\mathrm{K}} I' \subseteq J' \leq_{\mathrm{K}} J\};$$

then $\mathfrak{b}_I \leq \kappa_{I,J} \leq \lambda_{I,J} \leq \mathfrak{c}$ and $\mathfrak{b}_I \cdot \mathfrak{b}_J \leq \lambda_{I,J}$. If $I \not\leq_{\mathrm{K}} J$, then let

$$\kappa_{I,J} = \min\{\kappa_{I'\cap I,J} : I' \leq_{\mathrm{K}} J\},$$
$$\lambda_{I,J} = \min\{\lambda_{I'\cap I,J} : I' \leq_{\mathrm{K}} J\};$$

then $\kappa_{I,J} \leq \lambda_{I,J} \leq \mathfrak{c}$ and $\mathfrak{b}_J \leq \lambda_{I,J}$. By Corollary 2.6 all cardinals $\mathfrak{b}_I$ and $\mathfrak{b}_{I,J}$ are uncountable. The following holds:

Lemma 4.1. *Let I and J be arbitrary ideals on ω.*

(a) $\kappa_{I,J} = \min\{\kappa_{I'\cap I,J} : I' \leq_{\mathrm{K}} J\}$ *and* $\lambda_{I,J} = \min\{\lambda_{I'\cap I,J} : I' \leq_{\mathrm{K}} J\}$.

(b) $\omega_1 \leq \kappa_{I,J} \leq \lambda_{I,J} \leq \kappa_{\mathrm{Fin},J} = \lambda_{\mathrm{Fin},J} \leq \mathfrak{c}$.

(c) *If $I \subseteq J$, then* $\mathfrak{b}_{I,J} \cdot \mathfrak{b}_J \leq \nu_{I,J} \leq \kappa_{I,J} \leq \lambda_{I,J}$.

(d) *If $I \leq_{\mathrm{K}} J$, then* $\mathfrak{b}_I \leq \kappa_{I,J}$ *and* $\mathfrak{b}_I \cdot \mathfrak{b}_J \leq \lambda_{I,J}$.

(e) *If $I \not\leq_{\mathrm{K}} J$, then* $\mathfrak{b}_J \leq \lambda_{I,J}$. □

Obviously, if $I \leq_{\mathrm{K}} J$, then $\mathfrak{k}_{I,J} = \infty$ and $\mathfrak{k}_{I,J}$ can be omitted from the bounds in the next lemma.

Lemma 4.2. *Let I and J be ideals on ω.* (a) *If $I \subseteq J$, then* $\nu_{I,J} \leq \mathfrak{b}(\mathrm{B}_{I,J})$. (b) $\kappa_{\mathrm{Fin},J} \leq \mathfrak{b}(\mathrm{B}_{\mathrm{Fin},J})$. (c) $\min\{\mathfrak{k}_{I,J}, \kappa_{I,J}\} \leq \mathfrak{b}(\mathrm{C}_{I,J})$. (d) $\min\{\mathfrak{k}_{I,J}, \lambda_{I,J}\} \leq \mathfrak{b}(\mathrm{D}_{I,J})$.

PROOF. (a) If $I \subseteq I' \subseteq J$, then by Lemma 2.4 and Lemma 2.7 (b), $\mathfrak{b}_{I',J} = \mathfrak{b}(\mathrm{A}_{I',J}) \leq \mathfrak{b}(\mathrm{B}_{I',J}) \leq \mathfrak{b}(\mathrm{B}_{I,J})$ and hence, $\nu_{I,J} \leq \mathfrak{b}(\mathrm{B}_{I,J})$.

(b)–(c) Assume $I \leq_{\mathrm{K}} J$. By Lemma 2.3, Lemma 2.4, and Lemma 2.7 (b), (d), if $I \subseteq I' \subseteq J' \leq_{\mathrm{K}} J$, then $\mathfrak{b}_{I',J'} = \mathfrak{b}(\mathrm{A}_{I',J'}) \leq \mathfrak{b}(\mathrm{B}_{I',J'}) \leq \mathfrak{b}(\mathrm{C}_{I',J'}) \leq \mathfrak{b}(\mathrm{C}_{I,J})$ and $\mathfrak{b}(\mathrm{B}_{I',J'}) \leq \mathfrak{b}(\mathrm{B}_{\mathrm{Fin},J'}) \leq \mathfrak{b}(\mathrm{B}_{\mathrm{Fin},J})$. It follows that $\kappa_{I,J} \leq \mathfrak{b}(\mathrm{C}_{I,J})$ and $\kappa_{\mathrm{Fin},J} \leq \mathfrak{b}(\mathrm{B}_{\mathrm{Fin},J})$. This finishes the proof of (b).

Assume $I \not\leq_{\mathrm{K}} J$. Let $X \subseteq F(I)$ and $|X| < \min\{\mathfrak{k}_{I,J}, \kappa_{I,J}\}$ be arbitrary. Since $|X| < \mathfrak{k}_{I,J}$ and $\mathfrak{k}_{I,J} \geq \omega_1$, there is $\psi \in F(J)$ such that $\beta^{-1}(\{n\}) \in \psi^{\rightarrow}(J)$ for all $\beta \in X$ and $n \in \omega$. Denote $I' = \psi^{\rightarrow}(J)$. Then $\kappa_{I,J} \leq \kappa_{I'\cap I,J}$ because $I' \cap I \leq_{\mathrm{K}} J$ and by previous case, $\kappa_{I'\cap I,J} \leq \mathfrak{b}(\mathrm{C}_{I'\cap I,J})$. Since $X \subseteq F(I' \cap I)$ and $|X| < \kappa_{I,J} \leq \mathfrak{b}(\mathrm{C}_{I'\cap I,J})$ there is $(\varphi, \alpha) \in F_2(J)$ such that for all $\beta \in$

X, $\varphi^{-1}(\|\beta < \alpha\|) \in J$. Since $X \subseteq F(I)$ was arbitrary this proves that $\min\{\mathfrak{k}_{I,J}, \kappa_{I,J}\} \leq \mathfrak{b}(\mathrm{C}_{I,J})$.

(d) Similar to (c): If $I \leq_{\mathrm{K}} J$, consider $I \leq_{\mathrm{K}} I' \subseteq J' \leq_{\mathrm{K}} J$ and in all arguments replace $\mathrm{C}_{I,J}$ and $\kappa_{I,J}$ by $\mathrm{D}_{I,J}$ and $\lambda_{I,J}$. □

In the following theorem the arrow $\to$ denotes the inequality $\leq$ between the cardinals:

Theorem 4.3. *For any ideals I and J on ω the following holds:*

$$\begin{array}{ccccccc} & & \mathrm{non}((I, \leq_{\mathrm{K}} J\mathrm{QN})\text{-}space) & \to & \mathrm{non}(\mathrm{w}(I, J\mathrm{QN})\text{-}space) & \to \mathfrak{k}_{I,J} & \to \infty \\ & & \uparrow & & \uparrow & \uparrow & \\ \omega_1 & \to & \min\{\mathfrak{k}_{I,J}, \kappa_{I,J}\} & \to & \min\{\mathfrak{k}_{I,J}, \lambda_{I,J}\} & \to \min\{\mathfrak{k}_{I,J}, \mathfrak{c}\} & \end{array}$$

Proof. By [13, Lemma 3.10 (a)], $\mathfrak{p} \leq \mathfrak{k}_{I,J}$ and by Lemma 4.1 (b), $\omega_1 \leq \kappa_{I,J} \leq \lambda_{I,J} \leq \mathfrak{c}$. This proves the bottom row of the diagram. Lemma 2.3, $\mathfrak{b}(\mathrm{C}_{I,J}) \leq \mathfrak{b}(\mathrm{D}_{I,J}) \leq \mathfrak{b}(\mathrm{E}_{I,J})$ which by Theorem 2.2 proves the top row. The vertical inequalities proves Lemma 4.2 (c) and (d). □

Theorem 4.4. *Let I and J be ideals on ω.*

(a) *If $I \subseteq J$ and J is a weak $P(I)$-ideal, then*

$$\mathfrak{b}_{I,J} \cdot \mathfrak{b}_J \leq \nu_{I,J} \leq \mathrm{non}((I, J\mathrm{QN})\text{-}space) \leq \mathfrak{d}_{I,J}.$$

(b) *If every ideal $\leq_{\mathrm{K}}$-below J is a weak $P(I)$-ideal, then*

$$\min\{\mathfrak{k}_{I,J}, \kappa_{I,J}\} \leq \mathrm{non}((I, \leq_{\mathrm{K}} J\mathrm{QN})\text{-}space) \leq \min\{\mathfrak{k}_{I,J}, \mathfrak{d}({}^{\omega}I, \leq^1_{\mathrm{Fin}})\}.$$

(c) *If J is a $W(I)$-ideal, then*

$$\min\{\mathfrak{k}_{I,J}, \lambda_{I,J}\} \leq \mathrm{non}(\mathrm{w}(I, J\mathrm{QN})\text{-}space) \leq \min\{\mathfrak{k}_{I,J}, \mathfrak{d}({}^{\omega}I, \leq^1_{\mathrm{Fin}})\}.$$

Proof. By Lemma 2.3, $\mathfrak{b}(\mathrm{C}_{I,J}) \leq \mathfrak{b}(\mathrm{D}_{I,J}) \leq \mathfrak{k}_{I,J}$. Therefore the upper bounds follow by Theorem 2.12 and by Theorem 2.2. By Theorem 2.2, the lower bounds follow by Lemma 4.2 (a), (c), (d) and Lemma 4.1 (c). □

For a pair of ideals I, J on ω consider the following conditions:

$H_1(I,J) \Leftrightarrow (\exists K$ a P-ideal$)\ I \subseteq K \subseteq J$.
$H_2(I,J) \Leftrightarrow (\forall I' \leq_{\mathrm{K}} J)(\exists K$ a P-ideal$)\ I' \cap I \subseteq K \leq_{\mathrm{K}} J$.
$H_3(I,J) \Leftrightarrow (\forall I' \leq_{\mathrm{K}} J)(\exists K$ a P-ideal$)\ I' \cap I \leq_{\mathrm{K}} K \leq_{\mathrm{K}} J$.

Obviously, $H_1(I,J) \Rightarrow H_2(I,J)$, $H_2(I,J) \Rightarrow H_3(I,J)$ and, if $I \leq_{\mathrm{K}} J$, then

$$H_2(I,J) \Leftrightarrow (\exists K \text{ a } P\text{-ideal})\ I \subseteq K \leq_{\mathrm{K}} J,$$
$$H_3(I,J) \Leftrightarrow (\exists K \text{ a } P\text{-ideal})\ I \leq_{\mathrm{K}} K \leq_{\mathrm{K}} J.$$

By [13, Lemma 3.10 (a)], $\mathfrak{k}_{I,J} \geq \mathfrak{p}$ and therefore $\min\{\mathfrak{k}_{I,J}, \mathfrak{b}\} \geq \mathfrak{p}$.

Lemma 4.5. *Let I and J be ideals on ω.*

(a) $H_1(I,J)$ *implies* $\mathfrak{b} \leq \nu_{I,J} \leq \mathrm{non}((I,J\mathrm{QN})$*-space*$)$.

(b) $H_2(I,J)$ *implies* $\mathfrak{b} \leq \kappa_{I,J}$ *and* $\min\{\mathfrak{k}_{I,J}, \mathfrak{b}\} \leq \mathrm{non}((I, \leq_{\mathrm{K}} J\mathrm{QN})$*-space*$)$.

(c) $H_3(I,J)$ *implies* $\mathfrak{b} \leq \lambda_{I,J}$ *and* $\min\{\mathfrak{k}_{I,J}, \mathfrak{b}\} \leq \mathrm{non}(\mathrm{w}(I,J\mathrm{QN})$*-space*$)$.

Proof. (a) By Lemma 2.8 (a), $\mathfrak{b} \leq \mathfrak{b}_{\mathrm{Fin},J}$. If K is a P-ideal such that $I \subseteq K \subseteq J$, then by Lemma 2.8 (c), $\mathfrak{b}_{\mathrm{Fin},J} = \mathfrak{b}_{I,J}$ while $\mathfrak{b}_{I,J} \leq \nu_{I,J}$. By Lemma 4.2 (a), $\nu_{I,J} \leq \mathrm{non}((I,J\mathrm{QN})\text{-space})$.

(b) The proof of $\mathfrak{b} \leq \kappa_{I,J}$ is similar to the proof of $\mathfrak{b} \leq \lambda_{I,J}$ in (c) and we leave it to the reader. Then by Lemma 4.2 (c), $\min\{\mathfrak{k}_{I,J}, \mathfrak{b}\} \leq \min\{\mathfrak{k}_{I,J}, \kappa_{I,J}\} \leq \mathfrak{b}(\mathrm{C}_{I,J})$. Apply Theorem 2.2 (c).

(c) To prove $\mathfrak{b} \leq \lambda_{I,J}$ it is enough to prove $\mathfrak{b} \leq \lambda_{I' \cap I,J}$ for all $I' \leq_{\mathrm{K}} J$ because by Lemma 4.1 (a), $\lambda_{I,J} = \min\{\lambda_{I' \cap I,J} : I' \leq_{\mathrm{K}} J\}$. Let $I' \leq_{\mathrm{K}} J$. By $H_3(I,J)$ there is a P-ideal K such that $I \cap I' \leq_{\mathrm{K}} K \leq_{\mathrm{K}} J$. Hence $\mathfrak{b}_{K,K} \leq \lambda_{I' \cap I,J}$ and by Lemma 2.8 (d), $\mathfrak{b} \leq \mathfrak{b}_{K,K}$. By Lemma 4.2 (d), $\min\{\mathfrak{k}_{I,J}, \mathfrak{b}\} \leq \min\{\mathfrak{k}_{I,J}, \lambda_{I,J}\} \leq \mathfrak{b}(\mathrm{D}_{I,J})$. Apply Theorem 2.2 (b). □

Corollary 4.6. *Let J be a capacitous ideal on ω.*

(a) $\nu_{I,J} \leq \mathfrak{b}$ *for every ideal $I \subseteq J$ and $\kappa_{I,J} \leq \lambda_{I,J} \leq \mathfrak{b}$ for every ideal I.*

(b) $H_1(I,J)$ *implies* $\mathrm{non}((I,J\mathrm{QN})$*-space*$) = \nu_{I,J} = \mathfrak{b}$.

(c) $H_2(I,J)$ *implies* $\kappa_{I,J} = \lambda_{I,J} = \mathfrak{b}$ *and*

$$\mathrm{non}((I, \leq_{\mathrm{K}} J\mathrm{QN})\text{-}space) = \mathrm{non}(\mathrm{w}(I,J\mathrm{QN})\text{-}space) = \min\{\mathfrak{k}_{I,J}, \mathfrak{b}\}.$$

(d) $H_3(I,J)$ *implies* $\lambda_{I,J} = \mathfrak{b}$ *and* $\mathrm{non}(\mathrm{w}(I,J\mathrm{QN})$*-space*$) = \min\{\mathfrak{k}_{I,J}, \mathfrak{b}\}$.

Proof. (a) Each of the cardinals $\nu_{I,J}$, $\kappa_{I,J}$, $\lambda_{I,J}$ is supremum of a subset of the set $\{\mathfrak{b}_{I',J'} : I' \subseteq J' \leq_{\mathrm{K}} J\}$. By Lemma 3.2 (3) every ideal $J' \leq_{\mathrm{K}} J$ is capacitous and, by Corollary 3.6 (b), $\mathfrak{b}_{I',J'} \leq \mathfrak{b}$ for every $I' \subseteq J'$. Therefore $\nu_{I,J} \leq \mathfrak{b}$ for every ideal $I \subseteq J$ and $\kappa_{I,J} \leq \lambda_{I,J} \leq \mathfrak{b}$ for every ideal I on ω.

(b) By Lemma 4.5 (a) and Corollary 3.6 (b).

(c) The equalities $\kappa_{I,J} = \lambda_{I,J} = \mathfrak{b}$ follow by (a) and Lemma 4.5 (b). By Lemma 4.5 (b), $\min\{\mathfrak{k}_{I,J}, \mathfrak{b}\}$ is a lower bound of $\mathrm{non}(\dots)$, by Corollary 3.6 (b), $\mathfrak{b}$ is an upper bound, and by Theorem 4.3, $\mathfrak{k}_{I,J}$ is an upper bound, too.

(d) In the proof of (c) replace Lemma 4.5 (b) by Lemma 4.5 (c). □

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