THE JOURNAL OF SYMBOLIC LOGIC
Volume 60, Number 2, June 1995

HECHLER REALS

GRZEGORZ ŁABĘDZKI AND MIROSLAV REPICKÝ

Abstract. We define a σ-ideal $\mathscr{I}_{\mathscr{D}}$ on the set of functions ${}^{\omega}\omega$ with the property that a real $x \in {}^{\omega}\omega$ is a Hechler real over **V** if and only if x omits all Borel sets in $\mathscr{I}_{\mathscr{D}}$. In fact we define a topology $\mathscr{D}$ on ${}^{\omega}\omega$ related to Hechler forcing such that $\mathscr{I}_{\mathscr{D}}$ is the family of first category sets in $\mathscr{D}$. We study cardinal invariants of the ideal $\mathscr{I}_{\mathscr{D}}$.

§0. Introduction. In the paper we perform a topological characterization of Hechler reals, i.e. generic reals added by Hechler forcing $\mathbb{D}$ (defined below). This is similar to the topological characterization which is known for Cohen reals. For this reason we shall introduce the so-called dominating topology $\mathscr{D}$ on the set of reals ${}^{\omega}\omega$ and show that a real is a Hechler real if and only if it omits all Borel sets coded in the ground model which are meager in the topology $\mathscr{D}$ (Theorem 4.3).

For any ideal $\mathscr{I}$ let us consider the following cardinal invariants:

$$\begin{aligned}
\operatorname{add}(\mathscr{I}) &= \min\{|\mathscr{I}_0| : \mathscr{I}_0 \subseteq \mathscr{I} \ \& \ \bigcup \mathscr{I}_0 \notin \mathscr{I}\},\\
\operatorname{cov}(\mathscr{I}) &= \min\{|\mathscr{I}_0| : \mathscr{I}_0 \subseteq \mathscr{I} \ \& \ \bigcup \mathscr{I}_0 = \bigcup \mathscr{I}\},\\
\operatorname{non}(\mathscr{I}) &= \min\{|A| : A \subseteq \bigcup \mathscr{I} \ \& \ A \notin \mathscr{I}\},\\
\operatorname{cof}(\mathscr{I}) &= \min\{|\mathscr{I}_0| : \mathscr{I}_0 \subseteq \mathscr{I} \ \& \ (\forall A \in \mathscr{I})(\exists B \in \mathscr{I}_0)(A \subseteq B)\}.
\end{aligned}$$

For the ideal $\mathscr{I}_{\mathscr{D}}$ of meager sets in the topology $\mathscr{D}$ we shall prove that $\operatorname{cov}(\mathscr{I}_{\mathscr{D}}) = \operatorname{add}(\mathscr{M})$ and $\operatorname{non}(\mathscr{I}_{\mathscr{D}}) = \operatorname{cof}(\mathscr{M})$ (Theorem 3.6), where $\mathscr{M}$ is the ideal of meager subsets of the set of reals. The same equalities hold for the ideal of meager subsets of the Stone space of the complete Boolean algebra r. o.($\mathbb{D}$), which is isomorphic to the factor algebra *Borel*/$\mathscr{I}_{\mathscr{D}}$. Using the ideas of [2] and [3], we easily get the equalities $\operatorname{add}(\mathscr{I}_{\mathscr{D}}) = \operatorname{add}(\mathscr{I}_{\mathscr{D}}^{\sigma}) = \omega_1$ and $\operatorname{cof}(\mathscr{I}_{\mathscr{D}}) = \operatorname{cof}(\mathscr{I}_{\mathscr{D}}^{\sigma}) = 2^{\omega}$ (Corollary 6.4). The ideal $\mathscr{I}_{\mathscr{D}}$ is orthogonal to the ideal of meager sets of reals as well as to the ideal of measure zero sets (Theorem 5.5), but the intersection $\mathscr{I}_{\mathscr{D}} \cap \mathscr{M}$ is still large in some sense (Theorem 5.4).

The dominating topology $\mathscr{D}$ is a completely regular c.c.c. topology and the density number for $\mathscr{D}$ is equal to the cardinal number $\mathfrak{d}$—minimal cardinality of a dominating family of functions in ${}^{\omega}\omega$ with respect to the (pre)ordering $\leq^*$ defined by $f \leq^* g$ iff $(\forall^{\infty} k)(f(k) \leq g(k))$, for $f, g \in {}^{\omega}\omega$. Here $(\forall^{\infty} k)$ stands for

Received February 1, 1994; revised August 1, 1994.
1991 *Mathematics Subject Classification*. Primary 03E40.
Key words and phrases: Hechler real, dominating topology, meager sets, cardinal invariants.
The second author was supported by GA SAV 2/1224/94 of Slovak Academy of Sciences.

0022-4812/95/6002-0006/$02.50

$(\exists n)(\forall k > n)$. Similarly, $(\exists^\infty k)$ abbreviates $(\forall n)(\exists k > n)$. Further let us recall that $\mathfrak{b}$ is the minimal cardinality of an unbounded family of functions. We use standard set-theoretical notation.

We choose the following form of Hechler forcing:

$$\mathbb{D} = \{\langle s, f\rangle : s \in {}^{<\omega}\omega \ \& \ f \in {}^{\omega}\omega\},$$

$$\langle s, f\rangle \leq \langle t, g\rangle \quad \text{iff} \quad t \subseteq s \ \& \ (\forall k)[(\mathrm{lh}(t) \leq k < \mathrm{lh}(s) \rightarrow g(k) \leq s(k)) \\ \& \ (k \geq \mathrm{lh}(s) \rightarrow g(k) \leq f(k))].$$

Note that the set $\{\langle s, f\rangle \in \mathbb{D} : s \subseteq f\}$ is a separative dense subset of $\mathbb{D}$, and usually this set with the same ordering is denoted as Hechler forcing. The reason for taking the set $\mathbb{D}$ is to simplify some of our future notation.

§**1. Dominating topology** $\mathscr{D}$**.** Let $\dot{d}$ be the canonical name for a generic real over $\mathbb{D}$ and let $\langle s, f\rangle \in \mathbb{D}$. Then $\langle s, f\rangle \Vdash$ "$s \subseteq \dot{d}$ & $(\forall k \geq \mathrm{lh}(s))(f(k) \leq \dot{d}(k))$". We put $U_{s,f} = \{x \in {}^{\omega}\omega : s \subseteq x \ \& \ (\forall k \geq \mathrm{lh}(s))(f(k) \leq x(k))\}$. Then $\langle s, f\rangle \Vdash$ "$\dot{d} \in U_{s,f}$" when looking at the definition of the set $U_{s,f}$, and clearly, this is the whole information that $\langle s, f\rangle$ can give about $\dot{d}$.

LEMMA 1.1. *The family* $\{U_{s,f} : s \in {}^{<\omega}\omega \ \& \ f \in {}^{\omega}\omega\}$ *is a base of a topology. Moreover, every set* $U_{s,f}$ *is clopen in this topology.*

PROOF. The first part follows from the fact that whenever $U_{s,f} \cap U_{t,g} \neq \emptyset$ then $U_{s,f} \cap U_{t,g} = U_{s\cup t, \max\{f,g\}}$. Let o be the function constantly equal to zero on ω. Now since

$${}^{\omega}\omega - U_{s,f} = \bigcup\{U_{t,o} : (\exists k \in \mathrm{dom}(s) \cap \mathrm{dom}(t))(s(k) \neq t(k)) \\ \vee (\exists k \in \mathrm{dom}(t) - \mathrm{dom}(s))(t(k) < f(k))\},$$

these sets are all clopen in the generated topology. □

The topology described in the above lemma will be denoted by $\mathscr{D}$, and let $X_{\mathscr{D}}$ denote the topological space $\langle {}^{\omega}\omega, \mathscr{D}\rangle$. The family of all first category (meager) sets in $X_{\mathscr{D}}$ is denoted by $\mathscr{I}_{\mathscr{D}}$. Note that the sets $U_{s,o} = [s]$, for $s \in {}^{<\omega}\omega$, are just basic clopen sets of the Baire space ${}^{\omega}\omega$. Hence the sets $U_{s,f}$ are all closed subsets of the Baire space. We will work permanently with both the dominating and the Baire topology on the set of functions, and to avoid misunderstandings the following convention will be useful. When we speak about notions concerning the dominanting topology we will use $\mathscr{D}$ in the prefix of the words in the absence of any other such description, e.g., $\mathscr{D}$-open, $\mathscr{D}$-meager, $\mathscr{D}$-dense, or when we use notions of both topologies for a single object, e.g., open $\mathscr{D}$-dense set. In the opposite case the topological notions on the set of functions will refer to the Baire topology.

Let *Borel* denote the family of Borel subsets of the Baire space, and let *Baire* denote the family of sets having the Baire property in the Baire space. Let *Borel*$_{\mathscr{D}}$, *Baire*$_{\mathscr{D}}$ denote the corresponding families for the space $X_{\mathscr{D}}$.

Note that *Borel* $\subsetneq$ *Borel*$_{\mathscr{D}}$. The inclusion is obvious since each open set is $\mathscr{D}$-open. To see the inequality let us choose an almost disjoint family $\mathscr{X}$ of subsets of ω of size 2^ω and let $X \subseteq {}^{\omega}2$ be the set of all characteristic functions of sets in $\mathscr{X}$. There is a set $Y \subseteq X$ which is not $\mathbf{\Pi}^1_1$. As $Y \subseteq {}^{\omega}\omega$ we can define a $\mathscr{D}$-open set $U = \bigcup_{f \in Y} U_{\emptyset,f}$. Now as

$$Y = \{x \in {}^{\omega}\omega : x \in U \ \& \ (\forall y \in U)[(\forall n)(y(n) \leq x(n)) \to y = x]\}$$

is not $\mathbf{\Pi}^1_1$, U cannot be Borel.

LEMMA 1.2. (a) *The topology $\mathscr{D}$ is stronger than the Baire topology.*

(b) *The topological space $X_{\mathscr{D}}$ is a zero-dimensional (and therefore Tychonoff) c.c.c. space.*

(c) *$X_{\mathscr{D}}$ is a Baire space, i.e. no open set in $X_{\mathscr{D}}$ is meager.*

(d) *The ideal $\mathscr{I}_{\mathscr{D}}$ has a Borel base (in sense of the Baire topology).*

(e) *The factor algebras Borel$/\mathscr{I}_{\mathscr{D}}$, Borel$_{\mathscr{D}}/\mathscr{I}_{\mathscr{D}}$, Baire$/\mathscr{I}_{\mathscr{D}}$ and Baire$_{\mathscr{D}}/\mathscr{I}_{\mathscr{D}}$ are all complete Boolean algebras isomorphic to the Boolean algebra* r. o.$(X_{\mathscr{D}})$ *of regular open sets in $X_{\mathscr{D}}$.*

(f) *The minimal size of a dense subset of $X_{\mathscr{D}}$ is $\mathfrak{d}$.*

PROOF. The assertions (a), (b) are clear.

(c) Let U_n be open dense in $X_{\mathscr{D}}$ for $n \in \omega$ and let $U_{s,f}$ be any basic clopen set. Without loss of generality we can assume that $U_{n+1} \subseteq U_n$ for all n. By induction on $n \in \omega$ let us define a sequence of basic clopen sets U_{s_n,f_n} such that $U_{s_0,f_0} = U_{s,f}$ and $U_{s_{n+1},f_{n+1}} \subseteq U_{s_n,f_n} \cap U_n$. Let $x \in {}^{\omega}\omega$ be defined so that $s_n \subseteq x$ for all n. Clearly, $x \in U_n$ for every $n \in \omega$, and so $\bigcap_{n\in\omega} U_n$ is dense in $X_{\mathscr{D}}$.

(d) All basic clopen sets of $X_{\mathscr{D}}$ are closed. Hence by c.c.c. of $X_{\mathscr{D}}$, every $\mathscr{D}$-open dense set contains an F_σ $\mathscr{D}$-open dense set and every $\mathscr{D}$-meager set can be covered by a $G_{\delta\sigma}$ $\mathscr{D}$-meager set.

(e) It can easily be seen that regular open sets in $X_{\mathscr{D}}$ are representatives of the factors of all mentioned factor algebras and that all these correspondences preserve order.

(f) If $X \subseteq X_{\mathscr{D}}$ is $\mathscr{D}$-dense then for any $f \in {}^{\omega}\omega$ there is $x \in X \cap U_{\emptyset,f}$, and consequently, $f(k) \leq x(k)$ for all $k \in \omega$. In the opposite direction, let $X \subseteq {}^{\omega}\omega$ be any dominating family in ${}^{\omega}\omega$ such that $|X| = \mathfrak{d}$. Then

$$X^* = \{x \in {}^{\omega}\omega : (\exists y \in X)(\forall^{\infty} k)\, x(k) = y(k)\}$$

is $\mathscr{D}$-dense. □

Note that there is a more general way yet to define an ideal $\mathscr{I}_P$ on the set of reals for a c.c.c. forcing notion P so that r. o.$(P) \simeq$ *Borel*$/\mathscr{I}_P$ holds. The only condition which has to be put on P is that a generic filter on P is definable from a real or, equivalently, that r. o.(P) is countably generated. Let τ be the canonical P-name for a generic real on P. The ideal $\mathscr{I}_P$ can be defined by means of the σ-complete homomorphism h : *Borel* $\to$ r. o.(P) defined by $h(B) = \|\tau \in B^*\|_{\text{r.o.}(P)}$, for $B \in$ *Borel*, where B^* is the forcing term for a Borel set with the same Borel code as the set B has.

A similar approach using the notion of category base was already developed for amoeba reals in [10].

$\mathscr{D}$-meager sets and F_σ $\mathscr{D}$-meager sets. Let $\mathscr{I}^{\sigma}_{\mathscr{D}}$ denote the ideal generated by all F_σ subsets of the Baire space which are $\mathscr{D}$-meager. The family $\mathscr{I}^{\sigma}_{\mathscr{D}}$ is a σ-ideal with the following description.

LEMMA 1.3. *Let $A \subseteq {}^{\omega}\omega$. The following conditions are equivalent.*

(i) $A \in \mathscr{I}^{\sigma}_{\mathscr{D}}$.

(ii) *There exists a sequence of open $\mathscr{D}$-dense sets $U_n \subseteq {}^\omega\omega$ such that $A \cap \bigcap_{n\in\omega} U_n = \emptyset$.*

(iii) *There exists a G_δ $\mathscr{D}$-dense set $G \subseteq {}^\omega\omega$ such that $A \cap G = \emptyset$.*

(iv) *There exists a sequence $\langle s_n : n \in \omega\rangle$ of elements of ${}^{<\omega}\omega$ such that $A \cap \bigcap_{m\in\omega} \bigcup_{n>m}[s_n] = \emptyset$ and*

$$(*)\quad (\forall t \in {}^{<\omega}\omega)(\forall f \in {}^\omega\omega)(\exists^\infty k)[t \subseteq s_k \;\&\; (\forall i \in \mathrm{dom}(s_k - t))(s_k(i) \geq f(i))].$$

PROOF. (i)$\leftrightarrow$(ii)$\leftrightarrow$(iii), since the intersection of countably many open $\mathscr{D}$-dense sets is $\mathscr{D}$-dense.

(iii)$\to$(iv). Let G be a G_δ $\mathscr{D}$-dense subset of the Baire space such that $A \cap G = \emptyset$. The set G is the intersection of a decreasing sequence of open sets U_n for $n \in \omega$, and every open set U_n is the union of a sequence of disjoint basic clopen sets in ${}^\omega\omega$, say $[t_{n,m}]$, $m \in \omega$, such that $\mathrm{lh}(t_{n,m}) \geq n$ for all $m \in \omega$. Let $\{s_n : n \in \omega\}$ be an enumeration of the family $\{t_{n,m} : n, m \in \omega\}$. We prove that this is a sequence such as we are looking for. Let $t \in {}^{<\omega}\omega$, $f \in {}^\omega\omega$. Every U_n is $\mathscr{D}$-dense; hence for $n \geq \mathrm{lh}(t)$ there is $x \in U_n \cap [t]$ such that $x(i) \geq f(i)$ for all $i \in \omega - \mathrm{dom}(t)$. Choose $m, k \in \omega$ such that $x \in [t_{n,m}]$ and $t_{n,m} = s_k$. Then $s_k(i) \geq f(i)$ for all $i \in \mathrm{dom}(s_k - t)$, since $\mathrm{lh}(s_k) \geq n$ and for every n we can find such k there are infinitely many k's with the same property. We will finish this part of the proof by showing that $G = \bigcap_{m\in\omega}\bigcup_{n>m}[s_n]$. Let $x \in \bigcap_m \bigcup_{n>m}[s_n]$, i.e. for infinitely many $n \in \omega$, $x \in [s_n]$. Since every set U_n was partitioned into disjoint clopen sets, it follows that $x \in U_n$ for infinitely many $n \in \omega$, and by monotonicity of this sequence we get $x \in G$. The inverse inclusion is obvious too.

(iv)$\to$(ii). The sets $U_m = \bigcup_{n>m}[s_n]$ are open $\mathscr{D}$-dense. □

We will need the following observation. For $f \in {}^\omega\omega$ the set

$$D_f = \{x \in {}^\omega\omega : (\forall^\infty k)(x(k) \geq f(k))\}$$

is open dense in $X_\mathscr{D}$ and D_f is the disjoint union of basic clopen sets $U_{s,f}$ for $s \in S_f$, where

$$S_f = \{\emptyset\} \cup \bigcup_{k\in\omega}\{s \in {}^{k+1}\omega : s(k) < f(k)\}.$$

For every $s \in S_f$, $s \in {}^k\omega$ for some k, let $\pi_{s,f} : U_{s,f} \to {}^\omega\omega$ be the bijective mapping defined by

$$\pi_{s,f}(x)(i) = x(k+i) - f(k+i).$$

Notice that if the sets $U_{s,f}$ and ${}^\omega\omega$ are both endowed with the topology induced either by the Baire topology or by $\mathscr{D}$, then $\pi_{s,f}$ is a homeomorphism.

LEMMA 1.4. *Let D be an open dense set in $X_\mathscr{D}$ which is the union of a sequence of basic clopen sets $\{U_{s_n,f_n} : n \in \omega\}$, and let $f \in {}^\omega\omega$ eventually dominate all f_n, $n \in \omega$. Then for every $s \in S_f$ the set $\pi_{s,f}(D \cap U_{s,f}) \in \mathscr{D}$ is an open $\mathscr{D}$-dense subset of the Baire space.*

PROOF. Since $\pi_{s,f}$ is a homeomorphism with respect to the topology $\mathscr{D}$, the set $\pi_{s,f}(D \cap U_{s,f})$ is $\mathscr{D}$-dense. Since f dominates the functions f_n, $U_{s_n,f_n} \cap U_{s,f} =$

$\bigcup\{U_{t,f} : U_{t,f} \subseteq U_{s_n,f_n} \cap U_{s,f}\}$ for all $n \in \omega$. From the last equality it can easily be seen that $\pi_{s,f}(D \cap U_{s,f})$ is open. □

LEMMA 1.5. *There are mappings*

(i) $\alpha : \mathcal{I}_{\mathcal{D}} \to {}^\omega\omega$,

(ii) $\beta : \mathcal{I}_{\mathcal{D}} \times {}^\omega\omega \to \mathcal{I}_{\mathcal{D}}^\sigma$,

(iii) $\gamma : \mathcal{I}_{\mathcal{D}}^\sigma \times {}^\omega\omega \to \mathcal{I}_{\mathcal{D}}$,

(iv) $\delta : \mathcal{P}({}^\omega\omega) \times {}^\omega\omega \to \mathcal{P}({}^\omega\omega)$,

such that for all $A \in \mathcal{I}_{\mathcal{D}}$, $B \in \mathcal{I}_{\mathcal{D}}^\sigma$, $C \subseteq {}^\omega\omega$ *the following conditions hold*:

(1) $\alpha(A) \leq^* f \;\&\; \beta(A,f) \subseteq B \to A \subseteq \gamma(B,f)$;

(2) $\alpha(A) \leq^* f \;\&\; C \cap \beta(A,f) = \emptyset \to A \cap \delta(C,f) = \emptyset$;

(3) $\alpha(A) \leq^* f \;\&\; \delta(C,f) \subseteq A \to C \subseteq \beta(A,f)$;

(4) $C \neq \emptyset \to \delta(C,f) \neq \emptyset \;\&\; |\delta(C,f)| \leq \omega \cdot |C|$.

PROOF. Let $A \in \mathcal{I}_{\mathcal{D}}$, $f \in {}^\omega\omega$. There is a sequence U_n, $n \in \omega$, of open dense sets in $X_{\mathcal{D}}$ such that $A \cap \bigcap_{n\in\omega} U_n = \emptyset$. For each n let $\{U_{s^n_m, f^n_m} : m \in \omega\}$ be a maximal disjoint family of basic clopen sets, subsets of U_n. Let $\alpha(A) \in {}^\omega\omega$ be a function which eventually dominates all functions f^n_m, $n, m \in \omega$. Now if $\alpha(A) \leq^* f$ we set $\beta(A,f) = \bigcup_{s\in S_f} \pi_{s,f}(A \cap U_{s,f})$. Note that by Lemma 1.4 the set

$$\bigcap_{s\in S_f} \pi_{s,f}\left(\bigcap_{n\in\omega} U_n \cap U_{s,f}\right) = \bigcap_{s\in S_f}\bigcap_{n\in\omega} \pi_{s,f}(U_n \cap U_{s,f})$$

is G_δ $\mathcal{D}$-dense disjoint with $\beta(A,f)$. Hence $\beta(A,f) \in \mathcal{I}_{\mathcal{D}}^\sigma$. If $\alpha(A) \not\leq^* f$, we set $\beta(A,f) = \emptyset$.

For $B \in \mathcal{I}_{\mathcal{D}}^\sigma$, $C \subseteq {}^\omega\omega$, $f \in {}^\omega\omega$ we define

$$\gamma(B,f) = \bigcup_{s\in S_f} \pi_{s,f}^{-1}(B) \cup ({}^\omega\omega - D_f),$$

$$\delta(C,f) = \bigcup_{s\in S_f} \pi_{s,f}^{-1}(C).$$

Since the functions $\pi_{s,f}$ are homeomorphisms $\gamma(B,f)$ is in $\mathcal{I}_{\mathcal{D}}$, and since the sets $U_{s,f}$ for $s \in S_f$ form a partition of D_f, the conditions (1)–(4) can be easily verified. □

Using these four mappings we easily get the inequalities in the following lemma.

LEMMA 1.6. (a) $\min\{\mathfrak{b}, \mathrm{add}(\mathcal{I}_{\mathcal{D}}^\sigma)\} \leq \mathrm{add}(\mathcal{I}_{\mathcal{D}})$;

(b) $\min\{\mathfrak{b}, \mathrm{cov}(\mathcal{I}_{\mathcal{D}}^\sigma)\} \leq \mathrm{cov}(\mathcal{I}_{\mathcal{D}})$;

(c) $\mathrm{non}(\mathcal{I}_{\mathcal{D}}) \leq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{I}_{\mathcal{D}}^\sigma)\}$;

(d) $\mathrm{cof}(\mathcal{I}_{\mathcal{D}}) \leq \max\{\mathfrak{d}, \mathrm{cof}(\mathcal{I}_{\mathcal{D}}^\sigma)\}$.

PROOF. (a) Let $\mathcal{A} \subseteq \mathcal{I}_{\mathcal{D}}$, $|\mathcal{A}| < \min\{\mathfrak{b}, \mathrm{add}(\mathcal{I}_{\mathcal{D}}^\sigma)\}$. Then there is $f \in {}^\omega\omega$ such that f dominates all functions $\alpha(A)$ for $A \in \mathcal{A}$, and $B = \bigcup_{A\in\mathcal{A}} \beta(A,f) \in \mathcal{I}_{\mathcal{D}}^\sigma$. Hence by (1), $\bigcup\mathcal{A} \subseteq \gamma(B,f) \in \mathcal{I}_{\mathcal{D}}$.

(b) Let $\mathcal{A} \subseteq \mathcal{I}_{\mathcal{D}}$, $|\mathcal{A}| < \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{I}_{\mathcal{D}}^\sigma)\}$. There is $f \in {}^\omega\omega$ dominating all $\alpha(A)$, $A \in \mathcal{A}$, and by the same assumption, the family $\{\beta(A,f) : A \in \mathcal{A}\}$ is not a covering family. Hence, by (2) and (4), $\bigcup\mathcal{A} \neq {}^\omega\omega$.

(c) Let $X \subseteq {}^\omega\omega$ be a dominating family of functions, and let $C \subseteq {}^\omega\omega$ be such that $|X| = \mathfrak{d}$, $C \notin \mathcal{I}_{\mathcal{D}}^\sigma$ and $|C| = \mathrm{non}(\mathcal{I}_{\mathcal{D}}^\sigma)$. By (4), the set $A = \bigcup_{f\in X} \delta(C,f)$

has cardinality equal to the cardinal $\max\{\mathfrak{d}, \mathrm{non}(\mathcal{I}^{\sigma}_{\mathcal{D}})\}$. We show that $A \notin \mathcal{I}_{\mathcal{D}}$. Otherwise there would exist $f \in X$ such that $\alpha(A) \leq^* f$. Now $\delta(C, f) \subseteq A$, so, by (3), $C \subseteq \beta(A, f)$, which contradicts our assumption that $C \notin \mathcal{I}^{\sigma}_{\mathcal{D}}$. Therefore $A \notin \mathcal{I}_{\mathcal{D}}$.

(d) Let $X \subseteq {}^{\omega}\omega$ be a dominating family and $\mathcal{B}$ a cofinal family in $\mathcal{I}^{\sigma}_{\mathcal{D}}$ such that $|X| = \mathfrak{d}$ and $|\mathcal{B}| = \mathrm{cof}(\mathcal{I}^{\sigma}_{\mathcal{D}})$. Then the family $\mathcal{A} = \{\gamma(B, f) : B \in \mathcal{B} \ \&\ f \in X\}$ is cofinal in $\mathcal{I}_{\mathcal{D}}$ and $|\mathcal{A}| = \max\{\mathfrak{d}, \mathrm{cof}(\mathcal{I}^{\sigma}_{\mathcal{D}})\}$. □

§**2. Some remarks on Stone spaces.** The results of this section are at least implicitly known (see e.g. [8], [4], [1]).

Let X, Y be topological spaces. A function $f : X \to Y$ is said to be *δ-open* (see [8]) iff the inverse image of any nowhere dense subset of Y is a nowhere dense subset of X. We say that a function $f : X \to Y$ is *δ-continuous* iff the image of any nowhere dense set is nowhere dense.

We write $\mathrm{cov}(X)$ and $\mathrm{non}(X)$ for $\mathrm{cov}(\mathcal{M}_X)$ and $\mathrm{non}(\mathcal{M}_X)$, respectively, where $\mathcal{M}_X$ denotes the ideal of meager sets in X. The following lemma is easy.

LEMMA 2.1. (i) *If there is a δ-continuous function from X onto Y, then* $\mathrm{cov}(Y) \leq \mathrm{cov}(X)$ *and* $\mathrm{non}(X) \leq \mathrm{non}(Y)$.

(ii) *If there is a δ-open function from X into Y, then* $\mathrm{cov}(X) \leq \mathrm{cov}(Y)$ *and* $\mathrm{non}(Y) \leq \mathrm{non}(X)$. □

LEMMA 2.2. (i) *If $f : X \to Y$ is a continuous irreducible function, then f is δ-continuous.*

(ii) *If $f : X \to Y$ is a continuous open function, then f is δ-open.*

PROOF. (i) Let $N \subseteq X$ be a closed nowhere dense set, and assume that $\mathrm{Int}(f(N)) \neq \emptyset$. Choose a nonempty open set $U \subseteq f(N)$. Since $f^{-1}(U)$ is open, the set $F = (X - f^{-1}(U)) \cup N$ is a proper closed subset of X and so f is not irreducible.

(ii) is easy. □

For a Boolean algebra A, $\mathrm{St}(A)$ denotes the space of all ultrafilters in A with the clopen base for topology formed by sets $[a]_A = \{p \in \mathrm{St}(A) : a \in p\}$, for $a \in A$.

LEMMA 2.3. *Let B be a subalgebra of a Boolean algebra A and let $f : \mathrm{St}(A) \to \mathrm{St}(B)$ be the continuous mapping defined by $f(p) = p \cap B$.*

(a) *B is a regular subalgebra of A if and only if f is δ-open.*

(b) *B is a dense subalgebra of A if and only if f is irreducible.*

PROOF. (a) "B is a regular subalgebra of A" iff "whenever $\bigvee^B B_0 = \mathbf{1}$ for some $B_0 \subseteq B$ then $\bigvee^A B_0 = \mathbf{1}$" iff "whenever the set $U = \bigcup_{a \in B_0} [a]_B$ is open dense in $\mathrm{St}(B)$ for some $B_0 \subseteq B$ then $f^{-1}(U) = \bigcup_{a \in B_0} [a]_A$ is open dense in $\mathrm{St}(A)$".

(b) "f is not irreducible" iff "there is a proper closed set $N \subseteq \mathrm{St}(A)$ such that the function $f \restriction N$ is onto $\mathrm{St}(B)$" iff "there is an $a \in A$, $a \neq \mathbf{1}$, such that $f \restriction [a]_A$ is onto $\mathrm{St}(B)$" iff "there is an $a \in A$, $a \neq \mathbf{1}$, such that for every $p \in \mathrm{St}(B)$ there is $p' \in [a]_A$ extending p" iff "there is $a \in A$, $a \neq \mathbf{1}$, with no $b \in B$ such that $\mathbf{0} \neq b \leq a$". □

LEMMA 2.4. *Let $f : X \to Y$ be continuous open. Then the function $e : \mathrm{r.o.}(Y) \to \mathrm{r.o.}(X)$ defined by $e(U) = f^{-1}(U)$ is a complete embedding of complete Boolean algebras.*

PROOF. First note that f^{-1} commutes with the operations cl and Int: Let $A \subseteq Y$. $\mathrm{cl}(f^{-1}(A)) \subseteq f^{-1}(\mathrm{cl}(A))$ and if $x \in X - \mathrm{cl}(f^{-1}(A))$, then $U \cap f^{-1}(A) = \emptyset$ for some open $U \subseteq X$ such that $x \in U$. Hence $f(x) \in f(U)$ and $f(U) \cap A = \emptyset$.

As f is open, $x \notin f^{-1}(\mathrm{cl}(A))$ and so $\mathrm{cl}(f^{-1}(A)) = f^{-1}(\mathrm{cl}(A))$. Consequently, $f^{-1}(\mathrm{Int}(A)) = f^{-1}(Y - \mathrm{cl}(Y - A)) = X - \mathrm{cl}(X - f^{-1}(A)) = \mathrm{Int}(f^{-1}(A))$.

For $U \in \mathrm{r.o.}(Y)$, $f^{-1}(U) = f^{-1}(\mathrm{Int}(\mathrm{cl}(U))) = \mathrm{Int}(\mathrm{cl}(f^{-1}(U)))$ is regular open in X and so the function e is well-defined. It can easily be seen that e is a Boolean embedding. We show that e is complete: $e(\bigvee_\xi U_\xi) = f^{-1}(\mathrm{Int}(\mathrm{cl}(\bigcup_\xi U_\xi))) = \mathrm{Int}(\mathrm{cl}(\bigcup_\xi f^{-1}(U_\xi))) = \bigvee_\xi e(U_\xi)$. □

Lemma 2.5. *Let X be a zero-dimensional topological space. Then the mapping $F : X \to \mathrm{St}(\mathrm{clopen}(X))$ defined by $F(x) = \{U : x \in U \in \mathrm{clopen}(X)\}$ is a topological embedding.*

Proof. Since each point has a neighbourhood base consisting of clopen sets, the mapping F is one-to-one. There is a correspondence between the family of clopen sets in X and the family of clopen sets in $\mathrm{St}(\mathrm{clopen}(X))$ which ensures that F is an embedding. □

Lemma 2.6. *Let X be a zero-dimensional Baire space. Then*:

(i) $\mathrm{cov}(X) \leq \mathrm{cov}(\mathrm{St}(\mathrm{clopen}(X))) = \mathrm{cov}(\mathrm{St}(\mathrm{r.o.}(X)))$,

(ii) $\mathrm{non}(X) \geq \mathrm{non}(\mathrm{St}(\mathrm{clopen}(X))) = \mathrm{non}(\mathrm{St}(\mathrm{r.o.}(X)))$.

Proof. The equalities hold by 2.1 and 2.3 since $\mathrm{clopen}(X)$ is a dense regular subalgebra of $\mathrm{r.o.}(X)$. The inequalities are consequences of 2.5 since X is homeomorphic to a nonmeager (X is a Baire space) dense subset of $\mathrm{St}(\mathrm{clopen}(X))$. □

§3. **Cardinal coefficients.** Let $\varphi : {}^\omega\omega \to {}^\omega 2$ be the mapping defined by $\varphi(x)(k) = x(k) \bmod 2$, for $x \in {}^\omega\omega$. It is not hard to see that φ is continuous and open as a mapping from $X_{\mathscr{D}}$ to ${}^\omega 2$ as well as a mapping from the Baire space ${}^\omega\omega$ to ${}^\omega 2$. Hence we can apply for both topologies on ${}^\omega\omega$ the following lemma with an application for the ideal $\mathscr{I}^\sigma_{\mathscr{D}}$.

Lemma 3.1. *If a mapping $\varphi : X \to Y$ is continuous open, then $\varphi^{-1}(U)$ is an open dense subset of X whenever U is an open dense subset of Y.*

Theorem 3.2. $\mathrm{cov}(\mathscr{I}^\sigma_{\mathscr{D}}) = \mathrm{cov}(\mathscr{M})$, $\mathrm{non}(\mathscr{I}^\sigma_{\mathscr{D}}) = \mathrm{non}(\mathscr{M})$.

Proof. Since $\mathscr{I}^\sigma_{\mathscr{D}} \subseteq \mathscr{M}$, immediately we have $\mathrm{cov}(\mathscr{M}) \leq \mathrm{cov}(\mathscr{I}^\sigma_{\mathscr{D}})$, $\mathrm{non}(\mathscr{I}^\sigma_{\mathscr{D}}) \leq \mathrm{non}(\mathscr{M})$. For the proof of the reverse inequalities we use the fact that the cardinal invariants of the ideal of first category sets in all Polish spaces are the same. So, using the previous lemma and a remark before it, for any open dense set $U \subseteq {}^\omega 2$, $\varphi^{-1}(U)$ is an open $\mathscr{D}$-dense subset of ${}^\omega\omega$ and hence $\varphi^{-1}(A) \in \mathscr{I}^\sigma_{\mathscr{D}}$ whenever A is a meager subset of ${}^\omega 2$. It follows that $\mathrm{cov}(\mathscr{I}^\sigma_{\mathscr{D}}) \leq \mathrm{cov}(\mathscr{M})$ and $\mathrm{non}(\mathscr{M}) \leq \mathrm{non}(\mathscr{I}^\sigma_{\mathscr{D}})$. Therefore the equalities hold. □

Lemma 3.3. $\mathrm{cov}(\mathrm{St}(\mathrm{r.o.}(X_{\mathscr{D}}))) \leq \mathrm{cov}(\mathscr{M})$, $\mathrm{non}(\mathscr{M}) \leq \mathrm{non}(\mathrm{St}(\mathrm{r.o.}(X_{\mathscr{D}})))$.

Proof. The function $\varphi : X_{\mathscr{D}} \to {}^\omega 2$ defined by $\varphi(x)(k) = x(k) \bmod 2$ is continuous and open. Hence by 2.4 there is a complete embedding $e : \mathrm{r.o.}({}^\omega 2) \to \mathrm{r.o.}(X_{\mathscr{D}})$. By 2.3(a) there is a δ-open function $\psi : \mathrm{St}(\mathrm{r.o.}(X_{\mathscr{D}})) \to \mathrm{St}(\mathrm{r.o.}({}^\omega 2))$. By compactness, $\mathrm{St}(\mathrm{clopen}({}^\omega 2)) \simeq {}^\omega 2$, and since $\mathrm{clopen}({}^\omega 2)$ is a regular subalgebra of $\mathrm{r.o.}({}^\omega 2)$, by 2.3(a) there is a δ-open function $\theta : \mathrm{St}(\mathrm{r.o.}({}^\omega 2)) \to {}^\omega 2$. Now as the composition of the δ-open functions ψ, θ is δ-open, the inequalities of the lemma are consequences of 2.1(ii). □

Lemma 3.4. $\mathrm{cov}(\mathrm{St}(\mathrm{r.o.}(X_{\mathscr{D}}))) \leq \mathfrak{b}$, $\mathfrak{d} \leq \mathrm{non}(\mathrm{St}(\mathrm{r.o.}(X_{\mathscr{D}})))$.

Proof. Let $G = \{p \in \mathrm{St}(\mathrm{r.o.}(X_{\mathscr{D}})) : (\forall n)(\exists s \in {}^n\omega)([s] \in p)\}$. The set G is a G_δ dense subset of $\mathrm{St}(\mathrm{r.o.}(X_{\mathscr{D}}))$. Let us consider the mapping $\varphi : G \to {}^\omega\omega$

defined by $\varphi(p) = \bigcup\{s \in {}^{<\omega}\omega : [s] \in p\}$. For $f \in {}^{\omega}\omega$ let $G_f = \{p \in G$ $(\forall^\infty k)(\varphi(p)(k) \geq f(k))\}$ and let $U_f = \bigcup_{s\in {}^{<\omega}\omega}\{p \in \mathrm{St}(\mathrm{r.o.}(X_{\mathscr{D}})) : U_{s,f} \in p\}$ The set U_f is open dense in $\mathrm{St}(\mathrm{r.o.}(X_{\mathscr{D}}))$. Let $p \in U_f \cap G$. Then for som $s \in {}^{<\omega}\omega$ we have $U_{s,f} \in p$, and by the definition of $\varphi(p)$, $[\varphi(p)\restriction k] \in p$ for al $k \in \omega$. It follows that $U_{s,f} \cap [\varphi(p)\restriction k] \neq \emptyset$ for all k, and so $\varphi(p)(k) \geq f(k)$ fo all $k \geq \mathrm{lh}(s)$. Hence $p \in G_f$, and so the set G_f contains the G_δ set $U_f \cap G$.

Now if $X \subseteq {}^{\omega}\omega$ is an unbounded family of functions, then $\bigcap_{f\in X} G_f = \emptyset$, an if $A \subseteq \mathrm{St}(\mathrm{r.o.}(X_{\mathscr{D}}))$ has cardinality less than $\mathfrak{d}$, then $A \cap G_f = \emptyset$ for some $f \in {}^{\omega}\omega$ Hence the inequalities take place. □

Notice that the last two lemmata are based on the facts that the forcing $\mathbb{D}$ add a Cohen real and a dominating real. The proof of the next theorem can be foun in [6] (see also [12] and [11]), so we do not prove it here.

THEOREM 3.5 (J. Truss, A.W. Miller, J. Cichoń). (a) $\mathrm{add}(\mathscr{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathscr{M})\}$

(b) $\mathrm{cof}(\mathscr{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathscr{M})\}$. □

THEOREM 3.6. (a) $\mathrm{cov}(X_{\mathscr{D}}) = \mathrm{cov}(\mathrm{St}(\mathrm{clopen}(X_{\mathscr{D}}))) = \mathrm{cov}(\mathrm{St}(\mathrm{r.o.}(X_{\mathscr{D}}))) = \mathrm{add}(\mathscr{M})$

(b) $\mathrm{non}(X_{\mathscr{D}}) = \mathrm{non}(\mathrm{St}(\mathrm{clopen}(X_{\mathscr{D}}))) = \mathrm{non}(\mathrm{St}(\mathrm{r.o.}(X_{\mathscr{D}}))) = \mathrm{cof}(\mathscr{M})$.

PROOF. Put together the results 1.6(b)(c), 3.2, 2.6, 3.3, 3.4, and 3.5. □

COROLLARY 3.7. $\mathrm{add}(\mathscr{M})$ *is the least cardinal* κ *such that there is a family* $\mathscr{X}$ *o dense subsets of* $\mathbb{D}$, $|\mathscr{X}| = \kappa$, *and there is no* $\mathscr{X}$*-generic filter on* $\mathbb{D}$.

PROOF. It is an easy exercise to prove that the above cardinal κ is equal to th cardinal $\mathrm{cov}(\mathrm{St}(\mathrm{r.o.}(X_{\mathscr{D}})))$. □

Now we know that the additivity of any of the ideals $\mathscr{I}_{\mathscr{D}}$, $\mathscr{I}_{\mathscr{D}}^{\sigma}$ is between ω_1 an $\mathrm{add}(\mathscr{M})$, and the cofinality is between $\mathrm{cof}(\mathscr{M})$ and 2^ω. By Lemma 1.6, moreover $\mathrm{add}(\mathscr{I}_{\mathscr{D}}) = \omega_1$ implies $\mathrm{add}(\mathscr{I}_{\mathscr{D}}^{\sigma}) = \omega_1$, and $\mathrm{cof}(\mathscr{I}_{\mathscr{D}}) = 2^\omega$ implies $\mathrm{cof}(\mathscr{I}_{\mathscr{D}}^{\sigma}) = 2^\omega$ In §6 we shall prove that the additivity is equal to ω_1 and the cofinality is equa to 2^ω for both ideals.

§**4. Hechler reals.** Let us recall that Borel sets of reals can be represented b means of Borel codes (see [9]). The set BC of Borel codes is a Π^1_1 subset of ${}^{\omega}\omega$ and for every real $c \in BC$ a Borel set B_c is assigned in such a way that the relation $B_c \subseteq B_d$, $B_c = \emptyset$, $B_d = B_c \cap B_b$, $B_d = B_c - B_b$, $B_c = \bigcup_{n\in\omega} B_{c_n}$, etc., are all Π^1_1.

Let us fix an effective enumeration $\{t_m : m \in \omega\}$ of ${}^{<\omega}\omega$. We say that a rea $c \in {}^{\omega\times\omega}\omega$ codes the following $\mathscr{D}$-open set:

$$U_c = \bigcup\{U_{t_m,f_m} : c(m,0) = 1 \ \& \ (\forall k)(f_m(k) = c(m,k+1))\},$$

which is the union of a countable family of basic clopen sets in $X_{\mathscr{D}}$. One ca easily describe a Borel function $F : {}^{\omega\times\omega}\omega \to BC$ such that $U_c = B_{F(c)}$ for al $c \in {}^{\omega\times\omega}\omega$.

LEMMA 4.1. (a) *The predicate "*$c \in {}^{\omega\times\omega}\omega$ *codes an open dense subset of* $X_{\mathscr{D}}$ *which is the union of a disjoint family of basic clopen sets" is* Π^1_1.

(b) *The predicate "*$c \in {}^{\omega\times\omega}\omega$ *codes an open* $\mathscr{D}$*-dense subset of* $X_{\mathscr{D}}$*" is* Π^1_1.

PROOF. (a) Note that the predicate "U_c is $\mathscr{D}$-dense" is equivalent to the following Π^1_1 formula:

$$(\forall n)(\forall f \in {}^{\omega}\omega)(\exists m)(\exists k)[t_k = t_m \cup t_n \ \&\ c(m,0) = 1 \\ \&\ (\forall i \in \operatorname{dom}(t_k - t_m))(t_k(i) \geq c(m, i+1)) \\ \&\ (\forall i \in \operatorname{dom}(t_k - t_n))(t_k(i) \geq f(i))].$$

Whether the intersection of two basic clopen sets is empty or not depends only on finite initial parts of functions, the formula

$$(\forall k, l)((k \neq l \ \&\ c(k,0) = c(l,0) = 1) \rightarrow U_{t_k, f_k} \cap U_{t_l, f_l} = \emptyset)$$

is arithmetical. Hence the predicate "U_c is the disjoint union of basic clopen sets" is arithmetical.

(b) The proof of this part is similar. □

LEMMA 4.2. *The predicates $B_a \in \mathcal{I}_{\mathcal{D}}$, $B_a \in \mathcal{I}_{\mathcal{D}}^{\sigma}$ are Σ^1_2.*

PROOF. $B_a \in \mathcal{I}_{\mathcal{D}}$ iff

$$(\exists c \in {}^{\omega}({}^{\omega\times\omega}\omega)) \left(B_a \cap \bigcap_{n\in\omega} U_{c(n)} = \emptyset \ \&\ (\forall n)(U_{c(n)} \text{ is open dense in } X_{\mathcal{D}}) \right)$$

iff

$$(\exists c \in {}^{\omega}({}^{\omega\times\omega}\omega)) \left(B_a \cap \bigcap_{n\in\omega} B_{F(c(n))} = \emptyset \ \&\ (\forall n)(U_{c(n)} \text{ is open dense in } X_{\mathcal{D}}) \right).$$

Similarly, $B_a \in \mathcal{I}_{\mathcal{D}}^{\sigma}$ iff

$$(\exists c \in {}^{\omega}({}^{\omega\times\omega}\omega)) \left(B_a \cap \bigcap_{n\in\omega} B_{F(c(n))} = \emptyset \ \&\ (\forall n)(U_{c(n)} \text{ is open } \mathcal{D}\text{-dense}) \right). \quad \square$$

THEOREM 4.3. *Let $d \in {}^{\omega}\omega$. The following are equivalent.*

(i) *d is a Hechler real over $\mathbf{V}$;*

(ii) *for each $c \in {}^{\omega\times\omega}\omega \cap \mathbf{V}$, whenever U_c is open dense in $X_{\mathcal{D}}$ and whenever U_c is a disjoint union of basic clopen sets, then $d \in U_c$;*

(iii) *for each $a \in BC \cap \mathbf{V}$, whenever $B_a \in \mathcal{I}_{\mathcal{D}}$, then $d \notin B_a$.*

PROOF. By definition, the set U_c is dense in $X_{\mathcal{D}}$ if and only if the family

$$A_c = \{\langle t_m, f_m\rangle : f_m \in {}^{\omega}\omega \ \&\ c(m,0) = 1 \ \&\ (\forall k)(f_m(k) = c(m, k+1))\}$$

is a predense subset of $\mathbb{D}$. Moreover, for every countable predense subset in $A \subseteq \mathbb{D}$ there is some $c \in {}^{\omega\times\omega}\omega$ such that $A = A_c$. Hence it is easy to see that a real d is a Hechler real if and only if d satisfies condition (ii), i.e. (i)↔(ii).

The implication (iii)→(ii) is trivial, and we prove (i)→(iii).

Let d be a Hechler real, and for some $a \in BC \cap \mathbf{V}$ let $B_a \in \mathcal{I}_{\mathcal{D}}$. There is a sequence $\{c_n : n \in \omega\} \subseteq {}^{\omega\times\omega}\omega$ such that $B_a \cap \bigcap_{n\in\omega} B_{F(c_n)} = \emptyset$ and the U_{c_n} are open dense in $X_{\mathscr{D}}$. By absoluteness of $\mathbf{\Sigma}^1_2$ properties we can find the sequence $\{c_n : n \in \omega\}$ in $\mathbf{V}$ with the same property and hence $d \in U_{c_n}$ for all n, which means $d \notin B_a$. □

For $x, y \in {}^\omega\omega$ let $x + y \in {}^\omega\omega$ and $x - y \in {}^\omega\omega$ be defined by

$$(x+y)(i) = x(i) + y(i), \quad (x-y)(i) = x(i) \dot{-} y(i), \quad \text{for } i \in \omega,$$

where $n \dot{-} m = n - m$ if $n \geq m$, and $n \dot{-} m = 0$ otherwise.

THEOREM 4.4. *Let $d \in {}^\omega\omega$. The following are equivalent.*

(i) *d is a Hechler real over $\mathbf{V}$;*
(ii) *for every $y \in {}^\omega\omega \cap \mathbf{V}$, $d + y$ is a Hechler real over $\mathbf{V}$;*
(iii) *for every $y \in {}^\omega\omega \cap \mathbf{V}$, $d - y$ is a Hechler real over $\mathbf{V}$;*
(iv) *for every $y \in {}^\omega\omega \cap \mathbf{V}$, whenever $a \in BC \cap \mathbf{V}$ and $B_a \in \mathscr{I}^\sigma_{\mathscr{D}}$ then, $d - y \notin B_a$.*

PROOF. (i)→(ii). The mapping $\varphi_y : X_{\mathscr{D}} \to X_{\mathscr{D}}$ defined by $\varphi_y(x) = x + y$ is continuous open and hence the inverse image of an open dense set is open dense, i.e. $\varphi_y^{-1}(U_c) = \{x : x + y \in U_c\}$ is open dense whenever U_c is open dense. Moreover, for each $c \in \mathbf{V}$ we can effectively find some $c' \in \mathbf{V}$ such that $\varphi_y^{-1}(U_c) = U_{c'}$. Hence by Theorem 4.3(ii), if d is a Hechler real and U_c for some $c \in \mathbf{V}$ is open dense, then $d \in \varphi_y^{-1}(U_c)$ and so $d + y \in U_c$. Therefore $d + y$ is a Hechler real.

(i)→(iii). Let d be a Hechler real over $\mathbf{V}$ and let $y \in {}^\omega\omega \cap \mathbf{V}$. Then $d(k) \geq y(k)$ for all but finitely many $k \in \omega$, and so without loss of generality we can assume that $d(k) \geq y(k)$ for all $k \in \omega$ (otherwise take $y' = \min\{y, d\}$, which is also in $\mathbf{V}$ and $d - y = d - y'$). The mapping $\pi_{\emptyset,y} : U_{\emptyset,y} \to X_{\mathscr{D}}$ defined by $\pi_{\emptyset,y}(x) = x - y$ is a homeomorphism. Hence, whenever U_c is open dense, the set

$$\pi^{-1}_{\emptyset,y}(U_c) = \{x \in U_{\emptyset,y} : x - y \in U_c\} = U_{\emptyset,y} \cap (U_c + y)$$

is open dense in $U_{\emptyset,y}$. Since $d \in U_{\emptyset,y}$ (similarly to the previous part of the proof) we can derive $d \in U_c + y$ and hence $d - y \in U_c$ for $c \in \mathbf{V}$. It follows that $d - y$ is a Hechler real.

(ii)→(i) and (iii)→(i). Take y such that $y(i) = 0$, for all $i \in \omega$.

(iii)→(iv). This implication is a consequence of Theorem 4.3 and Lemma 4.2.

(iv)→(i). Condition (iv) implies that d is a dominating real. We verify condition (ii) in Theorem 4.3 to prove that d is a Hechler real. Let U_c be an open dense set in $X_{\mathscr{D}}$ which is a disjoint union of basic clopen sets $\{U_{t_m,f_m} : m \in \omega\}$ coded by some $c \in {}^{\omega\times\omega}\omega \cap \mathbf{V}$. Let $y \in {}^\omega\omega \cap \mathbf{V}$ be a function which dominates all functions f_m, i.e. $(\forall m)(\forall^\infty k)(f_m(k) \leq y(k))$ and such that for all $k \in \omega$, $y(k) \leq d(k)$. By Lemma 1.4, the set $\pi_{\emptyset,y}(U_c \cap U_{\emptyset,y})$ is an open $\mathscr{D}$-dense set coded in $\mathbf{V}$. Hence by the assumption, $d - y \in \pi_{\emptyset,y}(U_c \cap U_{\emptyset,y})$. Since $y \leq d$,

$$d \in \pi_{\emptyset,y}(U_c \cap U_{\emptyset,y}) + y = U_c \cap U_{\emptyset,y} \subseteq U_c.$$

The theorem is proved. □

§5. **The ideal $\mathscr{I}_{\mathscr{D}}$ versus $\mathscr{I}^\sigma_{\mathscr{D}}$.** While the ideal $\mathscr{I}^\sigma_{\mathscr{D}}$ is a subset of the ideal of meager sets $\mathscr{M}$, we will see that the ideal $\mathscr{I}_{\mathscr{D}}$ is far away from any such comparison, since the ideals $\mathscr{I}_{\mathscr{D}}$, $\mathscr{M}$ are orthogonal and the difference $\mathscr{I}_{\mathscr{D}} \cap \mathscr{M} - \mathscr{I}^\sigma_{\mathscr{D}}$ is still very large. Let us start with several examples of subsets of ${}^\omega\omega$.

EXAMPLE 5.1. Let $h \in {}^{\omega}\mathscr{P}(\omega)$ be such that $(\exists^{\infty}k)(|h(k)| = \omega)$. Then

$$A = \{x \in {}^{\omega}\omega : (\forall^{\infty}k)(x(k) \notin h(k))\} \in \mathscr{I}^{\sigma}_{\mathscr{D}}.$$

PROOF. For $m \in \omega$, the set $B_m = \{x \in {}^{\omega}\omega : (\exists k > m)(x(k) \in h(k))\}$ is an open subset of ${}^{\omega}\omega$. Since $A \cap \bigcap_{m \in \omega} B_m = \emptyset$, it is enough to prove that the B_m are $\mathscr{D}$-dense. Let $f \in {}^{\omega}\omega$ and $t \in {}^{<\omega}\omega$. There is $x \in U_{t,f}$ such that $x(k) \in h(k)$ for some $k > m$, and hence $x \in B_m \cap U_{t,f} \neq \emptyset$. □

EXAMPLE 5.2. For $f \in {}^{\omega}\omega$ the set

$$\{x \in {}^{\omega}\omega : (\forall^{\infty}k)(x(k) < f(k))\} \in \mathscr{I}^{\sigma}_{\mathscr{D}}.$$

PROOF. Take $h(k) = \omega - f(k)$ in Example (5.1). □

EXAMPLE 5.3. Let $f \in {}^{\omega}\omega$, let $a, b \subseteq \omega$ be two disjoint sets, and let $|a| = \omega$. Then

$$A = \{x \in {}^{\omega}\omega : (\exists^{\infty}k \in a)(x(k) < f(k)) \;\&\; (\forall k \in b)(x(k) \geq f(k))\} \in \mathscr{I}_{\mathscr{D}} - \mathscr{I}^{\sigma}_{\mathscr{D}}.$$

Moreover, the set A is nowhere dense in $X_{\mathscr{D}}$.

PROOF. The set A is disjoint with the set $B = \{x \in {}^{\omega}\omega : (\forall^{\infty}n \in \omega)(x(n) \geq f(n))\}$ which is open dense in $X_{\mathscr{D}}$. Hence $A \in \mathscr{I}_{\mathscr{D}}$. Now we prove that $A \notin \mathscr{I}^{\sigma}_{\mathscr{D}}$. To get a contradiction, let us assume that $A \in \mathscr{I}^{\sigma}_{\mathscr{D}}$. By Lemma 1.3 there is a sequence $\langle s_n : n \in \omega \rangle$ of elements of ${}^{<\omega}\omega$ such that $A \cap \bigcap_m \bigcup_{n>m}[s_n] = \emptyset$ and condition $(*)$ holds. By induction let us find a sequence $\langle t_m, t'_m : m \in \omega \rangle$ of elements of ${}^{<\omega}\omega$ such that $t'_0 = \emptyset$ and $t'_m \subseteq t_m \subseteq t'_{m+1}$ for all $m \in \omega$, by the following rules:

(a) There is $i \in \operatorname{dom}(t_m - t'_m) \cap a$ such that $t_m(i) < f(i)$.
(b) For all $i \in \operatorname{dom}(t_m - t'_m) \cap b$, $t_m(i) \geq f(i)$.
(c) $t'_{m+1} = s_n$, where n is minimal with the property that $t_m \subseteq s_n$ and, for all $i \in \operatorname{dom}(s_n - t_m)$, $s_n(i) \geq f(i)$.

Now let $x = \bigcup_{m \in \omega} t_m$. By conditions (a), (b) of the construction it follows that $x \in A$ while by (c), $x \in \bigcap_m \bigcup_{n>m}[s_n]$. This contradicts the choice of the sequence of s_n's. □

The following theorem is a contrast to Theorem 3.2.

THEOREM 5.4. *There are 2^{ω} many disjoint sets in $\mathscr{I}_{\mathscr{D}} \cap \mathscr{M} - \mathscr{I}^{\sigma}_{\mathscr{D}}$.*

PROOF. Let $\{a_{\alpha} : \alpha < 2^{\omega}\}$ be an almost disjoint family of subsets of ω (see [5]). Let $f \in {}^{\omega}\omega$ be any function such that for all but finitely many $k \in \omega$, $f(k) > 0$. For $\alpha < 2^{\omega}$ let

$$A_{\alpha} = \{x \in {}^{\omega}\omega : (\exists^{\infty}k \in a_{\alpha})(x(k) < f(k)) \;\&\; (\forall k \in \omega - a_{\alpha})(x(k) \geq f(k))\}.$$

By Example 5.3, $A_{\alpha} \in \mathscr{I}_{\mathscr{D}} - \mathscr{I}^{\sigma}_{\mathscr{D}}$ for all α and all these are pairwise disjoint meager subsets of ${}^{\omega}\omega$. □

Let the measure μ on ${}^{\omega}\omega$ be the product measure of ω copies of measure μ_0 on ω defined by $\mu_0(\{n\}) = 2^{-n-1}$. Note that there is a measure-preserving homeomorphism from ${}^{\omega}\omega$ to $\langle 0, 1\rangle - Q$, where Q is some countable dense subset of $\langle 0, 1\rangle$. Let $\mathscr{N}$ be the ideal of sets $A \subseteq {}^{\omega}\omega$ for which $\mu(A) = 0$. It is known that the ideals $\mathcal{M}$ and $\mathcal{N}$ are orthogonal, i.e. there is $A \in \mathcal{N}$ such that the complement of A is in $\mathcal{M}$.

THEOREM 5.5. (i) *The ideals* $\mathcal{M}$, $\mathcal{I}_{\mathcal{D}}$ *are orthogonal.*

(ii) *The ideals* $\mathcal{N}$, $\mathcal{I}_{\mathcal{D}}^{\sigma}$ *are orthogonal.*

(iii) *The ideals* $\mathcal{N}$, $\mathcal{I}_{\mathcal{D}}$ *are orthogonal.*

PROOF. (i) For an $f \in {}^{\omega}\omega$ the set $D_f = \{x \in {}^{\omega}\omega : (\forall^{\infty} n)(x(n) \geq f(n))\}$ is a meager subset of ${}^{\omega}\omega$ and open dense in $X_{\mathcal{D}}$.

The case (iii) follows immediately from (ii) since $\mathcal{I}_{\mathcal{D}}^{\sigma} \subseteq \mathcal{I}_{\mathcal{D}}$. We prove (ii). We shall use inequality

$$\frac{x}{1+x} \leq \ln(1+x), \quad \text{for } x \in (-1, \infty).$$

For each $n > 1$ let us choose $f_n \in {}^{\omega}\omega$ such that $\sum_{m\in\omega} \sum_{i \geq f_n(m)} 2^{-i-1} < 1/n$, and let $\alpha_n(m) = \sum_{i \geq f_n(m)} 2^{-i-1} < 1/n$. Set $A_n = \{x \in {}^{\omega}\omega : (\forall m \in \omega)(x(m) < f_n(m))\}$. By Example 5.2, the set $A = \bigcup_{n\in\omega} A_n$ is in $\mathcal{I}_{\mathcal{D}}^{\sigma}$. We shall prove that $\mu(A) = 1$. We have $\mu(A_n) = \prod_{m\in\omega}(1 - \alpha_n(m)) < 1$, and so

$$0 > \ln(\mu(A_n)) = \sum_m \ln(1 - \alpha_n(m)) \geq \sum_m \frac{-\alpha_n(m)}{1 - \alpha_n(m)}$$
$$\geq \frac{-n}{n-1} \sum_m \alpha_n(m) > \frac{-1}{n-1}.$$

Hence $1 \geq \mu(A) \geq \sup_{n>1} e^{-1/(n-1)} = 1$. □

§**6. The additivity and the cofinality of the ideals.** The aim of this part is to present some applications of the tools developed in [2], [3]. We prove that $\mathrm{add}(\mathcal{I}_{\mathcal{D}}) = \mathrm{add}(\mathcal{I}_{\mathcal{D}}^{\sigma}) = \omega_1$ and $\mathrm{cof}(\mathcal{I}_{\mathcal{D}}) = \mathrm{cof}(\mathcal{I}_{\mathcal{D}}^{\sigma}) = 2^{\omega}$. For this we will need the notion of rank introduced in [2] for modified Hechler forcing conditions of which are pairs of strictly increasing functions s, f. Our definition of rank is a slight modification of it and corresponds with the forcing notion

$$\mathbb{D}' = \{\langle s, f\rangle : (\exists x \in [\omega]^{<\omega})(s \in {}^{x}\omega \ \&\ f \in {}^{\omega}\omega)\},$$

$$\langle s, f\rangle \leq \langle t, g\rangle \quad \text{iff} \quad t \subseteq s \ \&\ (\forall k)[(k \in \mathrm{dom}(s-t) \rightarrow g(k) \leq s(k)) \ \&\ (k \in \omega - \mathrm{dom}(s) \rightarrow g(k) \leq f(k))].$$

Clearly, $\mathbb{D}$ is a dense subset of $\mathbb{D}'$, and for $\langle s, f\rangle \in \mathbb{D}'$ the set

$$U_{s,f} = \{x \in {}^{\omega}\omega : s \subseteq x \ \&\ (\forall k \in \omega - \mathrm{dom}(s))(f(k) \leq x(k))\}$$

is a clopen set in the dominating topology $\mathcal{D}$; we call it a basic clopen set too. In the following the symbol ${}^{\underline{\omega}}\omega$ denotes the set of all finite functions from ω to ω, i.e. ${}^{\underline{\omega}}\omega = \bigcup_{x\in[\omega]^{<\omega}} {}^{x}\omega$.

We say that a set $D \subseteq {}^{\underline{\omega}}\omega$ is open if with each $s \in D$, D contains all $t \in {}^{\underline{\omega}}\omega$ such that $s \subseteq t$; D is dense if for each $s \in {}^{\underline{\omega}}\omega$ there is $t \in D$ such that $s \subseteq t$; D is relatively dense if for each $s \in D$ and $t \in {}^{\underline{\omega}}\omega$ such that $s \subseteq t$ there is $s' \in D$ such that $t \subseteq s'$. Clearly, D is relatively dense whenever D is open or dense.

Following [2], for a set $D \subseteq {}^{\underline{\omega}}\omega$ we define a sequence $\langle D_\alpha : \alpha \in \omega_1 \rangle$ by induction as follows: $D_0 = D$, and, for $\alpha > 0$

$$D_\alpha = \{s \in {}^{\underline{\omega}}\omega : (\exists x \in [\omega]^{<\omega})(\forall k \in \omega)(\exists t \in \bigcup_{\beta<\alpha} D_\beta)$$
$$[s \subseteq t \ \&\ \mathrm{dom}(t) = x \ \&\ (\forall i \in \mathrm{dom}(t - s))(t(i) \geq k)]\}.$$

The minimal $\alpha < \omega_1$ such that $t \in D_\alpha$ we denote by $\mathrm{rank}_D(t)$. If there is no such α we write $\mathrm{rank}_D(t) = \infty$ (see [3]). Let us denote

$$R_D = \{t \in {}^{\underline{\omega}}\omega : \mathrm{rank}_D(t) < \omega_1\}, \qquad U_D = \bigcup_{s \in D} [s].$$

Since the set ${}^{\underline{\omega}}\omega$ is countable, there is $\alpha < \omega_1$ such that $R_D = D_\alpha$.

Note that $R_D = {}^{\underline{\omega}}\omega$ if and only if ${}^{<\omega}\omega \subseteq R_D$. We have the following characterization.

LEMMA 6.1. *Let $D \subseteq {}^{\underline{\omega}}\omega$ be relatively dense. Then $R_D = {}^{\underline{\omega}}\omega$ if and only if U_D is $\mathscr{D}$-dense.*

PROOF. Let the open set U_D be $\mathscr{D}$-dense, and to the contrary let us assume that we have some $s_0 \in {}^{\underline{\omega}}\omega - R_D$. We say that $t \in R_D$ is a minimal extension of s_0 if $s_0 \subseteq t$ and $t{\restriction}x \notin R_D$ for all sets x with $\mathrm{dom}(s_0) \subseteq x \subsetneqq \mathrm{dom}(t)$.

Claim. For every $x \in [\omega]^{<\omega}$ such that $\mathrm{dom}(s_0) \subseteq x$ there are only finitely many minimal extensions t of s_0 with $\mathrm{dom}(t) = x$.

Proof. Suppose that for some finite set x there is an infinite set T of minimal extensions t of s_0 with $\mathrm{dom}(t) = x$. By induction on $i \in x - \mathrm{dom}(s_0)$ we find a decreasing sequence of infinite sets $T_i \subseteq T$ so that for all i either there is $n_i \in \omega$ such that $t(i) = n_i$ for all $t \in T_i$ or $t_1(i) \neq t_2(i)$ for all $t_1, t_2 \in T_i$, $t_1 \neq t_2$. Let $T' = \bigcap_{i \in x - \mathrm{dom}(s_0)} T_i$. There is $u \in {}^{\underline{\omega}}\omega$ such that $s_0 \subseteq u$, $\mathrm{dom}(u) \subseteq x$, $u \subseteq t$ for all $t \in T'$ (i.e. $u(i) = n_i$ for $i \in \mathrm{dom}(u)$), and $t_1(i) \neq t_2(i)$ for $i \in x - \mathrm{dom}(u)$ and $t_1, t_2 \in T'$, $t_1 \neq t_2$. Hence for every $k \in \omega$ there is $t \in T'$ such that $t(i) \geq k$ for all $i \in x - \mathrm{dom}(u)$. Therefore $u \in R_D$, which contradicts the minimality of the elements of T'. The claim is proved. □

By the remark before the lemma, without loss of generality we can assume that the domain of the function s_0 is an integer. For each $n \in \omega$ let T_n be the finite set of all minimal extensions t of s_0 such that $\max(\mathrm{dom}(t)) = n$. Let us define $f \in {}^\omega\omega$ so that $f(n) > \max\{t(n) : t \in T_n\}$ for all $n \in \omega - \mathrm{dom}(s_0)$. Since the set U_D is $\mathscr{D}$-dense, there is $t \in D$ such that $U_{s_0,f} \cap [t] \neq \emptyset$. As D is relatively dense we can assume that $s_0 \subseteq t$. Hence $t \in R_D$, and so some function $t' \subseteq t$ is a minimal extension of s_0. Let $n = \max(\mathrm{dom}(t'))$. Then as $\mathrm{dom}(s_0)$ is an integer, $n \notin \mathrm{dom}(s_0)$ and $t' \in T_n$. Therefore $t(n) = t'(n) < f(n)$, which contradicts the fact that $U_{s_0,f} \cap [t] \neq \emptyset$. This contradiction proves that the equality $R_D = {}^{\underline{\omega}}\omega$ follows from the assumption that U_D is $\mathscr{D}$-dense.

Conversely, let $R_D = {}^{\underline{\omega}}\omega$ and let $f \in {}^\omega\omega$ be arbitrary. By induction on $\mathrm{rank}_D(s)$ we prove that $U_{s,f} \cap U_D \neq \emptyset$ for all $s \in R_D$. Let $\alpha > 0$ and let $s \in D_\alpha$. There is $t \in D_\beta$ for some $\beta < \alpha$ such that $s \subseteq t$ and $t(i) \geq f(i)$ for all $i \in \mathrm{dom}(t - s)$. Hence $U_{t,f} \subseteq U_{s,f}$ and by the induction hypothesis $U_{t,f} \cap U_D \neq \emptyset$. Therefore U_D is $\mathscr{D}$-dense. □

Note that if U is an open dense subset of $X_{\mathscr{D}}$ then the set

$$D = \{s \in {}^{\underline{\omega}}\omega : (\exists f \in {}^{\omega}\omega)(U_{s,f} \subseteq U)\}$$

is a dense subset of ${}^{\underline{\omega}}\omega$ and the set U_D is an open $\mathscr{D}$-dense subset of ${}^{\omega}\omega$. Therefore $R_D = {}^{\underline{\omega}}\omega$. The original formulation of Lemma 6.1 in [2] (conveniently modified) says that if $D \subseteq {}^{\underline{\omega}}\omega$ is the set of first coordinates s of conditions $\langle s, f\rangle$ of some open dense subset of Hechler forcing, then $R_D = {}^{\underline{\omega}}\omega$.

For an infinite set $A \subseteq \omega$ let

$$X_A = \{x \in {}^{\omega}\omega : \mathrm{rng}(x) \cap A = \emptyset\}.$$

Clearly, X_A is a closed $\mathscr{D}$-meager subset of ${}^{\omega}\omega$, i.e. $X_A \in \mathscr{I}_{\mathscr{D}}^{\sigma}$. Let $\mathscr{A} \subseteq [\omega]^{\omega}$ be an uncountable almost disjoint family (see [5]). The following result is implicitly proved in [3,Main Theorem].

THEOREM 6.2. *For any set $X \in \mathscr{I}_{\mathscr{D}}$ the family $\{A \in \mathscr{A} : X_A \subseteq X\}$ is at most countable.*

PROOF. Let $\langle U_n : n \in \omega\rangle$ be a sequence of open dense subsets of $X_{\mathscr{D}}$ such that $X \cap \bigcap_{n\in\omega} U_n = \emptyset$, and let $D_n = \{s \in {}^{\underline{\omega}}\omega : (\exists f \in {}^{\omega}\omega)(U_{s,f} \subseteq U_n)\}$. By Lemma 6.1, $R_{D_n} = {}^{\underline{\omega}}\omega$ for all $n \in \omega$. For every $n \in \omega$ and $t \in {}^{\underline{\omega}}\omega - D_n$ we will define a finite family $\mathscr{A}_{t,n}$ of "bad" elements of $\mathscr{A}$ in the following way.

Given $n \in \omega$, for each $t \in {}^{\underline{\omega}}\omega - D_n$ let us fix a finite set x_t and a sequence of $t_k = s(t,n,k)$ for $k \in \omega$ such that $\mathrm{dom}(t) \subseteq x_t = \mathrm{dom}(t_k)$, $\mathrm{rank}_{D_n}(t_k) < \mathrm{rank}_{D_n}(t)$ and $t_k(i) \geq k$ for all $i \in x_t - \mathrm{dom}(t)$ and all $k \in \omega$. By induction on $i \in x_t - \mathrm{dom}(t)$ we construct a decreasing sequence of infinite sets $B_i \subseteq \omega$ so that whenever for some $A \in \mathscr{A}$ the set $B_i' = \{k \in \bigcap_{j<i} B_j : t_k(i) \in A\}$ is infinite then we set $B_i = B_i'$ and if there is no such $A \in \mathscr{A}$ we set $B_i = \bigcap_{j<i} B_j$. Let $B_t = \bigcap_{i\in x_t-\mathrm{dom}(t)} B_i$ and let $\mathscr{A}_{t,n}$ be the family of $A \in \mathscr{A}$ for which there is $i \in x_t - \mathrm{dom}(t)$ such that the set B_i' is infinite and for all $k \in B_t$, $t_k(i) \in A$. Since $\mathscr{A}$ is almost disjoint, the family $\mathscr{A}_{t,n}$ is finite and for every $A \in \mathscr{A}$ either A is almost disjoint with the infinite sets $\{t_k(i) : k \in B_t\}$ for $i \in x_t - \mathrm{dom}(t)$ or A contains one of these sets and $A \in \mathscr{A}_{t,n}$. We prove that the countable family

$$\mathscr{A}_0 = \bigcup\{\mathscr{A}_{t,n} : n \in \omega \ \&\ t \in D_n\}$$

contains all bad elements of $\mathscr{A}$, i.e. for $A \in \mathscr{A} - \mathscr{A}_0$, $X_A \nsubseteq X$. The following result is crucial.

LEMMA 6.3. *Let $A \in \mathscr{A} - \mathscr{A}_0$, $n \in \omega$, $s \in {}^{\underline{\omega}}\omega$, $f \in \omega$. There is $t \in D_n$ such that $|\,\mathrm{rng}(t) \cap A| = |\,\mathrm{rng}(s) \cap A|$, $s \subseteq t$ and $t(i) \geq f(i)$ for all $i \in \mathrm{dom}(t - s)$.*

PROOF. Let $l = |\,\mathrm{rng}(s) \cap A|$ and let

$$Y = \{t \in {}^{\underline{\omega}}\omega : s \subseteq t \ \&\ |\,\mathrm{rng}(t) \cap A| = l \ \&\ (\forall i \in \mathrm{dom}(t - s))(t(i) \geq f(i))\}.$$

Choose $t \in Y$ with minimal $\mathrm{rank}_{D_n}(t)$. We prove that $t \in D_n$. To get a contradiction let us assume that $t \notin D_n$. Let us consider the sequence of $t_k = s(t,n,k)$ for $k \in \omega$ and the infinite set B_t defined above. Since $A \notin \mathscr{A}_0$, by the almost disjointness of $\mathscr{A}$, $A \cap \{t_k(i) : k \in B_t\}$ is finite for all $i \in x_t - \mathrm{dom}(t)$. Hence there is $k \in B_t$ such that $A \cap \{t_k(i) : i \in x_t - \mathrm{dom}(t)\} = \emptyset$ and hence $|\,\mathrm{rng}(t_k) \cap A| = |\,\mathrm{rng}(t) \cap A| = l$.

So $t_k \in Y$ and $\mathrm{rank}_{D_n}(t_k) < \mathrm{rank}_{D_n}(t)$, which is a contradiction. Hence $t \in D_n$, and obviously $U_{t,f} \subseteq U_{s,f}$. This establishes the lemma. □

Now we prove that for $A \in \mathscr{A} - \mathscr{A}_0$, $X_A \nsubseteq X$. Let $A \in \mathscr{A} - \mathscr{A}_0$. By induction on $n \in \omega$ we shall construct a decreasing sequence of basic clopen sets U_{s_n,f_n} such that $U_{s_{n+1},f_{n+1}} \subseteq D_n$ and such that $\mathrm{rng}(s_n) \cap A = \emptyset$. Let $s_0 = \emptyset$, and let $f_0 \in {}^\omega\omega$ be arbitrary. Assuming we have constructed s_n, f_n, by Lemma 6.3 we can find $s_{n+1} \in D_n$ such that $\mathrm{rng}(s_{n+1}) \cap A = \emptyset$, $n \in \mathrm{dom}(s_{n+1})$ and $U_{s_{n+1},f_n} \subseteq U_{s_n,f_n}$. Since $s_{n+1} \in D_n$ there is $g_n \in {}^\omega\omega$ such that $U_{s_{n+1},g_n} \subseteq U_n$. Let $f_{n+1} = \max\{f_n, g_n\}$. Then $U_{s_{n+1},f_{n+1}} \subseteq U_n \cap U_{s_n,f_n}$. Now let $x = \bigcup_{n\in\omega} s_n$. Then $x \in {}^\omega\omega$ and $x \in X_A \cap \bigcap_{n\in\omega} U_n$. Consequently, $X_A \nsubseteq X$. The theorem is proved. □

Since there is an almost disjoint family $\mathscr{A} \subseteq [\omega]^\omega$ of cardinality 2^ω, the following result is immediate.

COROLLARY 6.4. (a) $\mathrm{add}(\mathscr{I}_{\mathscr{D}}) = \mathrm{add}(\mathscr{I}^\sigma_{\mathscr{D}}) = \omega_1$.

(b) $\mathrm{cof}(\mathscr{I}_{\mathscr{D}}) = \mathrm{cof}(\mathscr{I}^\sigma_{\mathscr{D}}) = 2^\omega$. □

Acknowledgment. The authors would like to thank Jacek Cichoń for drawing their attention to the subject of the paper.

REFERENCES

[1] B. BALCAR, J. PELANT and P. SIMON, *The space of ultrafilters on N covered by nowhere dense sets*, ***Fundamenta Mathematicae***, vol. 110 (1980), pp. 11–24.

[2] J. E. BAUMGARTNER and P. DORDAL, *Adjoining dominating functions*, this JOURNAL, vol. 50 (1985), no. 1, pp. 94–101.

[3] J. BRENDLE, H. JUDAH and S. SHELAH, *Combinatorial properties of Hechler forcing*, ***Annals of Pure and Applied Logic***, vol. 59 (1992), pp. 185–199.

[4] W. COMFORT and S. NEGREPONTIS, ***Theory of ultrafilters***, Springer-Verlag, Berlin, 1974.

[5] E. K. VAN DOUWEN, *The integers and topology*, ***The handbook of set-theoretic topology*** (K. Kunen and J. Vaughan, editors), North-Holland, Amsterdam, 1984, pp. 111–167.

[6] D. H. FREMLIN, *Cichoń's diagram*, ***Publications Mathématiques de l'Université Pierre et Marie Curie***, vol. 66, 23eme année, 1983/84, No. 5 (1984), 13pp.

[7] ———, ***Consequences of Martin's axiom***, Cambridge University Press, Cambridge, 1984.

[8] R. C. HAWORTH and R. A. MCCOY, *Baire spaces*, ***Dissertationes Mathematicae/Rozprawy Matematyczne***, vol. 141 (1977).

[9] T. JECH, ***Set theory***, Academic Press, New York, 1978.

[10] H. JUDAH and M. REPICKÝ, *Amoeba reals*, preprint.

[11] A. W. MILLER, *Some properties of measure and category*, ***Transactions of the American Mathematical Society***, vol. 266 (1981), pp. 93–114.

[12] J. TRUSS, *Sets having caliber* $\aleph_1$, ***Logic Colloquium '76***, North-Holland, Amsterdam, 1977, pp. 595–612.

INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY
50-156 WROCŁAW, POLAND

E-mail: labedz@math.uni.wroc.pl

MATHEMATICAL INSTITUTE OF THE SLOVAK ACADEMY OF SCIENCES
041 54 KOŠICE, SLOVAKIA

E-mail: repicky@kosice.upjs.sk

LINKED CITATIONS

- Page 1 of 1 -

You have printed the following article:

Hechler Reals
Grzegorz #ab#dzki; Miroslav Repický
The Journal of Symbolic Logic, Vol. 60, No. 2. (Jun., 1995), pp. 444-458.
Stable URL:
http://links.jstor.org/sici?sici=0022-4812%28199506%2960%3A2%3C444%3AHR%3E2.0.CO%3B2-A

This article references the following linked citations. If you are trying to access articles from an off-campus location, you may be required to first logon via your library web site to access JSTOR. Please visit your library's website or contact a librarian to learn about options for remote access to JSTOR.

References

[2] **Adjoining Dominating Functions**
James E. Baumgartner; Peter Dordal
The Journal of Symbolic Logic, Vol. 50, No. 1. (Mar., 1985), pp. 94-101.
Stable URL:
http://links.jstor.org/sici?sici=0022-4812%28198503%2950%3A1%3C94%3AADF%3E2.0.CO%3B2-V

[11] **Some Properties of Measure and Category**
Arnold W. Miller
Transactions of the American Mathematical Society, Vol. 266, No. 1. (Jul., 1981), pp. 93-114.
Stable URL:
http://links.jstor.org/sici?sici=0002-9947%28198107%29266%3A1%3C93%3ASPOMAC%3E2.0.CO%3B2-9

NOTE: *The reference numbering from the original has been maintained in this citation list.*