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GOOD SEQUENCES FOR SACKS FORCING

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ABSTRACT. We introduce an ω-closed partially ordered set $\mathbb{P}^*_{\mathrm{good}}$ and prove that if it is κ-distributive then Sacks forcing is $(\kappa, \mathfrak{c}, \omega)$-distributive. Moreover, we prove that PFA implies that $\mathbb{P}^*_{\mathrm{good}}$ is $\mathfrak{c}$-distributive. We consider also some related partial orders, examine regularity properties for them, and find complete embeddings of the corresponding complete Boolean algebras.

1. Introduction

In [6] the authors proved that under Martin's axiom the least cardinal to which Sacks forcing $\mathbb{S}$ collapses the continuum is the additivity of Marczewski ideal and Martin's axiom does not prevent from collapsing of the continuum by $\mathbb{S}$. Namely, it is consistent with Martin's axiom that $\mathfrak{c} > \omega_1$ and $\mathfrak{c}$ is collapsed to ω_1. The collapse of the continuum can be expressed via $(\kappa, \mathfrak{c}, \mathfrak{c})$-distributivity of $\mathbb{S}$. We use the definition of the three-parameter distributivity from [1] and define

$$\mathrm{sh}(\lambda, \mathbb{P}) = \min\{\kappa \colon \mathbb{P} \text{ is nowhere } (\kappa, \mathfrak{c}, \lambda)\text{-distributive}\}.$$

If $\mathfrak{d} = \mathfrak{c}$, then $\mathrm{sh}(\lambda, \mathbb{S}) = \mathrm{sh}(\mathfrak{c}, \mathbb{S})$ for all $\omega_1 \le \lambda \le \mathfrak{c}$ because there are no small uncountable antichains in $\mathbb{S}$, see [8]. It is well known that $\mathrm{sh}(\omega, \mathbb{S}) \ge \omega_1$. We introduce an ω-closed partially ordered set $\mathbb{P}^*_{\mathrm{good}}$ and prove that $\mathrm{sh}(\omega, \mathbb{P}^*_{\mathrm{good}}) \le \mathrm{sh}(\omega, \mathbb{S})$ and, under PFA, $\mathrm{sh}(2, \mathbb{P}^*_{\mathrm{good}}) = \mathfrak{c} = \omega_2$. We consider also some other related partial orders, examine certain regularity properties for them and prove the existence of complete embeddings for the corresponding complete Boolean algebras.

If $\langle P, \le \rangle$ is a partial ordering, then $p, q \in P$ are said to be *compatible*, if there is $r \in P$ such that $r \le p$ and $r \le q$. If p, q are not compatible we say that they are *incompatible* and write $p \perp q$. A set $A \subseteq P$ is *predense* if for every $p \in P$ there is $q \in A$ compatible with p. A set $A \subseteq P$ is *predense below* $r \in P$ if for every $p \le r$ there is $q \in A$ compatible with p.

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A Boolean algebra is (κ, μ, λ)-distributive if for every system of maximal antichains $\{A_\alpha : \alpha < \kappa\}$ with each $|A_\alpha| \le \mu$ there is a maximal antichain $A \in B$ such that for each $p \in A$ and each $\alpha < \kappa$, $|\{q \in A_\alpha : p \not\perp q\}| < \lambda$. This definition can be applied to any "reasonable" partial ordering P which, in particular, can be viewed as a dense subset of the complete Boolean algebra $\mathrm{r.o.}(P)$. It is clear that the meaning of distributivity can change when passing to a dense subset but this is not the case when P has a dense subset of size $\le \mu$. This is one of the reasons why we consider only $(\kappa, \mathfrak{c}, \mu)$-distributivity for partial orderings of size $\mathfrak{c}$.

We denote $\mathrm{Seq} = {}^{<\omega}2$ (the set of finite sequences of 0's and 1's) and $\mathrm{Seq}^+ = \mathrm{Seq} \setminus \{\emptyset\}$. We say that a set $p \subseteq \mathrm{Seq}$ is a perfect tree if (i) $p \neq \emptyset$, (ii) $s{\restriction}k \in p$ for every $s \in p$ and $k \in \omega$, and (iii) for every $s \in p$ there is a splitting node $t \in p$ above s, i.e., a $t \supseteq s$ such that both $t^\frown 0$ and $t^\frown 1$ belong to p.

If p is a perfect tree and $s \in p$, then also $(p)_s = \{t \in p : s \subseteq t \text{ or } t \subseteq s\}$ is a perfect tree. Let $\mathbb{S}$ denote the set of all perfect trees in Seq ordered by $p \le q$ if and only if $p \subseteq q$. For $p \in \mathbb{S}$ let $\mathrm{stem}\, p$ be the minimal splitting node of p and inductively let us define

$$\mathrm{split}^0(p) = \{\mathrm{stem}\, p\},$$

$$\mathrm{split}^{k+1}(p) = \bigcup \{\mathrm{stem}((p)_{s^\frown i}) : s \in \mathrm{split}^k(p) \text{ and } i \in \{0,1\}\},$$

$$\mathrm{split}(p) = \bigcup_{k \in \omega} \mathrm{split}^k(p) = \{s \in p : s^\frown 0 \in p \text{ and } s^\frown 1 \in p\}.$$

For $p, q \in \mathbb{S}$ and $n \in \omega$ we say that $p \le_n q$ if $p \le q$ and $\mathrm{split}^n(p) = \mathrm{split}^n(q)$.

Let us recall that perfect trees correspond to perfect subsets of ${}^\omega 2$ via the equality $[p] = \{x \in {}^\omega 2 : (\forall n)\ x{\restriction}n \in p\}$ and Seq determines a basis of clopen sets in ${}^\omega 2$ consisting of $[s] = \{x \in {}^\omega 2 : s \subseteq x\}$ for $s \in \mathrm{Seq}$.

Let P, Q be partially ordered sets. A mapping $i : P \to Q$ is a *complete embedding* (see [7]) if

(1) $(\forall p, p' \in P)\ p' \le p \to i(p') \le i(p)$,
(2) $(\forall p, p' \in P)\ p' \perp p \to i(p') \perp i(p)$,
(3) $(\forall q \in Q)(\exists p \in P)(\forall p' \le p)\ i(p') \not\perp q$.

A function $\pi : Q \to P$ is *normal* (see [3]) if

(1) $(\forall q, q' \in Q)\ q' \le q \to \pi(q') \le \pi(q)$,
(2) $\pi``Q$ is dense in P,
(3) $\pi``\{p \in Q : p \le q\}$ is dense below $\pi(q)$.

Let us recall that for every partially ordered set P there is a canonical embedding into a complete Boolean algebra $\mathrm{r.o.}(P)$, i.e., an ordering preserving mapping $e : P \to \mathrm{r.o.}(P) \setminus \{0\}$, such that $e``P$ is dense in $\mathrm{r.o.}(P)$ and $p, q \in P$

are compatible if and only if $e(p)$, $e(q)$ are compatible (see [4]). If P is a separative partially ordered set (i.e., for $p \not\leq q$ there is $r \leq p$ incompatible with q), then the embedding e is one-to-one and hence P can be identified with a dense subset of $\mathrm{r.o.}(P)$.

Every complete embedding of Boolean algebras is a complete embedding in the above sense and the projection of a complete Boolean algebra on its subalgebra is a normal function. Conversely, if there is a complete embedding $i\colon P \to Q$ or if there is a normal function $\pi\colon Q \to P$, then $\mathrm{r.o.}(P)$ is (isomorphic to) a complete subalgebra of $\mathrm{r.o.}(Q)$. In such case we write $P \preceq Q$. If $P \preceq Q$ and $Q \preceq P$, then we write $P \approx Q$.

The main motivation for writing the present paper was a result in [6] saying that under Martin's axiom Miller forcing (i.e., superperfect tree forcing) does not collapse cardinals. In the proof of this result good sequences for Miller forcing were introduced and the crucial fact in the proof was that the ordering of good sequences is $\mathfrak{c}$-closed under Martin's axiom. In the present paper we show that there is an analogy between Sacks forcing and Miller forcing. We introduce good sequences for Sacks forcing, prove that an ordering of good sequences is ω-closed and under PFA it is $\mathfrak{c}$-closed, and we show that it has an effect on distributivity properties of Sacks forcing. At the same time we consider several related forcing notions, prove regularity properties for them, and prove the existence of complete embeddings between some of them.

2. Good sequences

We say that a function $f\colon \mathrm{Seq}^+ \to \mathrm{Seq}^+$ is a *good sequence* if $s \subseteq f(s)$ for every $s \in \mathrm{Seq}^+$. This notion is motivated by good sequences for superperfect trees in [6]. For a good sequence f and $s \in \mathrm{Seq}$ let us define inductively

$$\begin{aligned} S_s^0(f) &= \{s\}, \\ S_s^{k+1}(f) &= \{f(s^\frown i)\colon s \in S_s^k \text{ and } i \in \{0,1\}\}, \\ S_s(f) &= \bigcup_{k\in\omega} S_s^k(f). \end{aligned}$$

With a good sequence f we associate the sequence of perfect trees $\langle p_s(f)\colon s \in \mathrm{Seq}\rangle$ where $p_s(f)$ is the unique tree with $\mathrm{split}(p_s(f)) = S_s(f)$. We consider the following orderings of good sequences defined by

$$\begin{aligned} &f \leq g \quad \text{if } (\forall s \in \mathrm{Seq})\ p_s(f) \leq p_s(g), \\ &f \leq^* g \quad \text{if } (\forall^\infty s \in \mathrm{Seq})\ p_s(f) \leq p_s(g), \\ &f \leq^{**} g \text{ if } (\forall t \in \mathrm{Seq})(\forall^\infty s \in S_t(f))\ p_s(f) \leq p_s(g), \\ &f \leq^{\times} g \text{ if } (\forall t \in \mathrm{Seq})(\forall^\infty s \in S_t(f))\ p_s(f) = p_s(g). \end{aligned}$$

Clearly, $f \le g$ implies $f \le^* g$ and this implies $f \le^{**} g$, and $f \le^\times g$ implies $f \le^{**} g$. The transitivity of $\le$ and $\le^*$ is straightforward. To see that $\le^{**}$ is transitive let us assume that $f \le^{**} g$ and $g \le^{**} h$. Let $t \in \mathrm{Seq}$ be given. There is $n \in \omega$ such that $p_u(f) \le p_u(g)$ for all $u \in S^n_t(f)$. For every $u \in S^n_t(f)$ let $m_u \in \omega$ be such that $p_v(g) \le p_v(h)$ for all $v \in S^{m_u}_u(g)$ and let $m = \max\{m_u : u \in S^n_t(f)\}$. Then if $s \in S^k_t(f)$ with $k \le n + m$, then there are $u \in S^n_t(f)$ and $v \in S^{m_u}_u(f)$ such that $v \subseteq s$. As $p_u(f) \le p_u(g)$, $p_v(f) \le p_v(g) \le p_v(h)$. As $s \in S_v(f)$ it follows that $p_s(f) \le p_s(g) \le p_s(h)$. A similar proof shows that $\le^\times$ is transitive and that

$$f \le^{**} g \text{ and } g \le^{**} f \quad \text{if and only if} \quad f \le^\times g \text{ and } g \le^\times f .$$

Let P_{good} be the set of all good sequences and let $\mathbb{P}_{\mathrm{good}}$, $\mathbb{P}^*_{\mathrm{good}}$, and $\mathbb{P}^{**}_{\mathrm{good}}$ denote the partially ordered sets $\langle P_{\mathrm{good}}, \le \rangle$, $\langle P_{\mathrm{good}}, \le^* \rangle$, $\langle P_{\mathrm{good}}, \le^{**} \rangle$, respectively.

For $\mathcal{F} \subseteq P_{\mathrm{good}}$ and $f, g \in P_{\mathrm{good}}$ let us define

$$f \le_{\mathcal{F}} g \text{ if } (\forall h \in \mathcal{F})(\forall t \in \mathrm{Seq})\big(\forall^\infty s \in S_t(h)\big)\ p_s(h) \le p_s(g),$$
$$f =_{\mathcal{F}} g \text{ if } f \le_{\mathcal{F}} g \text{ and } g \le_{\mathcal{F}} f .$$

Then

$$f \le^* g \Leftrightarrow f \le_{\{\mathrm{id}\}} g, \quad f \le^{**} g \Leftrightarrow f \le_{\{f\}} g, \quad f \le^\times g \Leftrightarrow f =_{\{f\}} g,$$

and the above proof of transitivity of $\le^{**}$ shows that if $g \le^{**} h$ then $g \le_{\{f\}} h$ for all $f \le^{**} g$, i.e., $g \le_{\{f \in P_{\mathrm{good}} :\ f \le^{**} g\}} h$.

LEMMA 2.1. *Let $f, g \in P_{\mathrm{good}}$.*

(1) $f \le g$ *if and only if* $(\forall s \in \mathrm{Seq})\big(\forall i \in \{0,1\}\big)\ f(s^\frown i) \in S_s(g)$.

(2) *The following conditions are equivalent:*

(a) $f \le^* g$.

(b) *There exists $f' \in P_{\mathrm{good}}$ such that $f =^* f' \le g$.*

(c) $(\forall^\infty s \in \mathrm{Seq})\big(\forall i \in \{0,1\}\big)\ f(s^\frown i) \in S_s(g)$.

(3) *The following conditions are equivalent:*

(a) $f \le^{**} g$.

(b) *There exists $f' \in P_{\mathrm{good}}$ such that $f \le^\times f' \le g$.*

(c) $(\forall t \in \mathrm{Seq})\big(\forall^\infty s \in S_t(f)\big)\big(\forall i \in \{0,1\}\big)\ f(s^\frown i) \in S_s(g)$.

Proof. (1) is by an easy induction since $p_s(f) \le p_s(g)$ if and only if $S^n_s(f) \subseteq S_s(g)$ for every $n \in \omega$.

(2) The implications (b)$\to$(a)$\to$(c)$\to$(a) are obvious. For (a)$\to$(b) let us assume that $f \leq^* g$ and let us set

$$f'(s^\frown i) = \begin{cases} f(s^\frown i) & \text{if } p_s(f) \leq p_s(g), \\ g(s^\frown i) & \text{otherwise.} \end{cases}$$

Then $f =^* f'$ and $f'(s^\frown i) \in S_s(g)$ for all $s \in \mathrm{Seq}$. So, by (1), $f' \leq g$.

(3) We use the same arguments as for (2). For the implication (a)$\to$(b), if $f \leq^{**} g$ and $t \in \mathrm{Seq}$, then for all but finitely many $s \in S_t(f)$, $p_s(f) \leq p_s(g)$ and hence $f(s^\frown i) = f'(s^\frown i)$. Consequently, $p_s(f) = p_s(f')$ for all but finitely many $s \in S_t(f)$. □

Lemma 2.2.

(1) *Let* $\langle q_s : s \in \mathrm{Seq}\rangle$ *be a sequence of perfect trees such that* $\mathrm{stem}\, q_s = s$ *for every* s. *There exists* $h \in P_{\mathrm{good}}$ *such that for every* s *there exists* $r_s \subseteq s$ *such that* $p_s(h) \leq (q_{r_s})_s$. *Moreover, for* $f \in P_{\mathrm{good}}$ *we have:*

- (a) *If* $(\forall s \in \mathrm{Seq})\ q_s \leq p_s(f)$, *then* $h \leq f$.
- (b) *If* $(\forall^\infty s \in \mathrm{Seq})(\forall t \subseteq s,\ s \in \mathrm{split}(q_t))\ (q_t)_s \leq p_s(f)$, *then* $h \leq^* f$.
- (c) *If* $(\forall t \in \mathrm{Seq})(\forall^\infty s \in \mathrm{split}(q_t))\ (q_t)_s \leq p_s(f)$, *then* $h \leq^{**} f$.

(2) *Let* $S_{f,g} = \{s\colon$ *there is a perfect tree* $q_s \subseteq p_s(f) \cap p_s(g)$ *with* $s = \mathrm{stem}(q_s)\}$. *Then* f, g *are* $\leq$*-compatible* ($\leq^*$*-compatible*, $\leq^{**}$*-compatible, resp.) if and only if* $S_{f,g} = \mathrm{Seq}$ ($\mathrm{Seq} \setminus S_{f,g}$ *is finite,* $(\forall t \in \mathrm{Seq})$ $(\exists s \in S_{f,g})\ t \subseteq s$, *resp.*).

(3) *For a finite set* $A \subseteq P_{\mathrm{good}}$ *and* $g \in P_{\mathrm{good}}$ *let* $S_{A,g} = \{s\colon \{p_s(f)\colon f \in A\}$ *is predense below* $p_s(g)$ *in* $\mathbb{S}\}$. *If* A *is* $\leq$*-predense* ($\leq^*$*-predense,* $\leq^{**}$*-predense, resp.) below* g, *then* $S_{A,g} = \mathrm{Seq}$ ($\mathrm{Seq} \setminus S_{A,g}$ *is finite,* $(\forall t \in \mathrm{Seq})(\exists s \in S_{A,g})\ t \subseteq s$, *resp.*).

Proof. (1) There are perfect trees $q'_s \leq q_s$ so that the sets $\mathrm{split}(q'_s) \setminus \{s\}$, $s \in \mathrm{Seq}$, are pairwise disjoint. To see this, let $\langle s_n : n \in \omega\rangle = \mathrm{Seq}$ and on nth step of the inductive construction let us choose disjoint sets $\mathrm{split}^{n-i}(q'_{s_i}) \subseteq \mathrm{split}(q_{s_i})$, $i < n$, which are disjoint from $\bigcup_{j<n-1} \mathrm{split}^j(q'_{s_i})$. Now let $r_s \subseteq s$ be a minimal such that $s \in \mathrm{split}(q'_{r_s})$ and let $h(s^\frown 0)$ and $h(s^\frown 1)$ be minimal splitting nodes of q'_{r_s} above $s^\frown 0$ and $s^\frown 1$, respectively. By the disjointness property of q'_s, $s \in \mathrm{Seq}$, it follows that for every splitting node u of q'_{r_s}, $r_u = r_s$. Hence $p_s(h) = (q'_{r_s})_s \leq (q_{r_s})_s$. In the case (a), if $q_{r_s} \leq p_{r_s}(f)$, then s is a splitting node of $p_{r_s}(f)$, so $p_s(f) = \big(p_{r_s}(f)\big)_s$ and hence $p_s(h) \leq p_s(f)$. In the case (b), by the assumption, $(q_{r_s})_s \leq p_s(f)$ for all but finitely many $s \in \mathrm{Seq}$, and consequently, $h \leq^* f$. In the case (c), by the assumption, $(\forall t \in \mathrm{Seq})(\forall^\infty s \in \mathrm{split}\big((q'_{r_t})_t\big)$ $(q'_{r_t})_s \leq p_s(f)$ and hence $h \leq^{**} f$ (as $\big(\mathrm{split}(q'_{r_t})\big)_t = S_t(h)$ and $(q'_{r_t})_s = p_s(h)$).

(2) It is enough to prove only the "if" part of the equivalences. For each $s \in S_{f,g}$ let $q_s \subseteq p_s(f) \cap p_s(g)$ be a perfect tree with stem s. For $s \in \mathrm{Seq}$ and $i \in \{0,1\}$ choose $t_{s,i} \in S_{f,g}$ such that $s^\frown i \subseteq t_{s,i}$ and for $s \in \mathrm{Seq} \setminus S_{f,g}$ let $q_s = q_{t_{s,0}} \cup q_{t_{s,1}}$. Let $h \in P_{\mathrm{good}}$ be obtained by applying assertion (1) to the system $\langle q_s \colon s \in \mathrm{Seq}\rangle$. Then the assumption of the corresponding assertion (a), (b), or (c) is satisfied both for f and for g and so h is a lover bound for f and g in the corresponding ordering.

(3) For $s \in \mathrm{Seq} \setminus S_{A,g}$ there is $t_s \in S_s(g)$ such that for some i_s, $s^\frown i_s \subseteq t_s$ and $[t_s] \cap \bigcup_{f \in A} [p_s(f)] = \emptyset$. Let us define $h \le g$ by

$$h(s^\frown i) = \begin{cases} t_s & \text{if } s \in \mathrm{Seq} \setminus S_{A,g} \text{ and } i = i_s\,, \\ g(s^\frown i) & \text{otherwise}\,. \end{cases}$$

Now, if $s \in S_{f,h}$ for some $f \in A$, then the values $h(s^\frown 0)$ and $h(s^\frown 1)$ are not obtained by the first rule and hence $s \in S_{A,g}$. Therefore $\bigcup_{f \in A} S_{f,h} \subseteq S_{A,g}$ and the assertion follows by (2). □

Remark 2.3. The characterization for $\le^{**}$-compatibility in Lemma 2.2 (2) can be weakened by replacing $S_{f,g}$ by the set $S'_{f,g} = \{s \in \mathrm{Seq} \colon p_s(f) \text{ and } p_s(g) \text{ are compatible}\}$ because $S_{f,g}$ is cofinal in Seq if and only if $S'_{f,g}$ is. However, it is not possible to do this replacement in the characterizations for $\le$-compatibility and $\le^*$-compatibility.

P r o o f . To see the second assertion of the remark let us define for $s^\frown i \in \mathrm{Seq}^+$,

$$f(s^\frown i) = s^\frown i^\frown i\,, \qquad g(s^\frown i) = \begin{cases} s^\frown 0^\frown 1 & \text{if } i = 0 \text{ and } 1 \notin \mathrm{rng}(s)\,, \\ s^\frown i^\frown i & \text{otherwise}\,. \end{cases}$$

Then $p_s(f)$ and $p_s(g)$ are compatible for all s, namely, $p_s(f) \cap p_s(g) \supseteq p_{s^\frown 1^\frown 1}(f)$ for all s. But if $h \in P_{\mathrm{good}}$ then for every s of the form $s = 0^\frown \dots ^\frown 0$, $s^\frown 0^\frown 0 \in p_s(f) \setminus p_s(g)$ and hence if $p_s(h) \le p_s(f)$, then $p_s(h) \not\le p_s(g)$. □

THEOREM 2.4. *$\mathbb{P}_{\mathrm{good}}$ and $\mathbb{P}^*_{\mathrm{good}}$ are separative partially ordered sets.*

P r o o f . Let $f, g \in P_{\mathrm{good}}$ and let $S = \{s \in \mathrm{Seq} \colon p_s(f) \not\le p_s(g)\}$. For $\mathbb{P}_{\mathrm{good}}$, $\mathbb{P}^*_{\mathrm{good}}$, and $\mathbb{P}^{**}_{\mathrm{good}}$ we assume that $S \ne \emptyset$, $|S| = \omega$, and $|S \cap S_t(f)| = \omega$ for some $t \in \mathrm{Seq}$, respectively. For each $s \in S$ let us fix $v(s) \in S_s(f) \setminus S_s(g)$ and let us define $h \le f$ by

$$h(s^\frown i) = \begin{cases} v(s) & \text{if } s \in S \text{ and } s^\frown i \subseteq v(s)\,, \\ f(s) & \text{otherwise}\,. \end{cases}$$

It is obvious that for every $s \in S$ there is no perfect tree $q_s \subseteq p_s(h) \cap p_s(g)$ with $\text{stem}(q_s) = s$. Therefore h and g are incompatible (in the appropriate sense). □

Remark 2.5. $\mathbb{P}^{**}_{\text{good}}$ is not separative.

Proof. Let $p \subseteq \text{Seq}$ be a perfect tree with $\text{stem}(p) = \emptyset$ such that $[p]$ is nowhere dense in ${}^{\omega}2$. For each $s \in \text{Seq}$ choose $t_s \supseteq s$ such that $t_s \notin p$. Let us define $f, g \in P_{\text{good}}$ so that $p_\emptyset(f) = p$, $f(s^\frown i) = t_{s^\frown i}$, if $s \in p \setminus \text{split}(p)$, and $f(s^\frown i) = s^\frown i$, otherwise, and $g(s^\frown i) = t_{s^\frown i}$, if $s \in p$, and $g(s^\frown i) = s^\frown i$, otherwise. Then $[p_s(f)] = [p_s(g)] = [s]$ for all $s \in \text{Seq} \setminus p$ and $p_s(f) \not\leq p_s(g)$ for all $s \in S_\emptyset(f)$. Hence $f \not\leq^{**} g$. If $h \leq^{**} f$ is arbitrary, then $p_s(h) \leq p_s(f)$ for all s from some dense set S in $\langle \text{Seq}, \supseteq \rangle$. As $S \setminus p$ is again dense, by Lemma 2.2 (2), h and g are compatible. □

Lemma 2.6.

(1) *The set* $P_0 = \{h \in P_{\text{good}} \colon h$ *is a one-to-one function*$\}$ *is dense in* $\mathbb{P}_{\text{good}}$.

(2) *The set* $P_1 = \{f \in P_{\text{good}} \colon \text{Seq} \setminus \text{rng}(f)$ *is infinite*$\}$ *is open dense both in* P_{good} *and in* $\mathbb{P}^{*}_{\text{good}}$.

Proof. (1) For $f \in P_{\text{good}}$ define $h(s^\frown i) \in S_s(f)$ by induction on $|s|$ so that $s^\frown i \subseteq h(s^\frown i)$ and $h(s^\frown i) \neq h(t)$ for all $t \subseteq s$. Clearly, $h \leq f$ and $h \in P_0$.

(2) Let $f \in P_{\text{good}}$ be arbitrary. By (1) there is $f' \leq f$ such that $f' \in P_0$. Let $g(s^\frown i) = f'(f'(s^\frown i)^\frown 0)$ for $s^\frown i \in \text{Seq}^+$. Then $\{f'(f'(s^\frown i)^\frown 1) \colon s^\frown i \in \text{Seq}^+\}$ is an infinite subset of $\text{Seq} \setminus \text{rng}(g)$. Therefore $g \in P_1$ and $g \leq f' \leq f$. If $f \in P_1$ and $g \leq f$ then as $\text{rng}(g) \subseteq \text{rng}(f)$, $g \in P_1$. We have proved that P_1 is open dense in $\mathbb{P}_{\text{good}}$. Similarly, if $f \in P_1$ and $g \leq^* f$, then by Lemma 2.1, $\text{rng}(g) \setminus \text{rng}(f)$ is finite and consequently, $g \in P_1$. Therefore, P_1 is open dense in $\mathbb{P}^{*}_{\text{good}}$, too. □

Let $P|p = \{q \in P \colon q \leq p\}$ for a partially ordered set P and $p \in P$.

Lemma 2.7. *Let us assume that* $f \in P_0$ *and let* $T = \text{Seq} \setminus \text{rng}(f)$. *There is a function* $H \colon \{g \in P_{\text{good}} \colon g \leq^* f\} \to {}^{T}(P_{\text{good}})$ *with the following properties:*

(1) *The restriction* $H \restriction \{g \in P_{\text{good}} \colon g \leq f\}$ *is an isomorphism between* $\mathbb{P}_{\text{good}}|f$ *and* ${}^{T}(\mathbb{P}_{\text{good}})$.

(2) *For* $g_1, g_2 \leq^* f$ *the following conditions hold:*

(a) $g_1 =^* g_2$ *if and only if* $H(g_1) =^* H(g_2)$, *where* $H(g_1) =^* H(g_2)$ *means that* $H(g_1)(t)(s^\frown i) = H(g_2)(t)(s^\frown i)$ *for all but finitely many pairs* $(t, s^\frown i) \in T \times \text{Seq}^+$.

(b) $g_1 \leq^* g_2$ *if and only if* $H(g_1) \leq^* H(g_2)$, *where* $H(g_1) \leq^* H(g_2)$ *means that* $(\forall t \in T)\ H(g_1)(t) \leq^* H(g_2)(t)$ *and* $(\forall^\infty t \in T)\ H(g_1)(t) \leq H(g_2)(t)$.

Proof. For $t \in T$ let $\varphi_{f,t}\colon \mathrm{Seq} \to S_t(f)$ be the bijection defined by $\varphi_{f,t}(\emptyset) = t$, $\varphi_{f,t}(s^\frown i) = f(\varphi_{f,t}(s)^\frown i)$.

For $g \leq^* f$ let $H(g)\colon T \to P_{\mathrm{good}}$ be defined by

$$H(g)(t)(s^\frown i) = \begin{cases} \varphi_{f,t}^{-1}\big(g(\varphi_{f,t}(s)^\frown i)\big) & \text{if } g(\varphi_{f,t}(s)^\frown i) \in S_t(f)\,, \\ s^\frown i & \text{otherwise}. \end{cases}$$

To see that $H(g)(t) \in P_{\mathrm{good}}$ notice that either $g\big(\varphi_{f,t}(s)^\frown i\big) \in S_t(f)$ and then $g\big(\varphi_{f,t}(s)^\frown i\big) \supseteq f\big(\varphi_{f,t}(s)^\frown i\big) = \varphi_{f,t}(s^\frown i)$ and $H(g)(t)(s^\frown i) = \varphi_{f,t}^{-1}\big(g(\varphi_{f,t}(s)^\frown i)\big) \supseteq s^\frown i$, or, in the opposite case, $H(g)(t)(s^\frown i) = s^\frown i$.

For $g \leq f$, $H(g)(t)(s^\frown i) = \varphi_{f,t}^{-1}\big(g(\varphi_{f,t}(s)^\frown i)\big)$ for all $s^\frown i \in \mathrm{Seq}^+$. Since $p \in P_0$, the system $\mathcal{A} = \langle S_t(f)\colon t \in T\rangle$ is a partition of Seq. As $\mathcal{A}$ is disjoint, it follows that H is one-to-one on $\mathbb{P}_{\mathrm{good}}|f$. We show that H restricted to $\mathbb{P}_{\mathrm{good}}|f$ is onto ${}^T(\mathbb{P}_{\mathrm{good}})$.

Let $\langle f_t\colon t \in T\rangle \in {}^T(P_{\mathrm{good}})$. As $\mathcal{A}$ is a partition we can define a function $g\colon \mathrm{Seq}^+ \to \mathrm{Seq}^+$ by

$$g(s^\frown i) = \varphi_{f,t}\big(f_t(\varphi_{f,t}^{-1}(s)^\frown i)\big)\,, \quad \text{if } t \in T \text{ and } s \in S_t(f)\,.$$

For $t \in T$ and $s \in S_t(f)$, $g(s^\frown i) \in S_t(f)$. Since $f_t\big(\varphi_{f,t}^{-1}(s)^\frown i\big) \supseteq \varphi_{f,t}^{-1}(s)^\frown i$, $g(s^\frown i) \supseteq \varphi_{f,t}\big(\varphi_{f,t}^{-1}(s)^\frown i\big) = f(s^\frown i)$. Hence $g \in P_{\mathrm{good}}$, $g \leq f$ and $H(g)(t)(s^\frown i) = \varphi_{f,t}^{-1}\big(g(\varphi_{f,t}(s)^\frown i)\big) = f_t(s^\frown i)$ for all $s^\frown i \in \mathrm{Seq}^+$.

We claim that if $g \leq f$, $t \in T$, and $s \in S_t(f)$, then $\varphi_{f,t}``S_s^k\big(H(g)(t)\big) = S^k_{\varphi_{f,t}(s)}(g)$ for all $k \in \omega$. We prove this by induction on $k \in \omega$: For $k = 0$ both sets in the equality are equal to the singleton $\{\varphi_{f,t}(s)\}$, and using the induction hypothesis for $k$, $\varphi_{f,t}``S_s^{k+1}\big(H(g)(t)\big) = \{\varphi_{f,t}\big(H(g)(t)(u^\frown i)\big)\colon u \in S_s^k\big(H(g)(t)\big)\} = \{g\big(\varphi_{f,t}(u)^\frown i\big)\colon \varphi_{f,t}(u) \in S^k_{\varphi_{f,t}(s)}(g)\} = S^{k+1}_{\varphi_{f,t}(s)}(g)$.

If $g_1, g_2 \leq f$, then using this claim we can see that the following holds:

$$\begin{aligned} &(\forall t \in T)\ H(g_1)(t) \leq H(g_2)(t) \iff \\ &(\forall t \in T)(\forall s^\frown i \in \mathrm{Seq}^+)\ H(g_1)(t)(s^\frown i) \in S_s\big(H(g_2)(t)\big) \iff \\ &(\forall t \in T)(\forall s^\frown i \in \mathrm{Seq}^+)\ g_1\big(\varphi_{f,t}(s)^\frown i\big) \in S_{\varphi_{f,t}(s)}(g_2) \iff \\ &g_1 \leq g_2\,. \end{aligned}$$

This finishes the proof of assertion (1).

Assertion (2a) follows immediately by definition of the function H because in the definition of $H(g_1)$ and $H(g_2)$ for $g_1, g_2 \leq^* f$ the second rule is applied only finitely many times. To see (2b) let $g_1, g_2 \leq^* f$. There is $g_2' \leq f$ such that $H(g_2) = H(g_2')$ and hence $g_2' =^* g_2$. Then, using Lemma 2.1 (2),

$$g_1 \leq^* g_2 \Leftrightarrow g_1 \leq^* g_2' \Leftrightarrow (\exists g_1' \leq g_2')\ g_1 =^* g_1' \Leftrightarrow (\exists g_1' \leq f)\ g_1 =^* g_1' \leq g_2'$$
$$\Leftrightarrow (\exists g_1' \leq f)\ H(g_1) =^* H(g_1') \leq H(g_2') \Leftrightarrow H(g_1) \leq^* H(g_2)\,. \qquad \square$$

LEMMA 2.8. *Let B and C be complete Boolean algebras such that* $\mathrm{sat}(B) = \mathrm{sat}(C)$ *and let $D = \{b \in B\colon B{\restriction}b \simeq C\}$ be a dense subset of B. Then $B \simeq C$ and B, C are homogeneous.*

Proof. Let $a \in B \setminus \{0\}$ be arbitrary. Let us choose a maximal antichain $X \subseteq D$ such that for every $b \in X$ either $b \wedge a = 0$ or $b \leq a$. For every $b \in X$ there is a maximal antichain $X_b \subseteq D$ below b such that $|X_b| = |X|$ because $B{\restriction}b \simeq C$ and $\mathrm{sat}(B) \leq \mathrm{sat}(C)$. Let $Y = \bigcup\{X_b\colon b \leq a\}$. Then Y is a maximal antichain below a, $Y \subseteq D$, and $|Y| = |X|$. Let f be an arbitrary one-to-one function from Y onto X and for every $b \in Y$ let $e_b\colon B{\restriction}b \to B{\restriction}f(b)$ be an isomorphism of complete Boolean algebras. Then the function $h\colon B{\restriction}a \to B$, defined by $h(x) = \bigvee\{e_b(x \wedge b)\colon b \in Y\}$ for $x \leq a$, is a complete isomorphism. Therefore, B is homogeneous and the isomorphism $B \simeq C$ follows, too. $\square$

THEOREM 2.9. *The complete Boolean algebras* $\mathrm{r.o.}(\mathbb{P}_{\mathrm{good}})$ *and* $\mathrm{r.o.}(\mathbb{P}^*_{\mathrm{good}})$ *are homogeneous and* $\mathbb{P}_{\mathrm{good}} \approx \langle (P_{\mathrm{good}})^\omega, \leq\rangle$, $\mathbb{P}^*_{\mathrm{good}} \approx \langle (P_{\mathrm{good}})^\omega, \leq^*\rangle$.

Proof. Immediately by Lemmas (2.6), (2.7), and (2.8). $\square$

In Corollary 4.2 we show that Boolean algebra $\mathrm{r.o.}(\mathbb{P}^{**}_{\mathrm{good}})$ is homogeneous, too.

3. Amoeba for $\mathbb{S}$ and for $\mathbb{P}_{\mathrm{good}}$

Amoeba Sacks partial ordering $A(\mathbb{S})$ is the set $\mathbb{S} \times \omega$ ordered by $(p, n) \leq (q, m)$ if and only if $p \leq q$, $n \geq m$, and $p \cap {}^m2 = q \cap {}^m2$ (see, e.g., [6]). Forcing with it produces a perfect set of Sacks reals.

A partially ordered set $\langle P, \leq\rangle$ satisfies axiom A (see [2]) if there exist partial orderings $\leq_n$, $n \in \omega$, of P such that

(1) $p \leq_0 q$ if and only if $p \leq q$;

(2) if $p \leq_{n+1} q$, then $p \leq_n q$;

(3) if $A \subseteq P$ is a maximal antichain, $p \in P$, and $n \in \omega$, then $(\exists q \leq_n p)$ $(\exists A' \in [A]^{\leq\omega})$ A' is predense below q;

(4) if $p_{n+1} \leq_n p_n$ for all $n \in \omega$, then there is $q \in P$ such that $q \leq p_n$ (usually $q \leq_n p_n$ is required) for all n.

For $(p, n) \in A(\mathbb{S})$ and $m \geq n$ let $F(p, n, m) = \{S \subseteq p \cap {}^m2\colon S{\restriction}n = p \cap {}^n2\}$ and for $S \in F(p, n, m)$ let $p^S = \bigcup\{(p)_t\colon t \in S\}$. Then $(p^S, m) \leq (p, n)$.

LEMMA 3.1. *The family* $\mathcal{A} = \{(p^S, m) \colon S \in F(p,n,m)\}$ *is a finite maximal antichain below* (p,n).

Proof. For different $S_1, S_2 \in F(p,n,m)$, $p^{S_1} \cap {}^m2 = S_1 \neq S_2 = p^{S_2} \cap {}^m2$. Therefore $\mathcal{A}$ is an antichain. To prove that $\mathcal{A}$ is maximal below (p,n) let $(r,k) \leq (p,n)$ be arbitrary with $k \geq m$. Then $(r,k) \leq (r,m) \leq (p^{r \cap {}^m2}, m) \in \mathcal{A}$. □

The next theorem is well known and we include it here because its proof is much easier to read than the analogous proof of Theorem 3.4.

THEOREM 3.2. $A(\mathbb{S})$ *satisfies axiom A.*

Proof. Let us define

$$(p,n) \leq_0 (q,m) \text{ if and only if } (p,n) \leq (q,m),$$
$$(p,n) \leq_{k+1} (q,m) \text{ if and only if } (p,n) \leq (q,m),\ m = n, \text{ and } (\forall s \in p \cap {}^n2)\ \operatorname{split}^k((p)_s) = \operatorname{split}^k((q)_s).$$

Conditions (1), (2), and (4) of axiom A are clearly satisfied. It is enough to prove

(3′) if $D \subseteq A(\mathbb{S})$ is an open dense set, $(p,n) \in A(\mathbb{S})$, and $m \in \omega$, then there is $(q,n) \leq_m (p,n)$ and a countable set $D' \subseteq D$ predense below (q,n).

To prove (3′) we construct $(p_j, n_j) \in A(\mathbb{S})$ by induction on $j \in \omega$ so that

(i) $p_0 = p$, $n < n_0 < n_1 < \dots$,

(ii) $(\forall s \in p \cap {}^n2)\ \operatorname{split}^m((p)_s) \subseteq {}^{<n_0}2$,

(iii) $(\forall s \in p_{j+1} \cap {}^{n_j}2)\ \operatorname{split}((p_{j+1})_s) \cap {}^{<n_{j+1}}2 \neq \emptyset$,

(iv) $(p_{j+1}, n_j) \leq (p_j, n_j)$,

(v) $(\forall (r, n_j) \leq (p^S_{j+1}, n_j))\ (r, n_j) \notin D$ whenever $S \in F(p_{j+1}, n, n_j))$ and $(p^S_{j+1}, n_j) \notin D$.

Let us choose $n_0 > n$ so that (ii) holds. Let us assume that p_j, n_j have been constructed and we find p_{j+1} and n_{j+1}. Let $\{S_i \colon i < k\}$ be an enumeration of $F(p_j, n, n_j)$. By induction on $i \leq k$ we define $p_{j,i} \in \mathbb{S}$ as follows: $p_{j,0} = p_j$, and for $i < k$, if there is $(r, n_j) \leq (p^S_{j,i}, n_j)$ such that $(r, n_j) \in D$, then let

$$p_{j,i+1} = r \cup \bigcup \{(p_{j,i})_s \colon s \in (p_{j,i} \cap {}^{n_j}2) \setminus S_i\},$$

and if there is no such r, then let $p_{j,i+1} = p_{j,i}$. It follows that $(p_{j,i+1}, n_j) \leq (p_{j,i}, n_j)$ for $i < k$, hence, if we set $p_{j+1} = p_{j,k}$, then (iv) holds, and as $F(p_j, n, n_j) = F(p_{j+1}, n, n_j)$, also (v) holds. Now let us find $n_{j+1} > n_j$ so that (iii) holds.

Let $q = \bigcap_{j\in\omega} p_j$. By (iii) and (iv) it follows that q is a perfect tree and $(q, n_0) \leq (p, n_0)$. By (ii) then follows that $(q,n) \leq_m (p,n)$. Let us set

$$D' = \{(q^S, n_j) \in D\colon\ j \in \omega \ \text{ and } \ S \in F(q, n, n_j)\}.$$

The set D' is countable and we claim that it is predense below (q,n). Let $(r, n_j) \in D$ be arbitrary such that $(r, n_j) \leq (q,n)$. Let $S = r \cap {}^{n_j}2$. Then $S \in F(q,n,n_j) \subseteq F(p_j, n, n_j)$. As $(r, n_j) \leq (q^S, n_j) \leq (p_j^S, n_j)$, by (v) it follows that $(p_j^S, n_j) \in D$ and hence $(q^S, n_j) \in D'$. As D is open dense this proves that D' is predense below (q,n) and hence (3′) holds. □

We consider the set $A(\mathbb{P}_{\text{good}}) = P_{\text{good}} \times \omega$ ordered by

$$(f,n) \leq (g,m) \ \text{ if and only if } \ (\forall s \in \text{Seq}) \ \big(p_s(f), n\big) \leq \big(p_s(g), m\big).$$

Hence

$$(f,n) \leq (g,m) \ \text{ if and only if } \ f \leq g\,,\ \ n \geq m\,, \ \text{ and}$$
$$(\forall s^\frown i \in \text{Seq}^+)\big(|g(s^\frown i)| < m \to f(s^\frown i) = g(s^\frown i)\big).$$

Let $\mathbb{P}^{1-1}_{\text{good}}$ and $A(\mathbb{P}^{1-1}_{\text{good}})$ be the suborders of $\mathbb{P}_{\text{good}}$ and $A(\mathbb{P}_{\text{good}})$, respectively, formed by one-to-one good sequences. By Lemma 2.6 (1), $\mathbb{P}^{1-1}_{\text{good}} \approx \mathbb{P}_{\text{good}}$.

For $(f,n) \in A(\mathbb{P}_{\text{good}})$ and $m \geq n$ let $F(f,n,m)$ be the family of all functions $\varphi\colon {}^{\leq m}2 \setminus \{\emptyset\} \to {}^{\leq m}2$ such that for every $s^\frown i \in {}^{\leq m}2$ the following conditions hold:

(1) $s^\frown i \subseteq \varphi(s^\frown i)$.

(2) If $|f(s^\frown i)| < n$, then $\varphi(s^\frown i) = f(s^\frown i)$.

(3) If $|\varphi(s^\frown i)| < m$, then $\varphi(s^\frown i) \in S_s(f)$.

(4) If $|\varphi(s^\frown i)| = m$, then $\varphi(s^\frown i) \in p_s(f) \cap {}^m2$.

For $\varphi \in F(f,n,m)$ let $f^\varphi \in P_{\text{good}}$ be defined by

$$f^\varphi(s^\frown i) = \begin{cases} \min_\subseteq\{t \in S_s(f)\colon \varphi(s^\frown i) \subseteq t\} & \text{if } |s| < m\,,\\ f(s^\frown i) & \text{if } |s| \geq m\,. \end{cases}$$

Now we are going to prove that $A(\mathbb{P}_{\text{good}})$ and $A(\mathbb{P}^{1-1}_{\text{good}})$ satisfy axiom A. We do not know to tell whether $\mathbb{P}_{\text{good}}$ itself satisfies axiom A. However, in Theorem 4.1 (7) we prove that $\mathbb{P}_{\text{good}}$ can be embedded into $A(\mathbb{P}^{1-1}_{\text{good}})$. Therefore $\mathbb{P}_{\text{good}}$ inherits its nice forcing properties and in particular it is proper.

We start with an auxiliary lemma.

LEMMA 3.3.

(1) *$(f^\varphi, m) \le (f, n)$ for every $\varphi \in F(f,n,m)$.*

(2) *The family $\mathcal{A} = \{(f^\varphi, m) \colon \varphi \in F(f,n,m)\}$ is a finite maximal antichain below (f,n) in $A(\mathbb{P}_{\mathrm{good}})$.*

Proof. (1) By definition, $f^\varphi(s^\frown i) \in S_s(f)$ for all $s^\frown i \in \mathrm{Seq}^+$ and so $f \le g$. If $|f(s^\frown i)| < n$, then $\varphi(s^\frown i) = f(s^\frown i)$ and hence $f^\varphi(s^\frown i) = f(s^\frown i)$.

(2) Let $\varphi_1(s^\frown i) \ne \varphi_2(s^\frown i)$ for some $s^\frown i \in {}^{\le m}2$, $\varphi_1, \varphi_2 \in F(f,n,m)$. Now, either $\varphi_1(s^\frown i)$, $\varphi_2(s^\frown i)$ are not compatible, then $f^{\varphi_1}(s^\frown i)$, $f^{\varphi_2}(s^\frown i)$ are not compatible in Seq, so there is no perfect tree $q \subseteq p_s(f^{\varphi_1}) \cap p_s(f^{\varphi_2})$ with $s = \operatorname{stem} q$, and hence f^{φ_1}, f^{φ_2} are not compatible in $\mathbb{P}_{\mathrm{good}}$. Or, $\varphi_1(s^\frown i)$, $\varphi_2(s^\frown i)$ are compatible and without loss of generality let $\varphi_1(s^\frown i) \subsetneq \varphi_2(s^\frown i)$. Then $|\varphi_1(s^\frown i)| < m$, $f^{\varphi_1}(s^\frown i) = \varphi_1(s^\frown i) \subsetneq \varphi_2(s^\frown i) \subseteq f^{\varphi_2}(s^\frown i)$, and clearly, (f^{φ_1}, m), (f^{φ_2}, m) are not compatible in $A(\mathbb{P}_{\mathrm{good}})$.

To prove that $\mathcal{A}$ is maximal below (f,n) let $(g,k) \le (f,n)$ be arbitrary with $k \ge m$. Let $\varphi \colon {}^{\le m}2 \setminus \{\emptyset\} \to {}^{\le m}2$ be defined by $\varphi(s^\frown i) = g(s^\frown i) \restriction m$. Then $\varphi \in F(f,n,m)$ and $(g,k) \le (g,m) \le (f^\varphi, m) \in \mathcal{A}$. □

THEOREM 3.4. *$A(\mathbb{P}_{\mathrm{good}})$ and $A(\mathbb{P}^{1-1}_{\mathrm{good}})$ satisfy axiom A.*

Proof. Let us define

$$(f,n) \le_0 (g,m) \text{ if and only if } (f,n) \le (g,m),$$

$$(f,n) \le_{k+1} (g,m) \text{ if and only if } (f,n) \le (g,m),\ m = n, \text{ and } (\forall s^\frown i \in {}^{\le k+1}2)\ f(s^\frown i) = g(s^\frown i).$$

Verification of conditions (1), (2), and (4) of axiom A is the same for $A(\mathbb{P}_{\mathrm{good}})$ and $A(\mathbb{P}^{1-1}_{\mathrm{good}})$.

Conditions (1) and (2) of axiom A are clearly satisfied.

(4) Let us assume that $(f_{k+1}, n_{k+1}) \le_k (f_k, n_k)$. Then $n_{k+1} = n_1 \ge n_0$, and $|f_{k+1}(s^\frown i)| < n_1$ if and only if $|f_1(s^\frown i)| < n_1$ and $f_{k+1}(s^\frown i) = f_1(s^\frown i)$. Let us define $f(s^\frown i) = f_j(s^\frown i)$ if $|s^\frown i| = j+1$. We show that $(f, n_1) \le_{k+1} (f_{k+1}, n_{k+1})$ for all $k \in \omega$. Let $s^\frown i \in \mathrm{Seq}^+$ be arbitrary, $|s^\frown i| = j+1$. If $j > k$, then as $(f_{j+1}, n_{j+1}) \le_{k+1} (f_{k+1}, n_{k+1})$, $f(s^\frown i) = f_{j+1}(s^\frown i) \in S_s(f_{k+1})$; if $j \le k$, then as $(f_{k+1}, n_{k+1}) \le_{j+1} (f_{j+1}, n_{j+1})$, $f(s^\frown i) = f_{j+1}(s^\frown i) = f_{k+1}(s^\frown i)$. Finally, if $|f_{k+1}(s^\frown i)| < n_1$ and $|s^\frown i| = j+1$, then $|f_{j+1}(s^\frown i)| < n_1$ and $f(s^\frown i) = f_{j+1}(s^\frown i) = f_1(s^\frown i) = f_{k+1}(s^\frown i)$.

(3) for $A(\mathbb{P}_{\mathrm{good}})$: We prove that if $D \subseteq A(\mathbb{P}_{\mathrm{good}})$ is open dense, $(f,n) \in A(\mathbb{P}_{\mathrm{good}})$, and $m \in \omega$, then there is $(g,n) \le_{m+1} (f,n)$ and a countable $D' \subseteq D$ predense below (g,n).

We construct $(f_j, n_j) \in A(\mathbb{P}_{\text{good}})$ by induction on $j \in \omega$ so that

(i) $f_0 = f$, $m < n_0 < n_1 < \dots$.

(ii) $(\forall s^\frown i \in {}^{\leq m+1}2)\ |f(s^\frown i)| < n_0$.

(iii) $(\forall s^\frown i \in {}^{\leq n_j+1}2)\ |f_{j+1}(s^\frown i)| < n_{j+1}$.

(iv) $(f_{j+1}, n_{j+1}) \leq (f_j, n_j)$.

(v) $\big(\forall (h, n_j) \leq (f^\varphi_{j+1}, n_j)\big)\ (h, n_j) \notin D$ whenever $\varphi \in F(f_{j+1}, n, n_j)$ and $(f^\varphi_{j+1}, n_j) \notin D$.

We choose n_0 so that (ii) holds. Let us assume that f_j, n_j have been constructed and we find f_{j+1} and n_{j+1}. Let $\{\varphi_l : l < k\}$ be an enumeration of $F(f_j, n, n_j)$. By induction on $l \leq k$ we define $f_{j,l} \in P_{\text{good}}$. We set $f_{j,0} = f_j$. If there is $(h, n_j) \in D$ such that $(h, n_j) \leq \big(f^{\varphi_l}_{j,l}, n_j\big)$ then let

$$f_{j,l+1}(s^\frown i) = \begin{cases} h(s^\frown i) & \text{if } |h(s^\frown i)| \geq n_j, \\ f_{j,l}(s^\frown i) & \text{otherwise}, \end{cases}$$

and if there is no such h then let $f_{j,l+1} = f_{j,l}$. Let us set $f_{j+1} = f_{j,k}$ and let $n_{j+1} > n_j$ be such that (iii) holds. As $(f_{j,l+1}, n_j) \leq (f_{j,l}, n_j)$ for all $l < k$, (iv) holds and as $F(f_j, n, n_j) = F(f_{j+1}, n, n_j)$ also (v) holds. Hence all conditions (i)–(v) hold.

Let us define $g(s^\frown i) = f_l(s^\frown i)$ where l is minimal such that $|f_l(s^\frown i)| < n_l$. We claim that $(g, n_j) \leq (f_j, n_j)$ for all $j \in \omega$. To prove this let us fix $j \in \omega$, $s^\frown i \in \text{Seq}^+$ and let l be minimal such that $|f_l(s^\frown i)| < n_l$. If $|f_l(s^\frown i)| < n_j$, then $l \leq j$ and as $(f_j, n_j) \leq (f_l, n_l)$, $f_j(s^\frown i) = f_l(s^\frown i) = g(s^\frown i)$. If $f_l(s^\frown i) \geq n_j$, then $l > j$, $f_l \leq f_j$ and hence $g(s^\frown i) \in S_s(f_j)$.

By (ii) then follows that $(g, n) \leq_{m+1} (f, n)$. Let us set

$$D' = \big\{(g^\varphi, n_j) \in D : j \in \omega \text{ and } \varphi \in F(g, n, n_j)\big\}.$$

The set D' is countable and we claim that it is predense below (g, n).

Let $(h, n_j) \in D$ be arbitrary such that $(h, n_j) \leq (g, n)$. Let $\varphi\colon {}^{\leq n_j}2 \setminus \{\emptyset\} \to {}^{\leq n_j}2$ be defined by $\varphi(s^\frown i) = h(s^\frown i){\restriction}n_j$. Then $\varphi \in F(g, n, n_j) = F(f_j, n, n_j)$. As $(h, n_j) \leq (g^\varphi, n_j) \leq (f_j^\varphi, n_j)$, by (v) it follows that $(f_j^\varphi, n_j) \in D$ and hence $(g^\varphi, n_j) \in D'$. As D is open dense this proves that D' is predense below (g, n).

(3) for $A(\mathbb{P}^{1-1}_{\text{good}})$: Let $D \subseteq A(\mathbb{P}^{1-1}_{\text{good}})$ be predense, $(f, n) \in A(\mathbb{P}^{1-1}_{\text{good}})$, and $m \in \omega$. Let

$$E = \big\{x \in A(\mathbb{P}_{\text{good}}) : x \text{ refines an element of } D \text{ or } x \text{ is incompatible with all elements of } A(\mathbb{P}^{1-1}_{\text{good}})\big\}.$$

E is open dense in $A(\mathbb{P}_{\text{good}})$. Hence we can find $(g, n) \leq_m (f, n)$ and a countable predense set $E' \subseteq E$ below (g, n). We can find $g \in \mathbb{P}^{1-1}_{\text{good}}$. Now we define a countable set $D' \subseteq D$ by choosing one element above each element of E' if it exists. It is easy to verify that D' is predense below (g,n) in $A(\mathbb{P}^{1-1}_{\text{good}})$. □

For a filter G on $A(\mathbb{P}_{\text{good}})$ let ρ_G denote the relation

$$\rho_G = \{(s,t) \in (\mathrm{Seq}^+)^2 \colon \big(\exists (f,n) \in G\big)\ |f(s)| < n \quad \text{and} \quad f(s) = t\}.$$

Let us consider the following open dense subsets of $A(\mathbb{P}_{\text{good}})$:

$$D_n = \big\{(f,m) \in A(\mathbb{P}_{\text{good}}) \colon (\forall s \in {}^{<n}2)\ |f(s)| < m\big\}, \qquad n \in \omega.$$

Easily it can be verified that $\rho_G \in P_{\text{good}}$ if and only if G is a $\{D_n \colon n \in \omega\}$-generic filter on $A(\mathbb{P}_{\text{good}})$. If $\rho_G \in \mathbb{P}_{\text{good}}$, then $(\rho_G, m) \leq (f,m)$ for every $(f,m) \in G$.

LEMMA 3.5. *If $E \subseteq P_{\text{good}}$ is predense in $\mathbb{P}^*_{\text{good}}$, then the set*

$$D(E) = \big\{(f,n) \in A(\mathbb{P}_{\text{good}}) \colon (\exists h \in E)(\forall k \geq n)(\forall s \in {}^k2)\ p_s(f) \leq p_s(h)\big\}$$

is open dense in $A(\mathbb{P}_{\text{good}})$.

Proof. Let $(g,m) \in A(\mathbb{P}_{\text{good}})$. There is $h \in E$ which is $\leq^*$-compatible with g. Let $f \in P_{\text{good}}$ and $n \geq m$ be such that $p_s(f) \leq p_s(g)$ and $p_s(f) \leq p_s(h)$ whenever $|s| \geq n$. Let $f' \in P_{\text{good}}$ be defined by

$$f'(s^\frown i) = \begin{cases} g(s^\frown i) & \text{if } |s| < n, \\ f(s^\frown i) & \text{otherwise}. \end{cases}$$

Then, similarly as in Lemma 2.1 (2), $f =^* f' \leq g$, and moreover, $(f',n) \in D(E)$ where $(f',n) \leq (g,n) \leq (g,m)$. □

THEOREM 3.6. *$MA_\kappa\big(A(\mathbb{P}_{\text{good}})\big)$ implies $\mathrm{sh}(2,\mathbb{P}^*_{\text{good}}) > \kappa$. Consequently,* PFA *implies $\mathrm{sh}(2,\mathbb{P}^*_{\text{good}}) = \mathfrak{c} = \omega_2$.*

Proof. Let as assume that $\mathrm{MA}_\kappa\big(A(\mathbb{P}_{\text{good}})\big)$ holds. Let $\{E_\alpha \colon \alpha < \kappa\}$ be a family of open dense sets in $\mathbb{P}^*_{\text{good}}$. We prove that $\bigcap_{\alpha<\kappa} E_\alpha \neq \emptyset$.

By Lemma 3.5 for each $\alpha < \kappa$ the set $D(E_\alpha)$ is open dense in $A(\mathbb{P}_{\text{good}})$. Let G be a $\{D_n \colon n \in \omega\} \cup \big\{D(E_\alpha) \colon \alpha < \kappa\big\}$-generic filter on $A(\mathbb{P}_{\text{good}})$. Then $\rho_G \in \mathbb{P}_{\text{good}}$ and for every $\alpha < \kappa$ there is $(f_\alpha, m_\alpha) \in G \cap D(E_\alpha)$ and $h_\alpha \in E_\alpha$ such that $(\forall^\infty s)\ p_s(f_\alpha) \leq p_s(h_\alpha)$. Then $(\rho_G, m_\alpha) \leq (f_\alpha, m_\alpha)$ and hence $\rho_G \leq f_\alpha \leq^* h_\alpha$ for each $\alpha < \kappa$. It follows that $\rho_G \in \bigcap_{\alpha<\kappa} E_\alpha$.

$A(\mathbb{P}_{\text{good}})$ is proper by Theorem 3.4 and PFA is a generalization of Martin's axiom for proper partial orderings, but it implies $\mathfrak{c} = \omega_2$ (by Todorčević–Veličković's Theorem, see [5]). So, the second assertion follows. □

4. Good sequences and complete embeddings

In this section we prove the existence of several complete embeddings between considered forcing notions. We did not try to investigate quotients of complete Boolean algebras concerned in the embeddings.

For $f \in P_{\text{good}}$ let $F(f) = \bigcup\{[p_s(f)]: s \in \text{Seq}\}$. $F(f)$ is an F_σ set and $F(f) \cap [s]$ is uncountable for every $s \in \text{Seq}$. Let $\mathbb{F}$ be the family of all F_σ sets $F \subseteq {}^\omega 2$ such that $F \cap [s]$ is uncountable for every $s \in \text{Seq}$. The set $\mathbb{F}$ is ordered by inclusion $\subseteq$. Clearly, $F(f) \in \mathbb{F}$ for every $f \in P_{\text{good}}$.

THEOREM 4.1.

(1) $\mathbb{P}^*_{\text{good}} \preceq \mathbb{P}_{\text{good}}$.

(2) $\mathbb{P}^*_{\text{good}} \preceq A(\mathbb{P}_{\text{good}})$.

(3) $\mathbb{S} \preceq \mathbb{S}^\omega \preceq \mathbb{P}_{\text{good}}$.

(4) $\mathbb{S}^\omega \preceq A(\mathbb{P}_{\text{good}})$.

(5) $\mathbb{S} \preceq A(\mathbb{S}) \preceq A(\mathbb{P}_{\text{good}})$.

(6) $A(\mathbb{S}) \preceq A(\mathbb{P}^{1-1}_{\text{good}})$.

(7) $\mathbb{P}_{\text{good}} \preceq A(\mathbb{P}^{1-1}_{\text{good}})$.

(8) $\mathbb{P}^{**}_{\text{good}} \approx \mathbb{F}$.

Proof. (1) We prove that the function id: $\mathbb{P}_{\text{good}} \to \mathbb{P}^*_{\text{good}}$ is a normal function. Clearly it is monotone and surjective. It remains to verify the density condition for it, i.e., we have to prove that if $g \leq^* f$ then there is $h \leq f$ such that $h \leq^* g$. By Lemma 2.1 (2) there is $h \in P_{\text{good}}$ such that $g =^* h \leq f$ and hence $h \leq^* g$.

(2) The function $\varphi(f,n) = f$ is a normal function from $A(\mathbb{P}_{\text{good}})$ onto $\mathbb{P}^*_{\text{good}}$.

(3) The first inequality is trivial. We prove the second. For a perfect tree p and $s \in p$ let $(p)^s = \{t \in \text{Seq}: s^\frown t \in p\}$. Let $\langle s_k: k \in \omega\rangle$ be a sequence of pairwise incompatible elements of Seq. Let us define a normal function $\varphi: \mathbb{P}_{\text{good}} \to \mathbb{S}^\omega$ by $\varphi(f) = \langle (p_{s_k}(f))^{s_k^\frown 0}: k \in \omega\rangle$.

(4) We use the notation from (3) and define $\varphi(f,n) = \langle ((p_{s_k}(f))_{t_k})^{s_k^\frown 0}: k \in \omega\rangle$ where $t_k \in p_{s_k}(f) \cap {}^n 2$ is the leftmost element of $p_{s_k}(f) \cap {}^n 2$ in the lexicographical ordering, $(f,n) \in A(\mathbb{P}_{\text{good}})$. The function $\varphi: A(\mathbb{P}_{\text{good}}) \to \mathbb{S}^\omega$ is normal.

(5–6) The function $\pi: A(\mathbb{S}) \to \mathbb{S}$ defined by $\pi(p,n) = (p)_s$ where s is the leftmost element of $p \cap {}^n 2$ in the lexicographical ordering of ${}^n 2$ is normal. For $A(\mathbb{S}) \preceq A(\mathbb{P}_{\text{good}})$ and $A(\mathbb{S}) \preceq A(\mathbb{P}^{1-1}_{\text{good}})$ notice that $\varphi(f,n) = (p_\emptyset(f), n)$ is a normal function in both cases.

(7) We define a normal function $\varphi: A(\mathbb{P}^{1-1}_{\text{good}}) \to \mathbb{P}_{\text{good}}$. For $(f,n) \in A(\mathbb{P}^{1-1}_{\text{good}})$ we consider the sets

$$A_{f,n} = \{s^\frown i \in \text{Seq}^+: |f(s^\frown i)| < n\},$$
$$B_{f,n} = \{s^\frown i \in \text{Seq}^+: |s| < n \text{ and } |f(s^\frown i)| \geq n\}.$$

As f is one-to-one, $|A_{f,n}| \leq |B_{f,n}|$ for every $n \in \omega$. Moreover, it is possible to define one-to-one mappings $\varphi_{f,n}\colon A_{f,n} \to B_{f,n}$ in a uniform way so that the following three conditions are satisfied:

(i) If $\varphi_{f,n}(s^\frown i) = t^\frown j$, then $s^\frown i \subseteq t \in S_s(f)$.

(ii) If $(h,m) \leq (f,n)$ and $s^\frown i \in A_{f,n}$, then $h(\varphi_{h,m}(s^\frown i)) \in S_{f(\varphi_{f,n}(s^\frown i))}(f)$.

(iii) If $f\restriction A_{f,n} = h\restriction A_{h,n}$, then $\varphi_{f,n} = \varphi_{h,n}$.

The construction can be carried out by induction on $n \in \omega$.

Now let us define

$$\varphi(f,n)(s^\frown i) = \begin{cases} f(\varphi_{f,n}(s^\frown i)) & \text{if } s^\frown i \in A_{f,n}, \\ f(s^\frown i) & \text{otherwise}. \end{cases}$$

By the uniformity of functions $\varphi_{f,n}$ for $(f,n) \in A(\mathbb{P}^{1-1}_{\text{good}})$ it follows that φ is monotone. As $\varphi(f,n) \leq f$, the range of φ is dense in $\mathbb{P}_{\text{good}}$. Let $g \leq \varphi(f,n)$ be one-to-one and let

$$h(s^\frown i) = \begin{cases} f(s^\frown i) & \text{if } s^\frown i \in A_{f,n}, \\ g(t^\frown j) & \text{if } s^\frown i \in B_{f,n} \text{ and } \varphi_{f,n}(t^\frown j) = s^\frown i, \\ g(s^\frown i) & \text{otherwise}. \end{cases}$$

We show that $(h,n) \leq (f,n)$. There are three cases:

(a) If $s^\frown i \in A_{f,n}$, then $h(s^\frown i) = f(s^\frown i)$.

(b) If $h(s^\frown i) = g(s^\frown i)$, then $h(s^\frown i) = g(s^\frown i) \in S_s(\varphi(f,n)) \subseteq S_s(f)$.

(c) If $h(s^\frown i) = g(t^\frown j)$, where $\varphi_{f,n}(t^\frown j) = s^\frown i$ for some $t \in A_{f,n}$, then as $g \leq \varphi(f,n)$, $h(s^\frown i) = g(t^\frown j) \in S_{f(\varphi_{f,n}(t^\frown j))}(\varphi(f,n)) = S_{f(\varphi_{f,n}(t^\frown j))}(f) = S_{f(s^\frown i)}(f) \subseteq S_s(f)$.

Let $m > n$ be such that $m > |g(s^\frown i)|$ for all $s^\frown i \in B_{h,n}$. We prove that $\varphi(h,m) \leq g$. Notice that $A_{h,n} = A_{f,n}$, $B_{h,n} = B_{f,n}$, and $A_{h,n} \cup B_{h,n} \subseteq A_{h,m}$.

If $s^\frown i \notin A_{h,m}$, then $\varphi(h,m)(s^\frown i) = h(s^\frown i) = g(s^\frown i)$.

If $s^\frown i \in A_{h,m} \setminus A_{f,n}$ and $\varphi_{h,m}(s^\frown i) = t^\frown j$, then by (i), $t \in S_{h(s^\frown i)}(h)$. Therefore $\varphi(h,m)(s^\frown i) = h(\varphi_{h,m}(s^\frown i)) = h(t^\frown j) \in S_{h(s^\frown i)}(h) \subseteq S_s(g)$.

If $s^\frown i \in A_{f,n}$ and $\varphi_{f,n}(s^\frown i) = t'^\frown j'$, $\varphi_{h,m}(s^\frown i) = t^\frown j$, then $h(t'^\frown j') = h(\varphi_{f,n}(s^\frown i)) = g(s^\frown i)$. Now, as $(h,n) \leq (f,n)$, $\varphi_{f,n} = \varphi_{h,n}$ and, by (iii), $h(t^\frown j) \in S_{g(s^\frown i)}(h)$. Therefore $\varphi(h,m)(s^\frown i) = h(\varphi_{h,m}(s^\frown i)) = h(t^\frown j) \in S_{g(s^\frown i)}(h) \subseteq S_s(g)$.

(8) We prove that the function $F\colon \mathbb{P}^{**}_{\text{good}} \to \mathbb{F}$ is a normal function and a complete embedding as well. To show that F is monotone let us assume that

$f \leq^{**} g$. For $t \in \mathrm{Seq}$ let $n(t) \in \omega$ be such that $p_s(f) \leq p_s(g)$ for all $s \in S_s^{n(t)}(f)$. Then

$$F(f) = \bigcup_{t \in \mathrm{Seq}} [p_t(f)] = \bigcup_{t \in \mathrm{Seq}} \bigcup_{s \in S^{n(t)}(f)} [p_s(f)] \subseteq F(g)$$

and hence F is monotone. As $F(\mathrm{id}) = {}^{\omega}2$ it remains to prove that for $H \in \mathbb{F}$ and $f \in P_{\mathrm{good}}$ such that $H \subseteq F(f)$ there is $h \leq^{**} f$ such that $F(h) \subseteq H$. Given $s \in \mathrm{Seq}$, the set $[s] \cap F(f) = \bigcup_{t \supseteq s} [p_t(f)]$ is uncountable and hence for some $t_s \supseteq s$, $H \cap [p_{t_s}(f)]$ is uncountable and there is a perfect tree $p_s \subseteq \mathrm{Seq}$ such that $[p_s] \subseteq H \cap [p_{t_s}(f)]$. Let $q_s = p_{t_s\frown 0} \cup p_{t_s\frown 1}$ for $s \in \mathrm{Seq}$ and let h be obtained by applying Lemma 2.2 (1) to the sequence $\langle q_s \colon s \in \mathrm{Seq}\rangle$. Then for every $s \in \mathrm{Seq}$ there is $r_s \subseteq s$ with $p_s(h) \leq (q_{r_s})_s$ and hence $F(h) \subseteq H$. By assertion (c) of Lemma 2.2 (1), $h \leq^{**} f$. So we have proved that the function F is normal.

Now we prove that F is a complete embedding. First let us notice that for $f, g \in P_{\mathrm{good}}$, f, g are $\leq^{**}$-compatible if and only if $F(f)$ and $F(g)$ are compatible in $\mathbb{F}$. The "only if" part of the equivalence is by monotonicity of F. It remains to prove the "if" direction. Let f, g be $\leq^{**}$-incompatible. By Lemma 2.2 (2) and Remark 2.3 there is $t \in \mathrm{Seq}$ such that for every $s \supseteq t$, perfect trees $p_s(f)$ and $p_s(g)$ are incompatible. It follows that $F(f) \cap F(g) \cap [t]$ is countable and hence $F(f)$ and $F(g)$ are incompatible in $\mathbb{F}$. It follows that if A is a maximal antichain in $\mathbb{P}^{**}_{\mathrm{good}}$, then $\{F(f) \colon f \in A\}$ is a maximal antichain in $\mathbb{F}$. Finally let us notice that for $H \in \mathbb{F}$ there is $f \in P_{\mathrm{good}}$ such that $F(f) \subseteq H$. Then for every $g \leq^{**} f$, $F(g) \subseteq H$. Therefore F is a complete embedding. □

COROLLARY 4.2. *The Boolean algebra* $\mathrm{r.o.}(\mathbb{F}) = \mathrm{r.o.}(\mathbb{P}^{**}_{\mathrm{good}})$ *is homogeneous.*

Proof. Let D be the family of all sets $H \subseteq {}^{\omega}2$ such that there is a disjoint system of perfect nowhere dense sets $\langle P_s \colon s \in \mathrm{Seq}\rangle$ such that $H = \bigcup_{s \in \mathrm{Seq}} P_s$ and $P_s \subseteq [s]$ for all s. The set D is a dense subset of $\mathbb{F}$.

Let $H = \bigcup_{s \in \mathrm{Seq}} P_s$ and $H' = \bigcup_{s \in \mathrm{Seq}} P'_s$ be two elements of D and for each $s \in \mathrm{Seq}$ let us fix a homeomorphism $f_s \colon P_s \to P_s$. We claim that the function $\varphi \colon \mathbb{F}|H \to \mathbb{F}|H'$ defined by $\varphi(F) = \bigcup_{s \in \mathrm{Seq}} f_s(F \cap P_s)$ is an isomorphism.

To see this let us notice first that if $F \subseteq H$ and $F \cap [s]$ is uncountable for each $s \in \mathrm{Seq}$, then the set $S = \{s \in \mathrm{Seq} \colon F \cap P_s \text{ is uncountable}\}$ is cofinal in Seq. If not then there is $t \in \mathrm{Seq}$ such that $|F \cap P_s| \leq \omega$ for each $s \supseteq t$. As $\bigcup\{P_u \colon u \subseteq t\}$ is nowhere dense, there is $s \supseteq t$ such that $[s] \cap \bigcup\{P_u \colon u \subseteq t\} = \emptyset$. Then $[s] \cap F = [s] \cap \bigcup\{P_u \colon u \subseteq t \text{ or } t \subseteq u\} = [s] \cap \bigcup\{P_u \colon t \subseteq u\}$ and so $[s] \cap F$ is countable which is a contradiction. Hence S is cofinal in Seq and consequently the set $\varphi(F) \supseteq \bigcup\{f_s(F \cap [P_s]: s \in S)\}$ is in $\mathbb{F}|H'$. Similarly, if $F' \in \mathbb{F}|H'$, then $\varphi^{-1}(F') \in \mathbb{F}|H$, and hence φ is an isomorphism.

Now, using Lemma 2.8 it follows that $\mathrm{r.o.}(\mathbb{F})$ is homogeneous. □

Remark 4.3. The functions $\mathrm{id}\colon \mathbb{P}^*_{\mathrm{good}} \to \mathbb{P}^{**}_{\mathrm{good}}$ and $\mathrm{id}\colon \mathbb{P}_{\mathrm{good}} \to \mathbb{P}^{**}_{\mathrm{good}}$ are not normal functions.

Proof. We find $g \in P_{\mathrm{good}}$ such that the set $\{f\colon f \le^* g\}$ (and hence also the set $\{f\colon f \le g\}$) is not dense below g in $\mathbb{P}^{**}_{\mathrm{good}}$. Let $p \subseteq \mathrm{Seq}$ be a perfect tree with $\mathrm{stem}(p) = \emptyset$ such that $[p]$ is nowhere dense in ${}^\omega 2$. For $s \in \mathrm{Seq}$ let $v(s) \supseteq s$ be such that $v(s) \notin p$ if $s \in p$ and $v(s) = s$ if $s \notin p$. Let $f, g \in P_{\mathrm{good}}$ be defined by

$$f(s^\frown i) = v(s^\frown i)\,, \qquad g(s^\frown i) = \begin{cases} \mathrm{stem}\big((p)_{s^\frown i}\big) & \text{if } s \in \mathrm{split}(p)\,,\\ v(s^\frown i) & \text{otherwise}\,. \end{cases}$$

Then $F(g) = {}^\omega 2$, $p_\emptyset(g) = p$, $F(f) = {}^\omega 2 \setminus [p]$, and $f \le^{**} g$ as

$$(\forall t \in \mathrm{Seq})\big(\forall s \in S^1_t(f)\big)\ p_s(f) = p_s(g) = (\mathrm{Seq})_s\,.$$

But there is no $h \in P_{\mathrm{good}}$ such that $h \le^{**} f$ and $h \le^* g$ because otherwise, $F(h) \cap [p] \ne \emptyset$ and $F(h) \subseteq F(f)$. □

5. Sacks forcing and good sequences

Lemma 5.1. *Let $A \subseteq \mathbb{S}$ be a maximal antichain in $\mathbb{S}$.*

(1) *For every $f \in P_{\mathrm{good}}$ there is $h \le f$ such that for every $s \in \mathrm{Seq}$, $p_s(h)$ is compatible with at most two elements of A.*

(2) *If $f \in P_{\mathrm{good}}$ and for every $s \in \mathrm{Seq}$, $p_s(f)$ is compatible with at most finitely many elements of A, then every $g \le^{**} f$ has the same property.*

Proof. (1) Let $q_s \le p_s(f)$ be such that $\mathrm{stem}\, q_s = s$ and q_s is compatible with at most two elements of A (set $q_s = q^0_s \cup q^1_s$ where $q^i_s \le \big(p_s(f)\big)_{s^\frown i}$ refines an element of A). Let $h \le f$ be obtained by applying Lemma 2.2 (1) to the system $\langle q_s\colon s \in \mathrm{Seq}\rangle$. Then $p_s(h) \le (q_{r_s})_s \le q_{r_s}$ for every $s \in \mathrm{Seq}$.

(2) Let $g \le^{**} f$ and let $s \in \mathrm{Seq}$. There is $n \in \omega$ such that $p_t(g) \le p_t(f)$ for every $t \in S^n_s(g)$. As $\{p_t(g)\colon t \in S^n_s(g)\}$ is a finite predense set below $p_s(g)$ each element of which meets only a finite number of elements of A, then also $p_s(g)$ meets only a finite number of elements of A. □

THEOREM 5.2. *$\mathbb{P}^*_{\text{good}}$ is ω-closed.*

Proof. Let $f_0 \geq^* f_1 \geq^* \dots$ be a decreasing sequence in $\mathbb{P}^*_{\text{good}}$. We find $h \in P_{\text{good}}$ such that $h \leq^* f_k$ for all $k \in \omega$. Let $\{n_k\}_{k=0}^{\infty}$ be an increasing sequence of natural numbers such that $n_0 = 0$ and $f_k(s^\frown i) \in S_s(f_m)$ whenever $m \leq k$, $|s| \geq n_k$, and $i \in \{0,1\}$. Let $h \in P_{\text{good}}$ be defined by $h(s^\frown i) = f_k(s^\frown i)$ if $n_k \leq |s| < n_{k+1}$. Now let $m \in \omega$ be arbitrary. For every s with $|s| \geq n_m$ there is $k \geq m$ such that $n_k \leq |s| < n_k$ and hence $h(s^\frown i) = f_k(s^\frown i) \in S_s(f_m)$. Therefore, by Lemma 2.1, $h \leq^* f_m$ for all $m \in \omega$. □

Remark 5.3. $\mathbb{P}_{\text{good}}$, $\mathbb{P}^{**}_{\text{good}}$, and $\mathbb{F}$ are not ω-closed.

Proof. Forcing with $\mathbb{P}_{\text{good}}$ adds a Sacks real and hence $\text{r.o.}(\mathbb{P}_{\text{good}})$ is not ω-distributive (see [4, Theorem 58]). To show that $\mathbb{P}^{**}_{\text{good}}$ and $\mathbb{F}$ are not ω-closed let us consider $f \in P_{\text{good}}$ such that $[p_s(f)]$ are nowhere dense in ${}^\omega 2$ for all $s \in \text{Seq}$ and for all $s, t \in \text{Seq}$,

$$\text{if } s \subseteq t, \text{ then } S_s(f) \cap S_t(f) \neq \emptyset \text{ if and only if } t \in S_s(f) \qquad (*)$$

(see the construction at the beginning of the proof of Lemma 2.2). Let us recall that $F(f) = \bigcup_{t \in \text{Seq}} [p_t(f)]$. For $n \in \omega$ and $s \in \text{Seq}$ let us fix an $t^n_s \in (\text{Seq})_s \setminus \bigcup\{S_t(f) \colon t \in {}^{<n}2\}$ and let us define $f_n \in P_{\text{good}}$ by

$$f_n(s^\frown i) = \begin{cases} t^n_{s^\frown i} & \text{if } s \in \bigcup\{S_t(f) \colon t \in {}^{<n}2\}, \\ f(s^\frown i) & \text{otherwise}. \end{cases}$$

Then $f_0 = f$ and $f_{n+1} \leq^\times f_n$ because for every $s \in \text{Seq}$ either $s \in \text{Seq} \setminus \bigcup\{S_t(f) \colon t \in {}^{<n+1}2\}$ and then, by $(*)$, $p_s(f_{n+1}) = p_s(f_n) = p_s(f)$, or otherwise, as $t^{n+1}_{s^\frown i} \notin \bigcup\{S_t(f) \colon t \in {}^{<n+1}2\}$, again by $(*)$, for all $t \in S_s(f_{n+1}) \setminus \{s\}$, $p_t(f_{n+1}) = p_t(f_n) = p_t(f)$. At last, as $F(f_n) = F(f) \setminus \bigcup\{[p_t(f)] \colon t \in {}^{<n}2\}$, by $(*)$, it follows that $\bigcap_{n \in \omega} F(f_n) = \emptyset$ and hence there is a lower bound neither for $\langle f_n \colon n \in \omega \rangle$ in $\mathbb{P}^{**}_{\text{good}}$ nor for $\langle F(f_n) \colon n \in \omega \rangle$ in $\mathbb{F}$. □

It is well known that $\text{sh}(\omega, \mathbb{S}) \geq \omega_1$ (see, e.g., [4 proof of Lemma 26.4]). The following theorem provides an attempt to improve this inequality and to obtain another lower bound for $\text{sh}(\omega, \mathbb{S})$.

THEOREM 5.4.

(1) $\omega_1 \leq \text{sh}(2, \mathbb{P}^*_{\text{good}}) \leq \text{sh}(\omega, \mathbb{P}^*_{\text{good}}) \leq \text{sh}(\omega, \mathbb{S})$.

(2) $\text{sh}(2, \mathbb{P}^{**}_{\text{good}}) \leq \text{sh}(\omega, \mathbb{S})$.

(3) $\text{sh}(\omega, \mathbb{P}_{\text{good}}) \leq \text{sh}(\omega, \mathbb{P}^*_{\text{good}})$.

(4) $\text{sh}(\omega, \mathbb{P}_{\text{good}}) \leq \text{sh}(\omega, \mathbb{S}^\omega) \leq \text{sh}(\omega, \mathbb{S})$.

P r o o f . (1) The first two inequalities hold by Lemma 5.2 and monotonicity of the cardinal invariants $\mathrm{sh}(\kappa, \mathbb{P})$, respectively.

Let $\kappa < \mathrm{sh}(\omega, \mathbb{P}^*_{\mathrm{good}})$ and let $\langle A_\alpha : \alpha < \kappa\rangle$ be a sequence of maximal antichains in $\mathbb{S}$. We show that for every $q \in \mathbb{S}$ there is $r \leq q$ such that for every $\alpha < \kappa$ the set

$$A_{\alpha,r} = \{p \in A_\alpha : r \text{ is compatible with } p\}$$

is finite and hence $\kappa < \mathrm{sh}(\omega, \mathbb{S})$.

Let $q \in \mathbb{S}$ be given. Let $g \in P_{\mathrm{good}}$ be arbitrary such that $p_{\mathrm{stem}\, q}(g) = q$. By Lemma 5.1 for every $\alpha < \kappa$ there is a maximal antichain B_α in $\mathbb{P}^*_{\mathrm{good}}$ such that for every $f \in B_\alpha$ and for every $s \in \mathrm{Seq}$ the set

$$A_{\alpha,s,f} = \{p \in A_\alpha : p \text{ is compatible with } p_s(f)\}$$

is finite. As $\kappa < \mathrm{sh}(\omega, \mathbb{P}^*_{\mathrm{good}})$ there is $g' \leq^* g$ such that for every $\alpha < \kappa$ the set

$$B'_\alpha = \{f \in B_\alpha : f \text{ is } \leq^*\text{-compatible with } g'\}$$

is finite. Let $s_0 \in S_{\mathrm{stem}\, q}(g) = \mathrm{split}(q)$ be such that $p_{s_0}(g') \leq p_{s_0}(g)$. We show that $r = p_{s_0}(g')$ works.

Clearly, $r \leq q$ because $p_{s_0}(g) = (q)_{s_0}$. As B'_α is predense below g' in $\mathbb{P}^*_{\mathrm{good}}$, by Lemma 2.2 (3), there is $n_\alpha \in \omega$ such that for $|s| \geq n_\alpha$ the set $\{p_s(f) : f \in B'_\alpha\}$ is predense below $p_s(g')$ in $\mathbb{S}$. In particular, as the set $\{p_s(g') : s \in S^{n_\alpha}_{s_0}(g')\}$ is a finite maximal antichain below r in $\mathbb{S}$, the set $\{p_s(f) : f \in B'_\alpha$ and $s \in S^{n_\alpha}_{s_0}(g')\}$ is finite and predense below r. Therefore $A_{\alpha,r} \subseteq \bigcup\{A_{\alpha,s,f} : f \in B'_\alpha$ and $s \in S^{n_\alpha}_{s_0}(g')\}$ and so the set $A_{\alpha,r}$ is finite for every $\alpha < \kappa$.

(2) Let $\kappa < \mathrm{sh}(2, \mathbb{P}^{**}_{\mathrm{good}})$ and let $\langle A_\alpha : \alpha < \kappa\rangle$ be a sequence of maximal antichains in $\mathbb{S}$. We show that for every $q \in \mathbb{S}$ there is $r \leq q$ such that for every $\alpha < \kappa$ the set

$$A_{\alpha,r} = \{p \in A_\alpha : r \text{ is compatible with } p\}$$

is finite and hence $\kappa < \mathrm{sh}(\omega, \mathbb{S})$.

Let $q \in \mathbb{S}$ be given. Let $q_s = (s^\frown 0^\frown q) \cup (s^\frown 1^\frown q)$ for $s \in \mathrm{Seq}$ where $s^\frown q = \{u \in \mathrm{Seq} : (\exists t \in q)\ u \subseteq s^\frown t\}$ for $s \in \mathrm{Seq}$ and $q \in \mathbb{S}$. Hence q_s is a perfect tree with $\mathrm{stem}\, q_s = s$. Let $g \in P_{\mathrm{good}}$ be obtained by applying Lemma 2.2 (1) to the system $\langle q_s : s \in \mathrm{Seq}\rangle$. Then for every $s \in \mathrm{Seq}$ there is $r_s \subseteq s$ such that $p_s(g) \leq (q_{r_s})_s$. For each $\alpha < \kappa$ and $n \in \omega$ let $A^n_\alpha = \{s^\frown p : s \in {}^n2$ and $p \in A_\alpha\}$. Clearly, A^n_α is a maximal antichain in $\mathbb{S}$ for all $\alpha < \kappa$ and $n \in \omega$. By Lemma 5.1 the set B^n_α of those $f \in P_{\mathrm{good}}$ for which for every $s \in \mathrm{Seq}$, $p_s(f)$ is compatible with only a finite number of elements of A^n_α is open dense in $\mathbb{P}^{**}_{\mathrm{good}}$. As $\kappa < \mathrm{sh}(2, \mathbb{P}^{**}_{\mathrm{good}})$ there is $g' \leq^{**} g$ such that $g' \in \bigcap\{B^n_\alpha : \alpha < \kappa$ and $n \in \omega\}$. There is $s \in \mathrm{Seq}$ such that $p_s(g') \leq p_s(g')$. We can find s such that $|r_s| < |s|$ and let $n = |r_s|$. Let $t \in \mathrm{Seq}$ be such that $r_s{}^\frown t = s$. Then there is $r \leq (q)_t$ such that $p_s(g') = r_s{}^\frown r$. As $p_s(g')$ meets only a finite number of elements of A^n_α, $A_{\alpha,r}$ is finite.

(3) and (4) follow by Theorem 4.1. □

PROBLEM 5.5.

(1) *Is* $\mathrm{sh}(\omega, \mathbb{P}_{\mathrm{good}}) \geq \omega_1$ *?*

(2) *Is* $\mathrm{sh}(2, \mathbb{P}^{**}_{\mathrm{good}}) \geq \omega_1$ *?*

(3) *Is any of the inequalities* $\mathrm{sh}(\omega, \mathbb{P}_{\mathrm{good}}) > \omega_1$ *and* $\mathrm{sh}(2, \mathbb{P}^{**}_{\mathrm{good}}) > \omega_1$ *consistent with ZFC?*

THEOREM 5.6. $\mathrm{sh}(\omega, \mathbb{S}^\omega) \geq \omega_1$.

Proof. Let $\langle A_n : n \in \omega\rangle$ be a sequence of maximal antichains in $\mathbb{S}^\omega$. Given $q \in \mathbb{S}^\omega$ by induction on $n \in \omega$ we define $p_n = \langle p_{n,j} : j \in \omega\rangle \in \mathbb{S}^\omega$ so that $p_0 = q$ and the following three conditions hold:

(1) $p_{n+1} \leq p_n$, i.e., $(\forall j \in \omega)\ p_{n+1,j} \leq p_{n,j}$.

(2) $(\forall j < n)\ p_{n+1,j} \leq_n p_{n,j}$.

(3) The set $\{q \in A_n : q$ is compatible with $p_{n+1}\}$ is finite.

Now, if p is the fusion of the sequence $\langle p_n : n \in \omega\rangle$, then $p \leq q$ and p meets only a finite number of members of each antichain A_n.

To construct p_{n+1} let $S_j = \{s_{j,k} : k \leq 2^{n+1}\}$ for $j < n$ be an enumeration of the set $\{s^\frown i : s \in \mathrm{split}^n(p_{n,j})$ and $i \in \{0,1\}\}$, the set of successors of the nth splitting level of $p_{n,j}$. Let us define a finite decreasing sequence $\langle p^k_n : k < (2^{n+1})^n\rangle$ of elements of $\mathbb{S}^\omega$ so that $S_j \subseteq p^k_{n,j}$ for $j < n$ and for every function $\varphi \in {}^n(2^{n+1})$ there is k such that for some $p \in A_n$, $\langle (p^k_{n,j})_{s_{j,\varphi(j)}} : j < n\rangle \cup \langle p^k_{n,j} : j \geq n\rangle \leq p$. Set $p_{n+1} = p^k_n$ for $k = (2^{n+1})^n - 1$. Clearly, p_{n+1} meets at most $(2^{n+1})^n$ elements of A_n. □

The same proof shows that $\mathbb{S}^\omega$ satisfies axiom A.

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