ROSENTHAL FAMILIES, FILTERS, AND SEMIFILTERS

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Abstract. We continue the study of Rosenthal families initiated by Damian Sobota. We show that every Rosenthal filter is the intersection of a finite family of ultrafilters that are pairwise incomparable in the Rudin-Keisler partial ordering of ultrafilters. We introduce a property of filters, called an r-filter, properly between a selective filter and a p-filter. We prove that every r-ultrafilter is a Rosenthal family. We prove that it is consistent with ZFC to have uncountably many r-ultrafilters such that any intersection of finitely many of them is a Rosenthal filter.

1. Introduction

The notion of a Rosenthal family comes from a reformulation of a variant of Rosenthal's Lemma [12, Lemma 1.1] by Damian Sobota [13, 14] and is based on the observation that Rosenthal's Lemma is equivalent to the statement that $[\omega]^\omega$ is a Rosenthal family with this meaning:

Definition 1.1 ([14, Definition 1.3])**.** A family $\mathcal{F} \subseteq [\omega]^\omega$ is a *Rosenthal family* if for every matrix $\{c_{k,n}\}_{k,n\in\omega}$ of non-negative reals such that

$$(1.1)\qquad (\forall k \in \omega) \sum_{n\in\omega} c_{k,n} \leq 1$$

and for every $\varepsilon > 0$ there exists $A \in \mathcal{F}$ such that

$$(\forall k \in A) \sum_{n\in A\setminus\{k\}} c_{k,n} < \varepsilon.$$

A matrix satisfying (1.1) is called a *Rosenthal matrix.* A Rosenthal family that is a filter, semifilter, ultrafilter, etc., is called respectively, a *Rosenthal filter*, a *Rosenthal semifilter*, a *Rosenthal ultrafilter*, etc.

We start with some known results about Rosenthal families:

Theorem 1.2 (Rosenthal's Lemma)**.** *$[\omega]^\omega$ is a Rosenthal family.* □

Theorem 1.3 (Sobota [14, Theorem 3.6])**.** *A selective ultrafilter is a Rosenthal ultrafilter.* □

Theorem 1.4 (Sobota [14, Theorem 3.17])**.** *If Martin's Axiom for σ-centered forcing notions holds, then there is a Rosenthal P-point which is not a Q-point.* □

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The following duplicity can be observed: If there exists a selective ultrafilter, then the conclusion of Theorem 1.2 follows from the conclusion of Theorem 1.3 because every ultrafilter is a subset of $[\omega]^\omega$, otherwise these two results seem to be non-inclusive. On the other hand, the use of the notion of a semifilter can unify these two cases because $[\omega]^\omega$ is a selective semifilter. This is a reason why we consider some natural combinatorial properties of ultrafilters and filters also in the context of semifilters. Some questions about Rosenthal families can be restricted to upward closed families because $\mathcal{F} \subseteq [\omega]^\omega$ is a Rosenthal family if and only if its upward closure $\{A \subseteq \omega : (\exists B \in \mathcal{F})\ B \subseteq A\}$ is a Rosenthal family. From this point of view passing to semifilters seems not to be very limiting in the study of Rosenthal families.

In Section 2 we collect various properties of Rosenthal families, some of them considered in [14], and we prove relations between them. Perhaps three of these results are most important for later applications: First, the property "to be a Rosenthal family" is invariant under Rudin-Keisler isomorphism. This allows to consider Rosenthal families on arbitrary countable set. Second, a Rosenthal family $\mathcal{F}$ is not decomposable. This means that $\mathcal{F}$ is not a subfamily of an infinite sum of the form $\sum_{n\in\omega} \mathcal{F}_n$. Third, the semifilter of positive sets with respect to a filter on ω (i.e., a full semifilter) has no local diagonal.

In Section 3 we present several results on cardinal invariants related to properties considered in previous section; some of these invariants were considered in [14].

The main result of Section 4 says that every Rosenthal filter is Rudin-Keisler equivalent to the sum of a finite collection of Rosenthal ultrafilters pairwise incomparable in the Rudin-Keisler ordering. Equivalently, every Rosenthal filter is the intersection of a finite collection of Rosenthal ultrafilters pairwise incomparable in the Rudin-Keisler ordering.

In Section 5 we introduce s-, p-, and r- properties of semifilters as generalizations of a selective ultrafilter and a P-point. These generalizations to semifilters are not completely new and some germs of them can be found in [4, 10] for ideals on ω. Semifilters can be treated as forcing notions and the generic subset of a full s-semifilter, r-semifilter, p-semifilter is an s-ultrafilter, r-ultrafilter, p-ultrafilter, respectively. The main results of this section are Ramsey-like characterizations of the selective properties of semifilters.

In Section 6 we prove that if $\mathcal{F}$ is an r-semifilter which has no local diagonal, then $\mathcal{F}$ is a Rosenthal family. Note that Theorem 1.2 and Theorem 1.3 are consequences of this result. Moreover, if one takes a bit care in the proof of Theorem 1.4, then also the ultrafilter constructed there will be an r-ultrafilter. However, it is open whether this ultrafilter may be a non-r-ultrafilter.

A box product of full r-semifilters is an ω-closed forcing notion adding many r-ultrafilters. But the box product of any number of copies of the selective semifilter $[\omega]^\omega$ and of r-semifilters of the form $\mathcal{H}_{\boldsymbol{a}}^{>0} = \{A \subseteq \omega : \limsup_{n\in\omega} |a_n \cap A| = \infty\}$ (where $\boldsymbol{a} = \{a_n : n \in \omega\}$ is a partition of ω with $\limsup_{n\in\omega} |a_n| = \infty$) produces a generic set of ultrafilters such that any intersection of finitely many ultrafilters from this set is a Rosenthal filter.

2. Properties of Rosenthal families

In this paper, filters and ultrafilters on an infinite countable set always contain the Fréchet filter. The *Rudin-Keisler partial ordering* of filters on ω is defined by

$\mathcal{G} \leq_{\mathrm{RK}} \mathcal{F}$ if there is $f \in {}^\omega\omega$ such that $\mathcal{G} = f(\mathcal{F}) = \{A \subseteq \omega : f^{-1}(A) \in \mathcal{F}\}$. In a natural way we shall use this partial order for arbitrary families of subsets of ω. We write $\mathcal{G} =_{\mathrm{RK}} \mathcal{F}$, if $\mathcal{G} = f(\mathcal{F})$ for some bijection $f \in {}^\omega\omega$. By the next lemma the property of being a Rosenthal family is invariant under $=_{\mathrm{RK}}$.

Lemma 2.1. *Let $f : \omega \to \omega$ be one-to-one and let $\mathcal{F} \subseteq [\omega]^\omega$. $\mathcal{F}$ is a Rosenthal family if and only if $f[\mathcal{F}] = \{f(A) : A \in \mathcal{F}\}$ is a Rosenthal family.*

Proof. If $\{c_{k,n}\}_{k,n\in\omega}$ is a Rosenthal matrix, then $d_{k,n} = c_{f(k),f(n)}$ is a Rosenthal matrix because $\sum_{n\in\omega} d_{k,n} = \sum_{n\in\omega} c_{f(k),f(n)} \leq \sum_{n\in\omega} c_{f(k),n} \leq 1$ for all $k \in \omega$. Let $A \in \mathcal{F}$ be such that for all $k \in A$, $\sum_{n\in A\setminus\{k\}} d_{k,n} < \varepsilon$. Then for every $k = f(i)$ for an $i \in A$, $\sum_{n\in f(A)\setminus\{k\}} c_{k,n} = \sum_{n\in A\setminus\{i\}} c_{f(i),f(n)} = \sum_{n\in A\setminus\{i\}} d_{i,n} < \varepsilon$. Therefore, if $\mathcal{F}$ is a Rosenthal family, then $f[\mathcal{F}]$ is a Rosenthal family. The converse direction is similar. □

A family $\mathcal{F}$ of subsets of a set X is said to be an *upper family* (or an *upward closed family*) on X, if $B \in \mathcal{F}$ and $B \subseteq A \subseteq X$ imply $A \in \mathcal{F}$. We say that an upper family $\mathcal{F}$ on X is an *ultrafamily*, if for every set $A \subseteq X$, $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$. For an upper family $\mathcal{F}$ denote $\mathcal{F}^{=0} = \{X \setminus A : A \in \mathcal{F}\}$, $\mathcal{F}^{>0} = \mathcal{P}(X) \setminus \mathcal{F}^{=0}$, and $\mathcal{F}^{<1} = \mathcal{P}(X) \setminus \mathcal{F}$. The sets in $\mathcal{F}^{>0}$ are called *$\mathcal{F}$-positive sets.*

An upper family $\mathcal{F}$ on X is said to be a *semifilter on* X, if $X \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$, and $A \setminus a \in \mathcal{F}$ for all $A \in \mathcal{F}$ and $a \in [X]^{<\omega}$ (see Zdomskyy [16]). A semifilter is an *ultrasemifilter*, if it is an ultrafamily.

Natural examples of semifilters are $\mathcal{H}$ and $\mathcal{H}^{>0}$ when $\mathcal{H}$ is a filter on ω; in this case $\mathcal{H} \subseteq \mathcal{H}^{>0}$ and the equality holds if and only if $\mathcal{H}$ is an ultrafilter. If $\mathcal{F}$ is a semifilter on ω, then also $\mathcal{F}^{>0}$, $\mathcal{F}\cap\mathcal{F}^{>0}$, $\mathcal{F}\cup\mathcal{F}^{>0}$ are semifilters on ω, $[\omega]^{<\omega} \subseteq \mathcal{F}^{=0}$, $(\mathcal{F}^{>0})^{=0} = \mathcal{F}^{<1}$, $(\mathcal{F}^{>0})^{>0} = \mathcal{F}$, $(\mathcal{F}^{>0})^{<1} = \mathcal{F}^{=0}$, and $(\mathcal{F} \cap \mathcal{F}^{>0})^{>0} = \mathcal{F} \cup \mathcal{F}^{>0}$.

Definition 2.2. Let $\mathcal{F} \subseteq [\omega]^\omega$.

(1) $\mathcal{F}$ has the *antichain property*, if there exists a partition $\mathcal{A}$ of ω such that $(\forall A \in \mathcal{F})(\exists a \in \mathcal{A})\ a \subseteq A$ and $|a| \geq 2$ (this notion was introduced in [14]).
(2) Let $f \in {}^\omega\omega$ be a non-diagonal function, i.e., $(\forall k \in \omega)\ f(k) \neq k$.
   (a) We say that f is a *diagonal of* $\mathcal{F}$, if $(\forall A \in \mathcal{F})\ A \cap f^{-1}(A) \neq \emptyset$, i.e., $(\forall A \in \mathcal{F})(\exists k \in A)\ f(k) \in A$.
   (b) We say that f is a *local diagonal of* $\mathcal{F}$, if there exists $A \in \mathcal{F}$ such that $(\forall B \in \mathcal{F} \cap \mathcal{P}(A))\ B \cap f^{-1}(B) \neq \emptyset$.
   (c) A diagonal f of $\mathcal{F}$ is said to be a *subdiagonal of* $\mathcal{F}$ or a *superdiagonal of* $\mathcal{F}$, if $(\forall k > 0)\ f(k) < k$ or $(\forall k \in \omega)\ f(k) > k$, respectively.
(3) A function $f : \omega \to \mathcal{P}(\omega)$ is said to be a *set mapping*, if $k \notin f(k)$ for all $k \in \omega$. A set $A \subseteq \omega$ is called *free with respect to* f, if $(\forall k \in A)\ A \cap f(k) = \emptyset$ (see [3]). A set mapping $f : \omega \to [\omega]^{<\omega}$ is said to be *uniformly finite*, if there is $l \in \omega$ such that $|f(k)| \leq l$ for all $k \in \omega$. A uniformly finite set mapping f is said to be a *multi-diagonal of* $\mathcal{F}$, if no set in $\mathcal{F}$ is free with respect to f.
(4) $\mathcal{F}$ is *decomposable*, if there exists a partition $\{A_n : n \in \omega\}$ of ω such that $(\forall A \in \mathcal{F})(\forall n \in \omega)\ |A \cap A_n| = \omega$.

Obviously, if f is a diagonal of $\mathcal{F}$, then $g(n) = \{f(n)\}$ is a multi-diagonal of $\mathcal{F}$. Every non-diagonal finite-to-one function is a diagonal of the Fréchet filter. In the next we will show that no Rosenthal filter (family) has a diagonal.

We say that a set $\mathcal{F}_0 \subseteq \mathcal{F}$ is dense in $\mathcal{F}$, if $(\forall A \in \mathcal{F})(\exists B \in \mathcal{F}_0)\ B \subseteq A$; $\mathcal{F}_0$ is an open subset of $\mathcal{F}$, if $(\forall A \in \mathcal{F}_0)(\forall B \in \mathcal{F} \cap \mathcal{P}(A))\ B \in \mathcal{F}_0$.

Lemma 2.3. *Let $\mathcal{F} \subseteq [\omega]^\omega$.*

1. *$\mathcal{F}$ has no local diagonal if and only if for every uniformly finite set mapping $f : \omega \to [\omega]^{<\omega}$ the set $\{A \in \mathcal{F} : A$ is free w.r.t. $f\}$ is dense in $\mathcal{F}$.*
2. *If $\mathcal{F}$ has no local diagonal, then $\mathcal{F}$ has no multi-diagonal.*
3. *The following assertions are equivalent for a filter $\mathcal{F}$:* (i) *$\mathcal{F}$ has no diagonal.* (ii) *$\mathcal{F}$ has no local diagonal.* (iii) *$\mathcal{F}$ has no multi-diagonal.*

Proof. (1) Let $f : \omega \to [\omega]^l$ for some $l > 0$ be a uniformly finite set mapping and let $A_0 \in \mathcal{F}$. Let $f_i \in {}^\omega\omega$, $i < l$, be non-diagonal functions such that $f(k) = \{f_i(k) : i < l\}$ for all $k \in \omega$. If $\mathcal{F}$ has no local diagonal, then by induction on $i < l$, let $A_{i+1} \in \mathcal{F} \cap \mathcal{P}(A_i)$ be such that $(\forall k \in A_{i+1})\ f_i(k) \notin A_{i+1}$. Then $A_l \in \mathcal{F}$, $A_l \subseteq A_0$ and A_l is free with respect to f. The converse is obvious.

(2) A consequence of (1).

(3) (ii) $\to$ (iii) holds by (2) and (iii) $\to$ (i) is obvious. We show (i) $\to$ (ii). Let f be a non-diagonal function. By (i) there is $B_0 \in \mathcal{F}$ such that $B_0 \cap f^{-1}(B_0) = \emptyset$. Given $A \in \mathcal{F}$ for $B = A \cap B_0$ we have $B \cap f^{-1}(B) = \emptyset$ and, since $\mathcal{F}$ is a filter, $B \in \mathcal{F} \cap \mathcal{P}(A)$. Therefore f is not a local diagonal of $\mathcal{F}$. □

A family $\mathcal{F}$ is decomposable if and only if there exists a partition $\{A_n : n \in \omega\}$ of ω such that $(\forall A \in \mathcal{F})(\forall^\infty n \in \omega)\ A \cap A_n \neq \emptyset$ (by gluing infinite subfamilies of this partition together we obtain a partition satisfying the property in definition). If $\mathcal{F}$ is decomposable, then $\mathcal{F}$ is isomorphic to a subfamily of an infinite sum $\sum_{n\in\omega} \mathcal{F}_n$ defined in Section 4. This larger family is truly decomposable and by the following proposition neither this larger family is Rosenthal (we can say "larger" due to Lemma 2.1).

Proposition 2.4. *Let $\mathcal{F} \subseteq [\omega]^\omega$.*

1. *If $\mathcal{F}$ has the antichain property, then $\mathcal{F}$ has a superdiagonal and a subdiagonal.*
2. *If $\mathcal{F}$ is decomposable, then $\mathcal{F}$ has a subdiagonal.*
3. *If $\mathcal{F}$ has a diagonal or a multi-diagonal, then $\mathcal{F}$ is not Rosenthal.*

Proof. (1) Let $\mathcal{A}$ be a family of pairwise disjoint sets witnessing the antichain property of $\mathcal{F}$. By removing singletons we can assume that $|a| \geq 2$ for all $a \in \mathcal{A}$. For $a \in \mathcal{A}$ denote $m_a = \min a$ and $n_a = \min(a \setminus \{m_a\})$. The partial functions $f : m_a \mapsto n_a$ and $g : n_a \mapsto m_a$ for $a \in \mathcal{A}$ can be extended in an obvious way to a superdiagonal of $\mathcal{A}$ and a subdiagonal of $\mathcal{A}$, respectively.

(2) Let $\{A_n : n \in \omega\}$ be a partition of ω such that $(\forall A \in \mathcal{F})(\forall n \in \omega)\ |A \cap A_n| = \omega$ and let $f \in {}^\omega\omega$ be a subdiagonal function defined by $f(k) = n$, if $k \in A_n \setminus (n+1)$, and $f(k) = 0$, if $k \in A_n \cap (n+1)$, otherwise. The function f is a diagonal of $\mathcal{F}$ because given $A \in \mathcal{F}$ for any $n \in A$ and $k \in A \cap A_n \setminus (n+1)$ we have $f(k) = n$.

(3) Let $f \in {}^\omega\mathcal{P}(\omega)$ be a multi-diagonal of $\mathcal{F}$ and $l > 0$ be such that $(\forall k \in \omega)$ $|f(k)| \leq l$. The matrix $\{c_{k,n}\}_{k,n\in\omega}$ defined by

$$c_{k,n} = \begin{cases} 1/l, & \text{if } n \in f(k), \\ 0, & \text{otherwise} \end{cases}$$

is a Rosenthal matrix because $\sum_{n\in\omega} c_{k,n} = \sum_{n\in f(k)} c_{k,n} \leq |f(k)|/l \leq 1$ for all $k \in \omega$. Let $A \in \mathcal{F}$ and $k \in A$ be such that $f(k) \cap A \neq \emptyset$. Then $\sum_{n\in A\setminus\{k\}} c_{k,n} \geq 1/l$. Therefore $\mathcal{F}$ is not a Rosenthal family. □

Corollary 2.5 (Sobota [14])**.** *A family $\mathcal{F} \subseteq [\omega]^\omega$ with the antichain property is not Rosenthal.* □

Proposition 2.6. *There is a family $\mathcal{F} \subseteq [\omega]^\omega$ with a subdiagonal but with no superdiagonal.*

Proof. Let $\mathcal{D}^+ = \{f \in {}^\omega\omega : (\forall k \in \omega)\ k < f(k)\}$ and let $\mathcal{F} = \{A_f : f \in \mathcal{D}^+\}$ where $A_f = \{f^{(n)}(0) : n \in \omega\}$. The function f defined by $f(k) = 0$ for $k > 0$ and $f(0) = 1$ is a subdiagonal of $\mathcal{F}$ because $0 \in \bigcap \mathcal{F}$. We show that no $g \in \mathcal{D}^+$ is a diagonal of $\mathcal{F}$. For $g \in \mathcal{D}^+$ consider $A_f \in \mathcal{F}$ with $f = g + 1$. For every $n \in \omega$, $f^{(n)}(0) < g(f^{(n)}(0)) < f^{(n+1)}(0)$ because $f^{(n+1)}(0) = g(f^{(n)}(0)) + 1$. Therefore $A_f \cap g^{-1}(A_f) = \emptyset$. □

Corollary 2.7 (Sobota [14])**.** *There is a family $\mathcal{F} \subseteq [\omega]^\omega$ which is not Rosenthal and does not have the antichain property.* □

Sobota in [14, Remark 3.20] gave an argument based on a variant of Hajnal's free set theorem by which every ultrafilter is Rosenthal with respect to the class of all "uniformly finitely supported" Rosenthal matrices. We use the same argument in the proof of the following proposition:

Proposition 2.8. *If $\mathcal{H}$ is a filter on ω, then $\mathcal{H}^{>0}$ has no local diagonal.*

Proof. [6, Excercise 26.9] states: Assume that $f : S \to [S]^{\leq k}$ is a set mapping for some natural number k. Then S is the union of $2k + 1$ free sets with respect to f.

Fix a non-principal ultrafilter $\mathcal{G}$ that extends the filter $\mathcal{H}$. Let $f \in {}^\omega\omega$ be a non-diagonal function. By previous claim there is a partition of ω into three sets X such that $X \cap f^{-1}(X) = \emptyset$. One of these sets, say C, belongs to $\mathcal{G}$. Let $A \in \mathcal{H}^{>0}$ be arbitrary and let $B = A \cap C$. Then $B \in \mathcal{H}^{>0}$, $B \subseteq A$, and $B \cap f^{-1}(B) = \emptyset$. Therefore f is not a local diagonal of $\mathcal{H}^{>0}$. □

There are also other semifilters without local diagonals (see Theorem 6.6).

The next lemma extends the Talagrand's characterization of non-meager filters (see [1, Theorem 4.1.2], [15], [16]) for upper families. Its dual form is in [11, Lemma 1.3].

Lemma 2.9. *An upper family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is not meager if and only if for every finite-to-one function $f \in {}^\omega\omega$ there is $A \in [\omega]^\omega$ such that $\omega \setminus f^{-1}(A) \in \mathcal{F}$.* □

Lemma 2.10. *An upper family without diagonal is not meager. Consequently, every upward closed Rosenthal family is not meager, and in particular, every Rosenthal filter and every Rosenthal semifilter is not meager.*

Proof. We apply Lemma 2.9. Let $\mathcal{F} \subseteq [\omega]^\omega$ be an upper family with no diagonal. Let $f \in {}^\omega\omega$ be a finite-to-one function. Let $\pi \in {}^\omega\omega$ be a bijection such that $g(k) = \pi(f(k)) \neq k$ for all $k \in \omega$. Since g is not a diagonal of $\mathcal{F}$, there is $B \in \mathcal{F}$ such that $B \cap g^{-1}(B) = \emptyset$. The set $A = \pi^{-1}(B)$ is infinite and $\omega \setminus f^{-1}(A) = \omega \setminus g^{-1}(B) \supseteq B \in \mathcal{F}$. □

The converse of Lemma 2.10 does not hold because if $f \in {}^\omega\omega$ is an unbounded non-diagonal function, then $\{A \in [\omega]^\omega : A \cap f^{-1}(A) \neq \emptyset\}$, the largest upper family with diagonal f, is an open dense subset of $[\omega]^\omega$. For Rosenthal filters we get a stronger result in Section 4: Every Rosenthal filter is the intersection of finitely many ultrafilters pairwise incomparable in the Rudin-Keisler ordering. By Plewik's result, intersection of $< \mathfrak{c}$ ultrafilters is a non-meager filter ([1, 9]). The proof of Lemma 2.10 works also if the function f is not finite-to-one and we get the following:

Lemma 2.11. *Every upper family which is $\leq_{\mathrm{RK}}$-above the Fréchet filter has a diagonal and consequently is not Rosenthal.* □

3. Cardinal invariants

We prove that every family $\mathcal{F} \subseteq [\omega]^\omega$ of cardinality smaller than the *reaping number* $\mathfrak{r}$ is not Rosenthal, i.e., $\mathfrak{r} \leq \mathfrak{ros}$ with the following denotation.

Definition 3.1. Let

$$\mathrm{cov}(\mathcal{M}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{M} \text{ and } \mathbb{R} = \bigcup \mathcal{F}\},$$
$$\mathfrak{anti} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \text{ does not have the antichain property}\},$$
$$\overline{\mathfrak{diag}} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \text{ does not have a superdiagonal}\},$$
$$\underline{\mathfrak{diag}} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \text{ does not have a subdiagonal}\},$$
$$\mathfrak{diag} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \text{ does not have a diagonal}\},$$
$$\mathfrak{dcmp} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \text{ is not decomposable}\},$$
$$\mathfrak{ros} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \text{ is Rosenthal}\},$$
$$\mathfrak{r} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \text{ and } (\forall B \in [\omega]^\omega)(\exists A \in \mathcal{F})\ A \subseteq^* B \text{ or } A \subseteq^* \omega \setminus B\}.$$

A family $\mathcal{F}$ satisfying the property in the definition of $\mathfrak{r}$ is called a *reaping family.*

Proposition 3.2 (Sobota [14])**.** $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{anti} \leq \mathfrak{r}$. □

Proposition 3.3. $\mathfrak{anti} \leq \min\{\overline{\mathfrak{diag}}, \underline{\mathfrak{diag}}\} \leq \max\{\overline{\mathfrak{diag}}, \underline{\mathfrak{diag}}\} \leq \mathfrak{diag} \leq \mathfrak{ros} \leq \mathfrak{c}$ *and* $\mathfrak{dcmp} \leq \underline{\mathfrak{diag}}$.

Proof. $\mathfrak{ros} \leq \mathfrak{c}$ holds by Rosenthal's Lemma and the other inequalities follow by Proposition 2.4. □

Recall that the dominating number $\mathfrak{d}$ is the minimal cardinality of a family $D \subseteq {}^\omega\omega$ such that every $f \in {}^\omega\omega$ is dominated by a $g \in D$, i.e., $(\forall n \in \omega)\ f(n) \leq g(n)$; the dominance of functions can be replaced here by the eventual dominance. D. Sobota has proved $\mathfrak{anti} \leq \mathfrak{d}$ which follows also by the next proposition.

Proposition 3.4. $\overline{\mathfrak{diag}} \leq \mathfrak{d}$ *and* $\mathfrak{dcmp} = \mathfrak{r} \leq \underline{\mathfrak{diag}}$.

Proof. Proof of $\overline{\mathfrak{diag}} \leq \mathfrak{d}$. Let $\mathcal{D}^+ = \{f \in {}^\omega\omega : (\forall k \in \omega)\ k < f(k)\}$. We are going to define mappings $\varphi : {}^\omega\omega \to [\omega]^\omega$ and $\psi : \mathcal{D}^+ \to {}^\omega\omega$ such that for every $g \in {}^\omega\omega$, $f \in \mathcal{D}^+$, and $k \in \omega$,

$$k, f(k) \in \varphi(g) \to \psi(f)(k) > g(k)$$

It follows that $\overline{\mathfrak{diag}} \leq \mathfrak{d}$ because if $\mathcal{F} \subseteq {}^\omega\omega$ and $|\mathcal{F}| < \overline{\mathfrak{diag}}$, then for a superdiagonal f of $\{\varphi(g) : g \in \mathcal{F}\}$ the function $\psi(f)$ is not dominated by any $g \in \mathcal{F}$.

For $g \in {}^\omega\omega$ define $\varphi(g) = \{\bar{g}^{(n)}(0) : n \in \omega\}$ where $\bar{g}(k) = \max\{g(k), k+1\}$ for all $k \in \omega$. For $f \in \mathcal{D}^+$ let $\psi(f) = f + 1$. Assume $f \in \mathcal{D}^+$ and $k, f(k) \in \varphi(g)$. Then for some $i \geq 1$, $\psi(f)(k) > f(k) = \bar{g}^{(i)}(k) \geq g(k)$.

Proof of $\mathfrak{dcmp} = \mathfrak{r}$. If $\mathcal{F} \subseteq [\omega]^\omega$ and $|\mathcal{F}| < \mathfrak{r}$, then applying definition of $\mathfrak{r}$ one can construct inductively a sequence of pairwise disjoint sets A_n such that for every $n \in \omega$, $(\forall A \in \mathcal{F})\ |A \cap A_n| = |A \setminus A_n| = \omega$. Then obviously, $\mathcal{F}$ is decomposable. Therefore $\mathfrak{r} \leq \mathfrak{dcmp}$. On the other hand, a family $\mathcal{F}$ from the definition of $\mathfrak{r}$ witnessing the equality $|\mathcal{F}| = \mathfrak{r}$ is not decomposable and therefore $\mathfrak{r} \geq \mathfrak{dcmp}$. □

By the consistency of $\mathfrak{d} < \mathfrak{r}$ (forcing with a measure algebra preserves dominating families while random reals kill reaping families) we get:

Corollary 3.5. *It is consistent with ZFC that* $\overline{\mathfrak{diag}} < \underline{\mathfrak{diag}}$. □

Further information on these cardinal numbers is in Corollary 6.8.

4. Decomposable families

In this section we prove that every Rosenthal filter is the intersection of a finite family of pairwise $\leq_{\mathrm{RK}}$-incomparable Rosenthal ultrafilters.

Assume that $\mathcal{F}_i \subseteq \mathcal{P}(X_i)$ for $i \in I$. Define $\sum_{i\in I} X_i = \bigcup_{i \in I}\{i\} \times X_i$ and $\sum_{i\in I} \mathcal{F}_i = \{A \subseteq \sum_{i\in I} X_i : (\forall i \in I)\ \{n \in \omega : (i,n) \in A\} \in \mathcal{F}_i\}$. For a family $\mathcal{F} \subseteq \mathcal{P}(X)$ we write $\mathcal{F} =_{\mathrm{RK}} \sum_{i\in I} \mathcal{F}_i$ if there is a bijection $f : \sum_{i\in I} X_i \to X$ such that $\mathcal{F} = \{A \subseteq X : f^{-1}(A) \in \sum_{i\in I}\mathcal{F}_i\}$. For example, if $\{A_i : i \in I\}$ is a partition of ω and $\mathcal{F}_i \subseteq \mathcal{P}(A_i) \setminus \{\emptyset\}$ for $i \in I$, then $\sum_{i\in I}\mathcal{F}_i =_{\mathrm{RK}} \{A \subseteq \omega : (\forall i \in I)\ A \cap A_i \in \mathcal{F}_i\}$. Let $\mathcal{F}{\restriction}A = \{A \cap B : B \in \mathcal{F}\}$.

We say that a family $\{\mathcal{F}_i : i \in I\}$ of filters on ω is *separated* by a family $\{A_i : i \in I\}$ of pairwise disjoint subsets of ω if $A_i \in \mathcal{F}_i$ for all $i \in I$. In this case we say that the family of filters is *discrete*. Every finite family of ultrafilters is discrete.

Lemma 4.1. *Let $\{\mathcal{F}_i : i \in I\}$ be a family of filters on ω separated by a partition $\{A_i : i \in I\}$.*

1. $\bigcap_{i\in I}\mathcal{F}_i =_{\mathrm{RK}} \sum_{i\in I}\mathcal{F}_i{\restriction}A_i$ *and* $\bigcap_{i\in I}\mathcal{F}_i^{>0} =_{\mathrm{RK}} \sum_{i\in I}\mathcal{F}_i^{>0}{\restriction}A_i$.
2. $\bigcap_{i\in I}\mathcal{F}_i =_{\mathrm{RK}} \sum_{i\in I}\mathcal{F}_i$ *and* $\bigcap_{i\in I}\mathcal{F}_i^{>0} =_{\mathrm{RK}} \sum_{i\in I}\mathcal{F}_i^{>0}$ *whenever* $\mathcal{F}_i^{=0}{\restriction}A_i \neq [A_i]^{<\omega}$ *for all* $i \in I$.
3. $\bigcap_{i\in I}\mathcal{F}_i =_{\mathrm{RK}} \sum_{i\in I}\mathcal{F}_i$ *for every discrete family of ultrafilters on* ω.

Proof. (1) follows by definitions and (3) is a consequence of (2).

(2) For every $i \in I$ there is a bijection $f_i : A_i \to \omega$ such that $\mathcal{F}_i = \{B \subseteq \omega : f_i^{-1}(B) \in \mathcal{F}_i{\restriction}A_i\}$ and $\mathcal{F}_i^{>0} = \{B \subseteq \omega : f_i^{-1}(B) \in \mathcal{F}_i^{>0}{\restriction}A_i\}$ (choose $A_i' \in \mathcal{F}_i$ such that $A_i' \subseteq A_i$ and $|A_i \setminus A_i'| = \omega$ and let the bijection $f_i : A_i \to \omega$ be such that $f{\restriction}A_i'$ is the identity on A_i' and $f_i(A_i \setminus A_i') = \omega \setminus A_i'$). It follows that $\sum_{i\in I}\mathcal{F}_i =_{\mathrm{RK}} \sum_{i\in I}\mathcal{F}_i{\restriction}A_i$ and $\sum_{i\in I}\mathcal{F}_i^{>0} =_{\mathrm{RK}} \sum_{i\in I}\mathcal{F}_i^{>0}{\restriction}A_i$. Now apply (1). □

Lemma 4.2. *An upper family $\mathcal{F}$ on a set X is an ultrafamily if and only if there are no two disjoint $\mathcal{F}$-positive sets.*

Proof. $(\exists A \subseteq X)(A \notin \mathcal{F}$ and $X \setminus A \notin \mathcal{F}) \leftrightarrow (\exists A \subseteq X)(X \setminus A \notin \mathcal{F}^{=0}$ and $A \notin \mathcal{F}^{=0}) \leftrightarrow (\exists A \subseteq X)(X \setminus A \in \mathcal{F}^{>0}$ and $A \in \mathcal{F}^{>0}) \leftrightarrow$ there are two disjoint $\mathcal{F}$-positive sets. □

We say that $\mathcal{F}$ has the *finiteness property*, if $(\exists p \in \omega)(\forall A \in \mathcal{F})\ |A \cap p| \geq 2$.

Lemma 4.3. *Let $\mathcal{F} \subseteq [\omega]^\omega$ be an upper family.* (1) *If there are two disjoint finite $\mathcal{F}$-positive sets, then $\mathcal{F}$ has the finiteness property.* (2) *If $\mathcal{F}$ has the finiteness property, then $\mathcal{F}$ is not Rosenthal.*

Proof. (1) If $p \in \omega$ contains two disjoint finite $\mathcal{F}$-positive sets, then p witnesses that $\mathcal{F}$ has the finiteness property.

(2) Fix a Rosenthal matrix $\{c_{k,n}\}_{k,n\in\omega}$ with all $c_{k,n} > 0$. Let $p \in \omega$ be such that $(\forall A \in \mathcal{F})\ |A \cap p| \geq 2$ and let $\varepsilon = \min\{c_{k,m} : k, m < p\}$. Then $\mathcal{F}$ is not Rosenthal because every $A \in \mathcal{F}$ contains distinct $k, m < p$ with $\sum_{n\in A\setminus\{k\}} c_{k,n} \geq c_{k,m} \geq \varepsilon$. □

Lemma 4.4. *If an upper family $\mathcal{F} \subseteq [\omega]^\omega$ is a Rosenthal family, then there exists a finite partition $\mathcal{A} \subseteq \mathcal{F}^{>0}$ of ω such that*

(i) *for every $A \in \mathcal{A}$, $\mathcal{F}{\restriction}A = \{A \cap B : B \in \mathcal{F}\}$ is an upper ultrafamily on A,*
(ii) *at most one of the sets $A \in \mathcal{A}$ is finite and then $\mathcal{F}{\restriction}A$ contains at least one singleton,*
(iii) *for every $A \in \mathcal{A} \cap [\omega]^\omega$, $\mathcal{F}{\restriction}A$ is a Rosenthal ultrafamily on A, and*
(iv) *$\sum_{A\in\mathcal{A}} \mathcal{F}{\restriction}A$ is a Rosenthal family on ω.*

If $\mathcal{F}$ is a Rosenthal semifilter, then moreover,

(v) *$\mathcal{A} \subseteq [\omega]^\omega$ and $\mathcal{F}{\restriction}A$ is a Rosenthal ultrasemifilter on A for every $A \in \mathcal{A}$.*

Proof. We say that a partition $\mathcal{A}$ of ω is primitive, if $\mathcal{A} \subseteq \mathcal{F}^{>0}$ and no $A \in \mathcal{A}$ is union of two disjoint nonempty $\mathcal{F}$-positive sets. One can observe that, if no finite partition is primitive, then there exists an infinite partition of $\mathcal{F}$-positive sets and consequently, $\mathcal{F}$ is decomposable. Therefore, by Proposition 2.4, it follows that there is a finite primitive partition $\mathcal{A}$ of ω. Now, condition (i) is obviously satisfied and condition (ii) holds because by Lemma 4.3, $\mathcal{F}$ has not the finiteness property.

(iii) $\mathcal{F}{\restriction}A$ is a Rosenthal family on ω because for every $B \in \mathcal{F}$, $A \cap B \in \mathcal{F}{\restriction}A$ and $A \cap B \subseteq B$. Due to Lemma 2.1, $\mathcal{F}{\restriction}A$ is a Rosenthal family on A.

(iv) Since $\mathcal{A} \subseteq \mathcal{F}^{>0}$, $\sum_{A\in\mathcal{A}} \mathcal{F}{\restriction}A =_{\mathrm{RK}} \{B \subseteq \omega : (\forall A \in \mathcal{A})\ A \cap B \in \mathcal{F}{\restriction}A\} \supseteq \mathcal{F}$ and consequently, $\sum_{A\in\mathcal{A}} \mathcal{F}{\restriction}A$ is a Rosenthal family by Lemma 2.1.

(v) If $\mathcal{F}$ is a semifilter, then $\mathcal{F}^{>0} \subseteq [\omega]^\omega$ and for every $A \in \mathcal{F}^{>0}$, $\mathcal{F}{\restriction}A$ is a semifilter on A. Therefore the assertion holds by (iii). □

Theorem 4.5. *Every Rosenthal filter is Rudin-Keisler equivalent to the sum of a finite collection of Rosenthal ultrafilters pairwise incomparable in the Rudin-Keisler ordering.*

Proof. Let $\mathcal{F}$ be a Rosenthal filter on ω. Let $\mathcal{A} \subseteq \mathcal{F}^{>0}$ be a finite partition of ω satisfying conditions (i)–(v) of Lemma 4.4. For every $A \in \mathcal{A}$, $\mathcal{F}{\restriction}A$ is a filter on A and by (iii), $\mathcal{F}{\restriction}A$ is a Rosenthal ultrafilter on A. Obviously, $\mathcal{F} \subseteq \{B \subseteq \omega : (\forall A \in \mathcal{A})\ A \cap B \in \mathcal{F}{\restriction}A\}$. Since $\mathcal{F}$ is a filter and $\mathcal{A}$ is finite, the opposite inclusion also holds. Therefore $\mathcal{F} =_{\mathrm{RK}} \sum_{A\in\mathcal{A}} \mathcal{F}{\restriction}A$.

Assume that $A_0, A_1 \in \mathcal{A}$ are distinct such that $\mathcal{F}{\restriction}A_0 \leq_{\mathrm{RK}} \mathcal{F}{\restriction}A_1$, i.e., $\mathcal{F}{\restriction}A_0 = \{A \subseteq A_0 : f^{-1}(A) \in \mathcal{F}{\restriction}A_1\}$ for some $f \in {}^{A_1}A_0$. Since $A_1 \cap f(A_1) = \emptyset$, there is a non-diagonal function $h \in {}^\omega\omega$ such that $h(n) = f(n)$ for $n \in A_1$ and $h(n) \in A_1$ for $n \in \omega \setminus A_1$. By Proposition 2.4, $\mathcal{F}$ has no diagonal and hence there is $A \in \mathcal{F}$ such that $A \cap h^{-1}(A) = \emptyset$. Then $(A \cap A_1) \cap f^{-1}(A \cap A_0) = \emptyset$ because $f^{-1}(A \cap A_0) = h^{-1}(A \cap A_0) \subseteq h^{-1}(A)$. This is a contradiction because $A \cap A_1 \in \mathcal{F}{\restriction}A_1$ and $f^{-1}(A \cap A_0) \in \mathcal{F}{\restriction}A_1$ as $A \cap A_0 \in \mathcal{F}{\restriction}A_0$. Therefore the ultrafilters $\mathcal{F}{\restriction}A$ for $A \in \mathcal{A}$ are pairwise $\leq_{\mathrm{RK}}$-incomparable. □

By Lemma 4.1, Theorem 4.5 has the following consequences:

Corollary 4.6. *A Rosenthal filter is the intersection of a finite family of Rosenthal ultrafilters pairwise incomparable in the Rudin-Keisler ordering.* □

Corollary 4.7. *A filter contained in two different $\leq_{\mathrm{RK}}$-comparable ultrafilters is not Rosenthal.* □

By Lemma 5.4 (1) below, the factors of the decomposition of a Rosenthal semifilter into Rosenthal ultrasemifilters in Lemma 4.4 need not be Rudin-Keisler incomparable.

Question 4.8. Does the converse of Theorem 4.5 or Corollary 4.6 hold?

5. Combinatorics on semifilters

Comfort and Negrepontis in [2, Theorem 9.6] have proved the equivalence of ten distinct properties of ultrafilters including selective, Ramsey, weakly Ramsey, and quasi-normal ultrafilter. In the present section we generalize to semifilters some selectivity properties known for ultrafilters and filters and prove characterizations for them analogous to those in [2] (restricted to the four mentioned properties). Some of these results we apply in the next section.

For $A \subseteq \omega$, $n \in \omega$, and $\varphi \in {}^\omega\omega$ denote

$$[A]^n = \{a \subseteq A : |a| = n\},$$
$$[A]^{n,\varphi} = \{a \in [A]^n : \varphi\restriction a \text{ is one-to-one}\},$$
$$\mathrm{Mon} = \{\varphi \in {}^\omega\omega : \varphi \text{ is non-decreasing, finite-to-one, and surjective}\}.$$

Definition 5.1. Let $\mathcal{F}$ be a semifilter on ω and let $\rho \in {}^\omega\omega$ be non-decreasing and $\rho \geq 1$.

(i) (1) A partition $\{A_n : n \in \omega\}$ of ω is an *$\mathcal{F}$-partition*, if $\bigcup_{k\leq n} A_k \in \mathcal{F}^{=0}$ for all $n \in \omega$. A family $\{A_n : n \in \omega\} \subseteq \mathcal{F}$ is *$\mathcal{F}$-centered*, if $\bigcap_{k\leq n} A_k \in \mathcal{F}$ for all $n \in \omega$.

(2) $\mathcal{F}$ is an *$s(\rho)$-semifilter* (a *p-semifilter*, respectively), if for every $\mathcal{F}$-partition $\{A_n : n \in \omega\}$ of ω there is $A \in \mathcal{F}$ such that $|A \cap A_n| \leq \rho(n)$ ($|A \cap A_n| < \omega$, respectively) for all $n \in \omega$.

(ii) (1) We define a semifilter $\mathcal{F}^{(n)}$ on $[\omega]^n$ for $n \geq 1$ as follows:

(a) If $\mathcal{F}$ is a filter, then we define by induction on $n \geq 1$: $\mathcal{F}^{(1)} = \{P \subseteq [\omega]^1 : \bigcup P \in \mathcal{F}\}$ and $\mathcal{F}^{(n+1)} = \{P \subseteq [\omega]^{n+1} : \{m \in \omega : P_m \in \mathcal{F}^{(n)}\} \in \mathcal{F}\}$, where $P_m = \{a \in [\omega]^n : a \cup \{m\} \in P\}$.

(b) $\mathcal{F}^{(n)} = \bigcup\{\mathcal{F}_0^{(n)} : \mathcal{F}_0 \subseteq \mathcal{F} \text{ is a filter}\}$, if $\mathcal{F}$ is not a filter.

Let $P \subseteq [\omega]^n$ and $A \in [\omega]^\omega$. We say that P *is $\mathcal{F}$-big*, if $P \in \mathcal{F}^{(n)}$. We say that P *is A-big*, if P is $\{B \subseteq \omega : A \subseteq^* B\}$-big.

(2) $\mathcal{F}$ is *ρ-Ramsey* (*ω-Ramsey*, respectively), if for every $n \in \omega$,

$(*)_n$ for every $\mathcal{F}$-big set $P \subseteq [\omega]^n$ there exist $A \in \mathcal{F}$ and $\varphi \in \mathrm{Mon}$ such that $[A]^{n,\varphi} \subseteq P$ and for every $k \in \omega$, $1 \leq |A \cap \varphi^{-1}(\{k\})| \leq \rho(k)$ ($A \cap \varphi^{-1}(\{k\})$ is arbitrary finite, respectively).

$\mathcal{F}$ is *weakly ρ-Ramsey* (*weakly ω-Ramsey*, respectively), if $(*)_2$ holds.

(iii) $\mathcal{F}$ is an *s-semifilter*, a *Ramsey semifilter*, a *weakly Ramsey semifilter*, an *r-semifilter*, if $\mathcal{F}$ is $s(1)$-semifilter, 1-Ramsey semifilter, weakly 1-Ramsey semifilter, an $s(\rho)$-semifilter for $\rho(n) = n$, respectively.

(iv) Dual notions of [4, 10] with different notation. Let $\mathcal{F}$ be a filter on ω.
   (1) $\mathcal{F}$ is an *s-filter*, an *r-filter*, a *p-filter*, if $\mathcal{F}$ is an s-semifilter, an r-semifilter, a p-semifilter, respectively.
   (2) $\mathcal{F}$ is an *s^+-filter*, an *r^+-filter*, a *p^+-filter*, if $\mathcal{F}^{>0}$ is an s-semifilter, an r-semifilter, a p-semifilter, respectively.

(v) We say that $\mathcal{F}$ is *full* if for every $A \in \mathcal{F}$, $n \in \omega$, and $P \subseteq [\omega]^n$ at least one of the sets $[A]^n \cap P$ and $[A]^n \setminus P$ is $\mathcal{F}$-big.

Note that $[A]^{n,\varphi} \subseteq P$ implies P is A-big, i.e., $\{B \subseteq \omega : A \subseteq^* B\}$-big.

One can observe that every s-semifilter is an r-semifilter and every r-semifilter is a p-semifilter. For ultrafilters, an s-semifilter, a Ramsey semifilter, and a p-semifilter mean a selective ultrafilter, a Ramsey utrafilter, and a P-point, respectively. For ultrafilters, an s^+-filter as well as an s-filter means a selective ultrafilter, a p^+-filter as well as a p-filter means a P-point, and an r^+-filter and an r-filter have the same meaning.

For a monotone function $\rho \in {}^\omega\omega$ there are two possibilities. Either ρ is bounded and then there is $k \in \omega$ such that $\rho(n) = k$ for all but finitely many $n \in \omega$ and the property of an $s(\rho)$-semifilter coincides with the property of an $s(k)$-semifilter or ρ is unbounded and the property of an $s(\rho)$-semifilter coincides with the property of an r-semifilter. To see the latter note that if $\mathcal{F}$ is an $s(\rho)$-semifilter with ρ monotone and unbounded, then $\mathcal{F}$ is an $s(\rho')$-semifilter for arbitrary monotone unbounded function ρ' (as well as for arbitrary finite-to-one ρ'): By gluing finite subsystems in $\mathcal{F}$-partitions we obtain $\mathcal{F}$-partitions for which finite choices controlled by ρ produce finite choices for original $\mathcal{F}$-partitions that are controlled by ρ'.

The assertions "an ideal I is an s-ideal" and "an ideal I is an s^*-ideal" in [10] now mean "the semifilter of I-positive sets is an s-semifilter" and "the dual filter to I is an s-semifilter", respectively. The same translation applies for r- and p- properties.

Note that $[A]^{n,\varphi} \subseteq P$ implies P is A-big, i.e., $\{B \subseteq \omega : A \subseteq^* B\}$-big.

Proposition 5.2. *For a semifilter $\mathcal{F} \subseteq [\omega]^\omega$ the following conditions are equivalent:*
(1) *$\mathcal{F}$ is full.*
(2) $(\forall A \in \mathcal{F})(\forall X \subseteq \omega)$ $A \cap X \in \mathcal{F}$ *or* $A \setminus X \in \mathcal{F}$.
(3) *$\mathcal{F}^{>0}$ is a filter.*
(4) *There is a filter $\mathcal{H}$ on ω such that $\mathcal{F} = \mathcal{H}^{>0}$.*
(5) *$\mathcal{F}$ is a nonempty union of ultrafilters.*

Proof. (1) $\to$ (2) holds by definition for $n = 1$ and (3) $\to$ (4) holds due to the equality $\mathcal{F} = (\mathcal{F}^{>0})^{>0}$.

(2) $\to$ (3) Let $B, C \in \mathcal{F}^{>0}$, i.e., $B' = \omega \setminus B \notin \mathcal{F}$ and $C' = \omega \setminus C \notin \mathcal{F}$. By (2), $B' \cup C' \notin \mathcal{F}$. Then $B \cap C \in \mathcal{F}^{>0}$ because $B' \cup C' = \omega \setminus (B \cap C)$.

(4) $\to$ (5) If $\mathcal{H}$ is a filter, then $\mathcal{G} \subseteq \mathcal{H}^{>0}$ for every filter $\mathcal{G} \supseteq \mathcal{H}$ and for every $A \in \mathcal{H}^{>0}$ there is an ultrafilter $\mathcal{G} \supseteq \mathcal{H} \cup \{A\}$.

(5) $\to$ (1) It is enough to prove that an ultrafilter is full. Let $\mathcal{F}$ be an ultrafilter on ω. Since for every $A \in \mathcal{F}$, $\mathcal{F} \cap \mathcal{P}(A)$ is an ultrafilter on A, it is enough to prove by induction on $n \in \omega$ that for any $P \subseteq [\omega]^n$ at least one of the sets P and $[\omega]^n \setminus P$ is $\mathcal{F}$-big. Let $n > 0$. Denote $A = \{m \in \omega : P_m \text{ is } \mathcal{F}\text{-big}\}$ where $P_m = \{a \in [\omega]^{n-1} : a \cup \{m\} \in P\}$. If $A \in \mathcal{F}$, then P is $\mathcal{F}$-big. Assume that $\omega \setminus A \in \mathcal{F}$. Then for every $n \in \omega \setminus A$, P_m is not $\mathcal{F}$-big, and by inductive assumption, $([\omega]^n \setminus P)_m = [\omega]^{n-1} \setminus P_m$ is $\mathcal{F}$-big. Therefore $[\omega]^n \setminus P$ is $\mathcal{F}$-big. □

Corollary 5.3. (a) *A full semifilter is an ultrasemifilter.*

(b) *A filter is a full semifilter if and only if it is an ultrafilter.*

Proof. The assertions are consequences of (1) $\leftrightarrow$ (2) of Proposition 5.2. □

Obviously, a countable sum of semifilters on ω is a semifilter on a countable set.

Lemma 5.4. *Let $1 \leq m \leq \omega$ and let $\mathbb{P}$ denote the full product $\prod_{n<m}([\omega]^\omega, \subseteq^*)$.*

(1) *The semifilter $\sum_{n<m}[\omega]^\omega$ is an s-semifilter.*
(2) *If $\mathbb{G} = \prod_{n<m}\mathcal{G}_n$ is a generic subset of $\mathbb{P}$, then $\mathcal{G}_n$, $n < m$, are s-ultrafilters and $\sum_{n<m}\mathcal{G}_n$ and $\bigcap_{n<m}\mathcal{G}_n$ are s-filters* [10, Example 9, p. 105].

Proof. (1) Fix a partition $\{A_n : n < m\}$ of ω into infinite sets and let $\mathcal{F} = \{A \subseteq \omega : (\forall n < m)\ A \cap A_n \in [A_n]^\omega\}$. It is enough to prove that the semifilter $\mathcal{F}$ is an s-semifilter because $\sum_{n<m}[\omega]^\omega$ is isomorphic to $\mathcal{F}$. Let $\{B_n : n \in \omega\}$ be an $\mathcal{F}$-partition of ω, i.e., $A_n \setminus \bigcup_{k\leq l} B_k$ is infinite for all $l \in \omega$ and $n < m$. Then $a_n = \{k \in \omega : A_n \cap B_k \neq \emptyset\}$ is infinite for all $n < m$. Let $b_n \in [a_n]^\omega$ for $n < m$ be pairwise disjoint sets and let $b = \bigcup_{n<m} b_n$. Define a selector $B = \{m_k : k \in b\}$ of $\{B_k : k \in b\}$ so that $m_k \in A_n \cap B_k$, if $k \in b_n$. Then $B \in \mathcal{F}$ because $B \cap A_n = \{m_k : k \in b_n\} \in [A_n]^\omega$ for all $n < m$.

(2) Forcing with $\mathbb{P}$ adds no countable subsets because $\mathbb{P}$ is ω-closed. It is well-known that $\mathcal{G}_n$, $n < m$, are selective ultrafilters and an easy forcing argument shows that they are separated by a partition $\{A_n : n < m\}$ of ω with $A_n \in \mathcal{G}_n$. Due to Lemma 4.1 (2), the filters $\sum_{n<m}\mathcal{G}_n$ and $\bigcap_{n<m}\mathcal{G}_n$ are isomorphic and therefore it is enough to prove that the filter $\mathcal{G} = \bigcap_{n<m}\mathcal{G}_n$ is an s-filter.

Denote $p_0 = \langle A_n : n < m\rangle$; then $p_0 \in \mathbb{G}$. We write a condition $p \in \mathbb{P}$ by $p = \langle A_n^p : n < m\rangle$. The set $D = \{p \in \mathbb{P} : (\forall n < m)\ A_n^p \subseteq A_n\}$ is dense below p_0 in $\mathbb{P}$; the sets A_n^p, $n < m$, are pairwise disjoint for $p \in D$. Let a condition $p \in D$ forces that a partition $\{B_k : k \in \omega\}$ of ω is a $\dot{\mathcal{G}}$-partition, where $\dot{\mathcal{G}}$ is the canonical name for $\mathcal{G}$. For every $q \leq p$, $A_n^q \cap \bigcup_{k\leq l} B_k$ is finite for all $l \in \omega$ and $n < m$ (otherwise for some l and n there is $q \leq p$ with $A_n^q \subseteq \bigcup_{k\leq l} B_k$ and hence $q \Vdash \bigcup_{k\leq l} B_k \in \dot{\mathcal{G}}_n$ and $q \Vdash \bigcup_{k\leq l} B_k \notin \dot{\mathcal{G}}^{=0}$). Let $q \in D$ be arbitrary such that $q \leq p$. Then $a_n = \{k \in \omega : A_n^q \cap B_k \neq \emptyset\}$ is infinite for all $n < m$. Let $b_n \in [a_n]^\omega$ for $n < m$ be pairwise disjoint sets and let $b = \bigcup_{n<m} b_n$. Define a selector $B = \{m_k : k \in b\}$ of $\{B_k : k \in b\}$ so that $m_k \in A_n^q \cap B_k$, if $k \in b_n$. Define $r \in \mathbb{P}$ by $A_n^r = \{m_k : k \in b_n\}$ for $n < m$. Then $r \in D$, $r \leq q$, and $r \Vdash B \in \dot{\mathcal{G}}$ because $A_n^r \subseteq A_n^q$ and $A_n^r \subseteq B$ for all $n < m$. This density argument shows that every $\mathcal{G}$-partition has a selector in $\mathcal{G}$. Therefore $\mathcal{G}$ is an s-filter. □

Other s-, r-, p-semifilters can be obtained by the following lemma.

Lemma 5.5. *Let $\mathcal{F}$ be a semifilter on ω.*

(1) *If $\mathcal{F}$ is an s-semifilter, an r-semifilter, or a p-semifilter, then so is the semifilter $\mathcal{F} \cap \mathcal{P}(B)$ on B for $B \in \mathcal{F}$.*
(2) *If $\mathcal{F}$ is an s-semifilter, an r-semifilter, or a p-semifilter, then so is $\mathcal{F}(B) = \{A \subseteq \omega : A \cap B \in \mathcal{F}\}$ for $B \in \mathcal{F}$* [10, Lemma 1.5].
(3) *If $\mathcal{F}$ is an s-semifilter, an r-semifilter, or a p-semifilter, then so is $\mathcal{G} = \bigcap_{n\in\omega}\mathcal{F}(B_n)$ where $B_{n+1} \subseteq B_n \in \mathcal{F}$ for all $n \in \omega$* [10, Lemma 1.6].
(4) *If $\mathcal{F}_n$, $n < k$ $(k \leq \omega)$, are r-semifilters or p-semifilters, then so is $\sum_{n<k}\mathcal{F}_n$.*

Proof. (1) Denote $\mathcal{G} = \mathcal{F} \cap \mathcal{P}(B)$. For every $A \subseteq B$, $A \in \mathcal{G}^{=0} \leftrightarrow B \setminus A \in \mathcal{F} \leftrightarrow (\omega \setminus B) \cup A \in \mathcal{F}^{=0}$. Hence, if $\mathcal{A} = \{A_n : n \in \omega\}$ is a $\mathcal{G}$-partition of B, then

$\mathcal{A}' = \{\omega \setminus B\} \cup \mathcal{A}$ is an $\mathcal{F}$-partition of ω. If $A \in \mathcal{F}$ has finite intersections with sets from $\mathcal{A}'$, then $A \setminus B \in \mathcal{G}$ has the same intersections with sets from $\mathcal{A}$.

(2) Use these facts: (i) $A \in (\mathcal{F}(B))^{=0} \leftrightarrow A \cup (\omega \setminus B) \in \mathcal{F}^{=0}$. (ii) If $\{A_n : n \in \omega\}$ is an $\mathcal{F}(B)$-partition, then $A'_0 = A_0 \cup (\omega \setminus B)$, $A'_n = A_n \cap B$, $n \geq 1$, is an $\mathcal{F}$-partition. (iii) If $A \in \mathcal{F}$ and $|A \cap A'_n| < \omega$ for all n, then $A \subseteq^* B$ and hence, $A \cap B \in \mathcal{F}$, $A \cap B \in \mathcal{F}(B)$, and $|(A \cap B) \cap A_n| \leq |A \cap A'_n|$ for all $n \in \omega$.

(3) Use these facts: (i) $A \in \mathcal{G}^{=0} \leftrightarrow (\forall n \in \omega)\ A \cup (\omega \setminus B_n) \in \mathcal{F}^{=0}$. (ii) If $\{A_n : n \in \omega\}$ is a $\mathcal{G}$-partition, since $\mathcal{F}$ is a p-semifilter, there is $B \in \mathcal{F}$ such that $A_k \cup (\omega \setminus B_n) \subseteq^* \omega \setminus B$ for all $k, n \in \omega$; then $\{A_n : n \in \omega\}$ is an $\mathcal{F}(B)$-partition and $\mathcal{F}(B) \subseteq \mathcal{G}$. Therefore case (2) can be applied.

(4) We prove the lemma for r-semifilters and $k = \omega$; the case of p-semifilters and the case of $k < \omega$ is similar. Let $\{A_n : n \in \omega\}$ be a partition of ω into infinite sets, let $\mathcal{F}_n$ be an r-semifilter on A_n for every $n \in \omega$, and let $\mathcal{F} = \{A \subseteq \omega : (\forall n \in \omega)\ A \cap A_n \in \mathcal{F}_n\}$. Let $\{B_n : n \in \omega\}$ be an $\mathcal{F}$-partition of ω, i.e., $A_n \setminus \bigcup_{k \leq m} B_k \in \mathcal{F}_n$ for all $m, n \in \omega$. For every $n \in \omega$, $\{A_n \cap B_m : m \in \omega\}$ is an $\mathcal{F}_n$-partition. Hence, for every $n \in \omega$ there exists $C_n \subseteq A_n$ such that $C_n \in \mathcal{F}_n$ and $C_n \cap B_m \leq m$ for all $m \in \omega$. Let $C = \bigcup_{n \in \omega} \bigcup_{m > n} C_n \cap B_m$. Then $C \in \mathcal{F}$ because $C_n \setminus C$ is finite for all $n \in \omega$. For every $m \in \omega$, $|C \cap B_m| = |\bigcup_{n < m} C_n \cap B_m| \leq m^2$. □

Lemma 5.5 (4) does not hold for s-semifilters because for every filter $\mathcal{F}$ on ω the filter $\mathcal{F} \oplus \mathcal{F}$ is not an s-filter: Let $\{A_0, A_1\}$ be a partition of ω with the increasing enumerations $e_i : \omega \to A_i$, $i < 2$. The filter $\mathcal{F} \oplus \mathcal{F}$ is isomorphic to the filter $\mathcal{F}_2 = \{A \subseteq \omega : (\forall i < 2)\ e_i^{-1}(A \cap A_i) \in \mathcal{F}\}$ and the family $\{\{e_0(n), e_1(n)\} : n \in \omega\}$ of 2-element sets is an $\mathcal{F}_2$-partition of ω. If $A \subseteq \omega$ is a selector of this partition, then $A \notin \mathcal{F}_2$ because the sets $e_0^{-1}(A \cap A_0)$ and $e_1^{-1}(A \cap A_1)$ are disjoint and hence one of them does not belong to $\mathcal{F}$.

Proposition 5.6. *Let $\mathcal{F}$ be an x-semifilter, where $x \in \{p, r, s\}$.*

(1) *The generic subset of $(\mathcal{F}, \subseteq^*)$ is a filter.*
(2) *The following conditions are equivalent:*
 (a) *Every generic subset of $(\mathcal{F}, \subseteq^*)$ is an x-ultrafilter.*
 (b) *Every generic subset of $(\mathcal{F}, \subseteq^*)$ is an ultrafilter.*
 (c) *$\mathcal{F}$ is full.*

Proof. (1) As $\mathcal{F}$ is a p-semifilter, the forcing $(\mathcal{F}, \subseteq^*)$ is ω-closed; this forcing does not add new reals and the generic filter $\mathcal{G} \subseteq \mathcal{F}$ is a filter on ω (if $\mathcal{F}$ is a filter, the forcing with $\mathcal{F}$ is trivial and $\mathcal{G} = \mathcal{F}$).

(2) (a) $\to$ (b) is trivial and the implication (c) $\to$ (a) is known (see [4] and [10, Proposition 4.3]). By (b), for every $X \subseteq \omega$ the set $\{B \in \mathcal{F} : A \subseteq X$ or $B \subseteq \omega \setminus X\}$ is dense in $\mathcal{F}$. Then $\mathcal{F}$ is full because condition (2) in Proposition 5.2 holds. □

By Lemma 5.5 (1) the following condition is sufficient for a generic subset $\mathcal{G}$ of an x-semifilter $\mathcal{F}$ to be an x-filter: If $\{A_n : n \in \omega\}$ is a partition of ω and $B \Vdash_{\mathcal{F}}$ "$\{A_n : n \in \omega\}$ is a $\dot{\mathcal{G}}$-partition", then $\{A_n \cap B : n \in \omega\}$ is an $\mathcal{F} \cap \mathcal{P}(B)$-partition of B. This condition holds for countable sums of full semifilters.

We say that an upper family $\mathcal{F}$ is *hereditarily non-meager*, if $\mathcal{F} \cap \mathcal{P}(B)$ is not meager in $\mathcal{P}(\omega)$ for every $B \in \mathcal{F}$. We need this property in Proposition 5.9 through Lemma 5.8 and in Proposition 5.10. The adverb "hereditarily" has here another meaning than the same adverb in the notion of "hereditarily meager filters" in [8]. By Lemma 2.9 we get:

Lemma 5.7. *An upper family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is hereditarily non-meager if and only if for every finite-to-one function $f \in {}^{\omega}\omega$ and $B \in \mathcal{F}$ there exists $A \in [\omega]^{\omega}$ such that $B \setminus f^{-1}(A) \in \mathcal{F}$.* □

Lemma 5.8. *Every $s(\rho)$-semifilter $\mathcal{F}$ is hereditarily non-meager.*

Proof. We verify the condition in Lemma 5.7. Let a finite-to-one function $f \in {}^{\omega}\omega$ and $B \in \mathcal{F}$ be given. Let $\{a_k : k \in \omega\}$ be any partition of ω into finite sets such that $|a_k| = \rho(k) + 1$ for all $k \in \omega$. The system of sets $A_0 = (B \cap f^{-1}(a_0)) \cup (\omega \setminus B)$ and $A_k = B \cap f^{-1}(a_k)$, $k > 0$, is an $\mathcal{F}$-partition because all sets $f^{-1}(a_k)$ are finite. Since $\mathcal{F}$ is an $s(\rho)$-semifilter, there is a set $C \in \mathcal{F}$ such that $|C \cap A_k| \le \rho(k)$ for all $k \in \omega$. Let $B_0 = C \setminus A_0$. Then $B_0 \subseteq B$ and $B_0 \in \mathcal{F}$ because $|C \cap A_0| \le \rho(0)$. For every $k > 0$, $B_0 \cap A_k = B_0 \cap f^{-1}(a_k)$, $|a_k| > |B_0 \cap A_k|$, and hence there is $i_k \in a_k$ such that $B_0 \cap f^{-1}(\{i_k\}) = \emptyset$. Let $A = \{i_k : k > 0\}$. Then $B \setminus f^{-1}(A) \supseteq B_0 \in \mathcal{F}$. □

Lemma 5.8 does not hold for all p-semifilters (e.g., the Fréchet filter is meager).

An ultrafilter $\mathcal{F}$ on ω is said to be quasi-normal (see [2]), if for every family $\{A_n : n \in \omega\} \subseteq \mathcal{F}$ there is $A \in \mathcal{F}$ such that, if $n, m \in A$ and $n < m$, then $m \in A_n$. The following proposition is a generalization of [2, Theorem 9.6] for semifilters where conditions (2) and (3) replace quasi-normality:

An ultrafilter $\mathcal{F}$ on ω is said to be quasi-normal (see [2]), if for every family $\{A_n : n \in \omega\} \subseteq \mathcal{F}$ there is $A \in \mathcal{F}$ such that, if $n, m \in A$ and $n < m$, then $m \in A_n$. The following proposition generalizes the equivalences (b) ↔ (e) ↔ (f) ↔ (i) in [2, Theorem 9.6] for semifilters where conditions (2) and (3) modify quasi-normality:

Proposition 5.9. *Let $\mathcal{F}$ be a semifilter on ω and let $\rho \in {}^{\omega}\omega$ be non-decreasing and $\rho \ge 1$. The following statements are equivalent:*

1. *$\mathcal{F}$ is an $s(\rho)$-semifilter.*
2. *For every $\mathcal{F}$-centered system $\{A_n : n \in \omega\} \subseteq \mathcal{F}$ and $\{\varphi_n : n \in \omega\} \subseteq \mathrm{Mon}$ there exist $A \in \mathcal{F}$ and $\varphi \in \mathrm{Mon}$ such that $A \subseteq A_0$, $1 \le |A \cap \varphi^{-1}(\{k\})| \le \rho(k)$ for all $k \in \omega$, and for every $n, m \in A$ and $i \le n$, if $\varphi(n) < \varphi(m)$, then $m \in A_n$ and $\varphi_i(n) < \varphi_i(m)$.*
3. *For every $\mathcal{F}$-centered system $\{A_n : n \in \omega\} \subseteq \mathcal{F}$ there exists $A = \bigcup_{k\in\omega} a_k \in \mathcal{F}$ such that $A \subseteq A_0$, $1 \le |a_k| \le \rho(k)$, $\max a_k < \min a_{k+1}$, and $\bigcup_{l>k} a_l \subseteq \bigcap_{n\in a_k} A_n$ for all $k \in \omega$.*
4. *$\mathcal{F}$ is a ρ-Ramsey semifilter.*
5. *$\mathcal{F}$ is a weakly ρ-Ramsey semifilter.*

Proof. We prove (1) → (2) → (4) → (5) → (1) and (2) → (3) → (5).

(1) → (2) Let $\{A_n : n \in \omega\} \subseteq \mathcal{F}$ be $\mathcal{F}$-centered and $\{\varphi_n : n \in \omega\} \subseteq \mathrm{Mon}$. Without loss of generality we can assume that $\bigcap_{n\in\omega} A_n = \emptyset$ (otherwise consider $A'_n = A \setminus n$).

Define $f : \omega \to \omega$ by $f(m) = \min\{n \in \omega : m \notin A_n\}$. Then $\{f^{-1}(\{n\}) : n \in \omega\}$ is an $\mathcal{F}$-partition because $\{A_n : n \in \omega\}$ is $\mathcal{F}$-centered and $f^{-1}(\{n\}) \subseteq \omega \setminus A_n$ for all $n \in \omega$. Since $\mathcal{F}$ is a p-semifilter there is $B_0 \in \mathcal{F}$ such that $|f^{-1}(\{n\}) \cap B_0| < \omega$ for all $n \in \omega$ and $f^{-1}(\{0\}) \cap B_0 = \emptyset$; $B_0 \subseteq A_0$ because $f^{-1}(\{0\}) = \omega \setminus A_0$. For every $n \in \omega$ the set $x_n = \{m \in B_0 : f(m) \le n\}$ is finite by the choice of B_0 and $y_n = \{m \in \omega : (\exists i \le n)\ \varphi_i(m) \le \varphi_i(n)\}$ is finite because all φ_i are finite-to-one.

Define $g : \omega \to \omega$ by $g(n) = \max(\{n+1\} \cup x_n \cup y_n)$; $g(n) > n$ and g is monotone because $x_n \subseteq x_{n+1}$ and $y_n \subseteq y_{n+1}$ for all $n \in \omega$.

Define $d : \omega \to \omega$ by $d(0) = 0$ and $d(k+1) = g(d(k))$; d is strictly increasing.

Define $h : \omega \to \omega$ by $h(m) = \min\{k \in \omega : m \leq d(k)\}$. Then $\operatorname{rng}(h) = \omega$, $m \leq d(h(m))$, and $h^{-1}(\{k\}) = (d(k-1), d(k)]$, where we let $d(-1) = -1$.

By Lemma 5.8 there is $L \in [\omega]^\omega$ such that $B_0 \setminus h^{-1}(L) \in \mathcal{F}$. We can assume that $0 \in L$. Denote $B_1 = B_0 \setminus h^{-1}(L)$ and let $\{l_k\}_{k\in\omega}$ be the increasing enumeration of L. Since h is non-decreasing the sequence of $j_k = \min h^{-1}(\{l_k\})$, $k \in \omega$, is strictly increasing and $j_0 = 0$. We can find L so that $B_1 \cap [j_k, j_{k+1}) \neq \emptyset$ for all $k \in \omega$. Since $\mathcal{F}$ is an $s(\rho)$-semifilter and the system of sets $(B_1 \cap [j_0, j_1)) \cup (\omega \setminus B_1)$ and $B_1 \cap [j_k, j_{k+1})$, $k > 0$, is an $\mathcal{F}$-partition, there is $A \in \mathcal{F}$ such that $A \subseteq B_1$ and $1 \leq |A \cap [j_k, j_{k+1})| \leq \rho(k)$ for all $k \in \omega$. Then $A \cap h^{-1}(L) = \emptyset$ and $A \subseteq A_0$. Define $\varphi(n) = k$ for $n \in [j_k, j_{k+1})$. Then $\varphi \in \mathrm{Mon}$, $1 \leq |A \cap \varphi^{-1}(\{k\})| \leq \rho(k)$ for all $k \in \omega$, and if $n \in A$, then $h(n) \in [l_{\varphi(n)}, l_{\varphi(n)+1}) \setminus L = (l_{\varphi(n)}, l_{\varphi(n)+1})$.

Let $n, m \in A$ be such that $\varphi(n) < \varphi(m)$. Then $h(n) + 1 < h(m)$ because $h(n) < l_{\varphi(n)+1} < h(m)$. Then $d(h(n)+1) < m$ by definition of $h(m)$, and since $n \leq d(h(n))$, then $g(n) \leq g(d(h(n))) = d(h(n)+1) < m$ and hence $m \notin x_n \cup y_n$. Then $f(m) > n$ because $m \notin x_n$ and $\varphi_i(n) < \varphi_i(m)$ for all $i \leq n$ because $m \notin y_n$.

(2) $\to$ (4) First note that if $P \subseteq [\omega]^n$ is $\mathcal{F}$-big, then there is a filter $\mathcal{F}_0 \subseteq \mathcal{F}$ generated by countably many sets such that P is $\mathcal{F}_0$-big. Then by (2) there is $B \in \mathcal{F}$ such that $B \subseteq^* X$ for all $X \in \mathcal{F}_0$ and hence P is B-big.

By induction on $n \in \omega$ we prove that for every B-big set $P \subseteq [\omega]^n$ where $B \in \mathcal{F}$ there are $A \in \mathcal{F}$ and $\varphi \in \mathrm{Mon}$ such that $A \subseteq B$, $[A]^{n,\varphi} \subseteq P$, and $1 \leq |A \cap \varphi^{-1}(\{k\})| \leq \rho(k)$ for all $k \in \omega$. There is nothing to prove if $n = 0$. Assume that the assertion holds for an $n \in \omega$ and let $P \subseteq [\omega]^{n+1}$ be B-big. Let B_0 be the set of all $m \in \omega$ such that $P_m = \{a \in [\omega]^n : a \cup \{m\} \in P\}$ is B-big. Clearly, $B \subseteq^* B_0$. By the inductive assumption we can find $A_m \in \mathcal{F}$ and $\varphi_m \in \mathrm{Mon}$, satisfying $A_{m+1} \subseteq A_m \subseteq B \cap B_0$ for $m \in \omega$ and such that $[A_m]^{n,\varphi_m} \subseteq P_m$ for $m \in B_0$. By (2) there are $A \in \mathcal{F}$ and $\varphi \in \mathrm{Mon}$ such that $A \subseteq A_0$, $1 \leq |A \cap \varphi^{-1}(\{k\})| \leq \rho(k)$ for all $k \in \omega$, and

$$(*) \qquad (\forall m, k, l \in A)[\varphi(m) < \varphi(k) < \varphi(l) \to (k, l \in A_m \text{ and } \varphi_m(k) < \varphi_m(l))].$$

If $a \in [A]^{n+1,\varphi}$ and $m = \min(a)$, then $a \setminus \{m\} \in [A]^{n,\varphi}$, by $(*)$, $a \setminus \{m\} \in [A_m]^{n,\varphi_m} \subseteq P_m$, and hence $a \in P$. Therefore $A \subseteq B$ and $[A]^{n+1,\varphi} \subseteq P$.

(4) $\to$ (5) is trivial.

(5) $\to$ (1) Let $\{A_n : n \in \omega\}$ be an $\mathcal{F}$-partition of ω and let $\nu(n) = i$ for $n \in A_i$ and $i \in \omega$. The set $P = \{\{n, m\} \in [\omega]^2 : n < m \text{ and } m \in \omega \setminus \bigcup_{i \leq \nu(n)} A_i\}$ is $\mathcal{F}$-big and by (5) there are $A \in \mathcal{F}$ and $\varphi \in \mathrm{Mon}$ such that $1 \leq |A \cap \varphi^{-1}(\{k\})| \leq \rho(k)$ for all $k \in \omega$ and $[A]^{2,\varphi} \subseteq P$. Therefore

$$(**) \qquad (\forall n, m \in A)(\varphi(n) < \varphi(m) \to \nu(n) < \nu(m)).$$

Define $\mu(0) = 0$ and $\mu(k+1) = \max\{\nu(n) : n \in A \cap \varphi^{-1}(\{k\})\} + 1$. By $(**)$, μ is strictly increasing and $A \cap \bigcup\{A_i : \mu(k) \leq i < \mu(k+1)\} \subseteq A \cap \varphi^{-1}(\{k\})$ for all $k \in \omega$. If $\mu(k) \leq i < \mu(k+1)$, then $k \leq \mu(k) \leq i$ and $|A \cap A_i| \leq |A \cap \varphi^{-1}(\{k\})| \leq \rho(k) \leq \rho(i)$. Therefore $\mathcal{F}$ is an $s(\rho)$-semifilter.

(2) $\to$ (3) Find $A \in \mathcal{F}$ and $\varphi \in \mathrm{Mon}$ satisfying (2) for arbitrary φ_n and take $a_k = A \cap \varphi^{-1}(\{k\})$.

(3) $\to$ (5) Let $P \subseteq [\omega]^2$ be $\mathcal{F}$-big. Denote $P_n = \{m \in \omega : \{m, n\} \in P\}$ for $n \in \omega$. There is $B \in \mathcal{F}$ such that $P_n \in \mathcal{F}$ for all $n \in B$ and $\{P_n : n \in B\} \cup \{B\}$ is $\mathcal{F}$-centered. Let $A_n = P_n \cap B$ for $n \in B$ and $A_n = B$ for $n \in \omega \setminus B$. By (3) there is a set $A = \bigcup_{k\in\omega} a_k \in \mathcal{F}$ such that $A \subseteq B$, $1 \leq |a_k| \leq \rho(k)$ and $\max a_k < \min a_{k+1}$ for all $k \in \omega$, and $m \in A_n$ whenever $m \in a_l$ and $n \in a_k$ for some $l > k$. Let $\varphi \in \mathrm{Mon}$

be such that $n \in a_{\varphi(n)}$ for all $n \in A$. Then $1 \le |A \cap \varphi^{-1}(\{k\})| = |a_k| \le \rho(k)$ for all $k \in \omega$. We show that $[A]^{2,\varphi} \subseteq P$. Let $m, n \in A$ be arbitrary with $\varphi(m) > \varphi(n)$. Then $m \in A_n$ because $m \in a_{\varphi(m)}$ and $n \in a_{\varphi(n)}$. Since $n \in B$, $A_n \subseteq P_n$ and hence $m \in P_n$. Therefore $\{m, n\} \in P$. □

The next proposition is a similar characterization of non-meager p-filters and of hereditarily non-meager p-semifilters. By Lemma 2.9 and Lemma 5.7, due to Talagrand's characterization of non-meager filters, a filter is hereditarily non-meager if and only if it is non-meager.

Proposition 5.10. *Let $\mathcal{F}$ be a semifilter on ω. The following statements are equivalent:*

1. *$\mathcal{F}$ is a hereditarily non-meager p-semifilter.*
2. *For every $\mathcal{F}$-centered system $\{A_n : n \in \omega\} \subseteq \mathcal{F}$ and $\{\varphi_n : n \in \omega\} \subseteq \mathrm{Mon}$ there exist $A \in \mathcal{F}$ and $\varphi \in \mathrm{Mon}$ such that $A \subseteq A_0$ (if needed, $A \cap \varphi^{-1}(\{k\}) \neq \emptyset$ for all $k \in \omega$) and for every $n, m \in A$ and $i \le n$, if $\varphi(n) < \varphi(m)$, then $m \in A_n$ and $\varphi_i(n) < \varphi_i(m)$.*
3. *For every $\mathcal{F}$-centered system $\{A_n : n \in \omega\} \subseteq \mathcal{F}$ there exists $A = \bigcup_{k \in \omega} a_k \in \mathcal{F}$ such that $A \subseteq A_0$, $1 \le |a_k| < \omega$, $\max a_k < \min a_{k+1}$, and $\bigcup_{l > k} a_l \subseteq \bigcap_{n \in a_k} A_n$ for all $k \in \omega$.*
4. *$\mathcal{F}$ is ω-Ramsey.*
5. *$\mathcal{F}$ is weakly ω-Ramsey.*

Proof. The proof of implications (1) → (2) → (4) → (5) and (2) → (3) → (5) are same as in Proposition 5.9.

(5) → (1) Let $\{A_n : n \in \omega\}$ be an $\mathcal{F}$-partition of ω. Since the set $P = \{\{k, m\} \in [\omega]^2 : (\exists n \in \omega)\ k \in A_n \text{ and } m \notin A_n\}$ is $\mathcal{F}$-big there are $A \in \mathcal{F}$ and $\varphi \in \mathrm{Mon}$ such that $[A]^{2,\varphi} \subseteq P$. Then for every $n \in \omega$ there is $k \in \omega$ such that $A \cap A_n \subseteq \varphi^{-1}(\{k\})$ and hence $|A \cap A_n| < \omega$. Therefore $\mathcal{F}$ is a p-semifilter.

By Lemma 5.7 we prove that $\mathcal{F}$ is hereditarily non-meager. Let $f \in {}^\omega\omega$ be finite-to-one and $B \in \mathcal{F}$. Define $g(n) = \min\{m > n : (\exists k \in \omega)\ \emptyset \neq f^{-1}(\{k\}) \subseteq (n, m)\}$. The set $P = \{\{k, m\} \in [B]^2 : k > g(m)\}$ is $\mathcal{F}$-big and by (5) there are $C \in \mathcal{F}$ and $\varphi \in \mathrm{Mon}$ such that $[C]^{2,\varphi} \subseteq P$. The set $A = \{k \in \omega : C \cap f^{-1}(\{k\}) = \emptyset\}$ is infinite because $C \cap (n, g(n)) = \emptyset$ for all $n \in C_{\max}$ where $C_{\max} = \{n \in C : (\forall m \in C)\ m > n \to \varphi(m) > \varphi(n)\}$ is infinite. Now, $C \subseteq B$ because $[C]^{2,\varphi} \subseteq [B]^2$. Therefore $B \setminus f^{-1}(A) \supseteq C \in \mathcal{F}$. □

Corollary 5.11. *Let $\mathcal{F}$ be an $s(\rho)$-semifilter with a non-decreasing $\rho \in {}^\omega\omega$ and $\rho \ge 1$. For every $\mathcal{F}$-centered system $\{A_n : n \in \omega\} \subseteq \mathcal{F}$ and $\{\varphi_n : n \in \omega\} \subseteq \mathrm{Mon}$ there exist $A \in \mathcal{F}$ and $\varphi \in \mathrm{Mon}$ such that*

1. *$A \cap \bigcup_{k \ge n} \varphi^{-1}(\{k\}) \subseteq A_n$ for all $n \in \omega$,*
2. *$1 \le |A \cap \varphi^{-1}(\{k\})| \le \rho(k+1)$ for all $k \in \omega$, and*
3. *for every $n, m \in A$ and $i \le n$, if $\varphi(n) < \varphi(m)$, then $\varphi_i(n) < \varphi_i(m)$.*

Proof. Denote $A^*_n = \bigcap_{k \le n} A_k$. By Proposition 5.9 there are $B \in \mathcal{F}$ and $\psi \in \mathrm{Mon}$ such that for every $k \in \omega$, $1 \le |B \cap \psi^{-1}(\{k\})| \le \rho(k)$, and for every $n, m \in B$ and $i \le n$, if $\psi(n) < \psi(m)$, then $m \in A^*_n$ and $\varphi_i(n) < \varphi_i(m)$. Since ψ is monotone and surjective, for every $k \in \omega$, $k \le \max(B \cap \psi^{-1}(\{k\}))$. Define $A = B \setminus \psi^{-1}(\{0\})$ and $\varphi(n) = \max\{0, \psi(n) - 1\}$. For every $k \ge n$, $A \cap \varphi^{-1}(\{k\}) = B \cap \psi^{-1}(\{k+1\}) \subseteq A^*_{\max(B \cap \psi^{-1}(\{k\}))} \subseteq A^*_k \subseteq A_n$. Consequently (1) and (2) hold; (3) holds because $A \subseteq B$ and $\varphi(n) < \varphi(m)$ implies $\psi(n) < \psi(m)$. □

6. Selectivity and the Rosenthal property

A particular form of Rosenthal's Lemma states that the selective semifilter $[\omega]^\omega$ is a Rosenthal family. Sobota [14, Theorem 3.6] has proved that a selective ultrafilter is a Rosenthal family. Since the existence of selective ultrafiler is not provable in ZFC, this Sobota's result does not imply Rosenthal's Lemma although every ultrafilter is a subset of $[\omega]^\omega$. On the other hand, Sobota in [14, Proposition 3.14] introduced a full non-selective p-semifilter which is a Rosenthal family in ZFC. By Lemma 6.3 below this semifilter is in fact an r-semifilter. All these results are consequences of the following theorem because, by Proposition 2.8, every full semifilter has no local diagonal. Note that by Theorem 6.6, the property of not having a local diagonal is strictly weaker than fullness:

Theorem 6.1. *If $\mathcal{F} \subseteq [\omega]^\omega$ is an r-semifilter which has no local diagonal, then $\mathcal{F} \cap \mathcal{P}(A)$ is a Rosenthal family for every $A \in \mathcal{F}$.*

Proof. Let $\rho \in {}^\omega\omega$ be a fixed non-decreasing unbounded function such that $\rho \geq 1$. Then a semifilter is an r-semifilter if and only if it is an $s(\rho)$-semifilter.

For a Rosenthal matrix $\{c_{k,n}\}_{k,n\in\omega}$ and $\varepsilon > 0$ define

$$P(\varepsilon) = \{\{k,m\} \in [\omega]^2 : m < k \text{ and } c_{k,m} < \varepsilon\},$$
$$Q(\varepsilon) = \{\{k,m\} \in [\omega]^2 : k < m \text{ and } \textstyle\sum_{n\geq m} c_{k,n} < \varepsilon\}.$$

The set mapping $f_\varepsilon(k) = \{m < k : c_{k,m} \geq \varepsilon\}$ is uniformly finite because $|f_\varepsilon(k)| \leq 1/\varepsilon$ for all $k \in \omega$. Since $\mathcal{F}$ has no local diagonal, by Lemma 2.3, for every $A \in \mathcal{F}$ there is $B \in \mathcal{F} \cap \mathcal{P}(A)$ free with respect to f_ε, and consequently $[B]^2 \subseteq P(\varepsilon)$.

$Q(\varepsilon)$ is $\mathcal{F}$-big (in fact big with respect to any semifilter) because for every $k \in \omega$ the set $(Q(\varepsilon))_k = \{m \in \omega : \{k,m\} \in Q(\varepsilon)\} \supseteq \{m > k : \sum_{n\geq m} c_{k,n} < \varepsilon\}$ is co-finite. If $A \in \mathcal{F}$, then by Lemma 5.5 (1), $\mathcal{F} \cap \mathcal{P}(A)$ is an $s(\rho)$-semifilter, and by Proposition 5.9, $\mathcal{F}\cap\mathcal{P}(A)$ is weakly ρ-Ramsey. Therefore for every $A \in \mathcal{F}$ there exist $B \in \mathcal{F}\cap\mathcal{P}(A)$ and $\psi \in \mathrm{Mon}$ such that $[B]^{2,\psi} \subseteq Q(\varepsilon)$ and $1 \leq |B\cap\psi^{-1}(\{k\})| \leq \rho(k)$ for all $k \in \omega$.

Let $A \in \mathcal{F}$. By induction on $n \in \omega$ (using previous two observations) find $\psi \in \mathrm{Mon}$ and a decreasing sequence of sets $A_n \in \mathcal{F}$ such that $A_0 \subseteq A$, $[A_0]^{2,\psi} \subseteq Q(\varepsilon/2)$, and $[A_n]^2 \subseteq P(2^{-(n+2)}\varepsilon/\rho(n+1))$. Then by Corollary 5.11 there is $B \in \mathcal{F}$ and $\varphi \in \mathrm{Mon}$ such that

(1) $B \cap \varphi^{-1}(\{n\}) \subseteq A_n$ and $1 \leq |B \cap \varphi^{-1}(\{n\})| \leq \rho(n+1)$ for all $n \in \omega$, and
(2) for every $k, m \in B$, if $\varphi(k) < \varphi(m)$, then $\psi(k) < \psi(m)$.

Hence, $B \subseteq A_0$ and $[B]^{2,\varphi} \subseteq [A_0]^{2,\psi} \subseteq Q(\varepsilon/2)$. Let $k \in B$ be arbitrary. Denote $m_0 = \min\{m \in B : \varphi(m) > \varphi(k)\}$. Then $\{k, m_0\} \in [B]^{2,\varphi} \subseteq Q(\varepsilon/2)$ and so,

$$\sum_{m\in B,\varphi(m)>\varphi(k)} c_{k,m} \leq \sum_{m\geq m_0} c_{k,m} < \varepsilon/2.$$

By (1), if $m \in B \setminus \{k\}$ and $n = \varphi(m) \leq \varphi(k)$, then $\{k,m\} \in [A_n]^2$. Therefore

$$\begin{aligned}\sum_{m\in B\setminus\{k\},\varphi(m)\leq\varphi(k)} c_{k,m} &= \sum_{n\leq\varphi(k)} \sum_{m\in B\cap\varphi^{-1}(\{n\})\setminus\{k\}} c_{k,m} \\ &\leq \sum_{n\leq\varphi(k)} \sum_{m\in B\cap\varphi^{-1}(\{n\})\setminus\{k\}} 2^{-(n+2)}\varepsilon/\rho(n+1) \leq \sum_{n\leq\varphi(k)} 2^{-(n+2)}\varepsilon < \varepsilon/2.\end{aligned}$$

Now, $\sum_{m\in B\setminus\{k\}} c_{k,m} = \sum_{m\in B,\varphi(m)>\varphi(k)} c_{k,m} + \sum_{m\in B\setminus\{k\},\varphi(m)\leq\varphi(k)} c_{k,m} < \varepsilon$. □

By Proposition 2.8, $\mathcal{H}^{>0}$ has no local diagonal, if $\mathcal{H}$ is a filter. Therefore:

Corollary 6.2. *If $\mathcal{H}$ is an r^+-filter, then $\mathcal{H}^{>0}$ is a Rosenthal family.* □

The Fréchet filter is an s^+-filter (Lemma 5.4 (1) for $m = 1$). We show that the following filters $\mathcal{H}_{\boldsymbol{a}}$ are r^+-filters that are not s^+-filters. Let $\mathcal{P}_\infty$ be the set of all partitions $\boldsymbol{a} = \{a_n : n \in \omega\}$ of ω (into finite or infinite sets) such that $\limsup_{n\in\omega} |a_n| = \infty$. For $\boldsymbol{a} \in \mathcal{P}_\infty$ denote

$$\mathcal{H}_{\boldsymbol{a}} = \{A \subseteq \omega : \limsup_{n\in\omega} |a_n \setminus A| < \infty\},$$

Then $\mathcal{H}_{\boldsymbol{a}}$ is a filter on ω and

$$\mathcal{H}_{\boldsymbol{a}}^{=0} = \{A \subseteq \omega : \limsup_{n\in\omega} |a_n \cap A| < \infty\},$$
$$\mathcal{H}_{\boldsymbol{a}}^{>0} = \{A \subseteq \omega : \limsup_{n\in\omega} |a_n \cap A| = \infty\},$$
$$(\mathcal{H}_{\boldsymbol{a}}^{>0})^{=0} = \mathcal{P}(\omega) \setminus \mathcal{H}_{\boldsymbol{a}}.$$

By Lemma 3.15 of [14], $\mathcal{H}_{\boldsymbol{a}}^{>0}$ is a p-semifilter. We can say a bit more:

Lemma 6.3. *$\mathcal{H}_{\boldsymbol{a}}^{>0}$ is an r-semifilter, i.e., $\mathcal{H}_{\boldsymbol{a}}$ is an r^+-filter, and no filter $\mathcal{H} \supseteq \mathcal{H}_{\boldsymbol{a}}$ is an s^+-filter or an s-filter.*

Proof. Let $\{A_n : n \in \omega\}$ be an $\mathcal{H}_{\boldsymbol{a}}^{>0}$-partition of ω. By induction on $n \in \omega$ define $m_0 = 0$, $m_{n+1} > m_n$, and $b_n \subseteq a_{m_{n+1}} \setminus \bigcup_{k\le n} A_n$ so that $|b_n| = 2^n$. Then $B = \bigcup_{n\in\omega} b_n \in \mathcal{H}_{\boldsymbol{a}}^{>0}$ because $\limsup_{m\in\omega} |a_m \cap B| \ge \lim_{n\in\omega} |b_n| = \infty$ and $|A_n \cap B| \le |\bigcup_{k<n} b_k| = \sum_{k<n} 2^k = 2^n - 1$ for all $n \in \omega$. Hence $\mathcal{H}_{\boldsymbol{a}}^{>0}$ is an $s(\rho)$-semifilter for $\rho(n) = 2^n$.

If $\mathcal{H} \supseteq \mathcal{H}_{\boldsymbol{a}}$ is a filter, then $\mathcal{H}$ is not an s^+-filter and not an s-filter because all sets in $\boldsymbol{a}$ as well as all selectors of $\boldsymbol{a}$ belong to the ideal $\mathcal{H}_{\boldsymbol{a}}^{=0}$ and $\mathcal{H}_{\boldsymbol{a}}^{=0} \subseteq \mathcal{H}^{=0}$. □

Lemma 6.4. *The generic set $\mathcal{G}_{\boldsymbol{a}} \subseteq \mathcal{H}_{\boldsymbol{a}}^{>0}$ is a non-selective r-ultrafilter extending $\mathcal{H}_{\boldsymbol{a}}$.*

Proof. By Proposition 5.6 (2) the generic subset $\mathcal{G}_{\boldsymbol{a}}$ of $(\mathcal{H}_{\boldsymbol{a}}^{>0}, \subseteq^*)$ is an r-ultrafilter and $\mathcal{G}_{\boldsymbol{a}}$ is not selective by Lemma 6.3. □

By Proposition 2.4 infinite sums of semifilters are not Rosenthal families because they are decomposable. By Theorem 6.1 and Lemma 5.5 it follows that the infinite sums of r-semifilters are r-semifilters with local diagonals. We show that the finite sums of semifilters of the form $\sum_{i<s_0} [\omega]^\omega \oplus \sum_{i<s_1} \mathcal{H}_{\boldsymbol{a}_i}^{>0}$ with $s_0, s_1 \in \omega$ and with $\boldsymbol{a}_i \in \mathcal{P}_\infty$ for all $i < s_1$ are Rosenthal semifilters. They are r-semifilters due to Lemma 6.3, Lemma 5.4 (1), and Lemma 5.5 (4). By Theorem 6.1 it remains to prove that they have no local diagonals. For this we need the following lemma which is an adaptation of Lemma 3.13 from [14] for non-diagonal functions.

Lemma 6.5. *Let $\mathcal{X} = \{X_i : i < s\}$ be a finite partition of ω into infinite sets, for every $i < s$ let $\mathcal{H}_i$ be a filter on X_i, let $\mathcal{F} = \sum_{i<s} \mathcal{H}_i^{>0} = \{A \subseteq \omega : (\forall i < s)\ A \cap X_i \in \mathcal{H}_i^{>0}\}$, and let $f \in {}^\omega\omega$ be a non-diagonal function. Then*

$$(\forall m \ge 2)(\exists r_m > m)(\forall a \in [\omega]^{r_m})(\forall A \in \mathcal{F})(\exists b \in [a]^m)(\exists B \in \mathcal{F} \cap \mathcal{P}(A))$$
$$b \cap f^{-1}(b) = \emptyset \textit{ and } B \cap f^{-1}(b) = \emptyset \textit{ and } b \cap f^{-1}(B) = \emptyset.$$

Proof. Recall that for $p \ge 2$ the Ramsey number $R(p)$ is the minimal $r \in \omega$ such that the partition relation $r \to [p]^2_2$ holds, i.e, for every partition $\{c_0, c_1\}$ of $[r]^2$ there exists $a \in [r]^p$ such that $[a]^2 \subseteq c_0$ or $[a]^2 \subseteq c_1$. Obviously, $R(p) > p$ for $p \ge 3$.

Let $c_0 = \{\{k,n\} \in [\omega]^2 : n \neq f(k) \text{ and } k \neq f(n)\}$ and $c_1 = [\omega]^2 \setminus c_0$. If a set $b \subseteq \omega$ is homogeneous for this partition and $|b| \geq 4$, then $[b]^2 \subseteq c_0$ because $[b]^2 \cap c_0 \neq \emptyset$ (either (i) there are distinct $n, k \in b \setminus f[b]$, or (ii) there is exactly one $n \in b \setminus f[b]$ and then let $k \in b \cap f[b] \setminus \{f(n)\}$, or (iii) $f{\restriction}b$ is a permutation of b and then choose any $n \in b$ and $k \in b \setminus \{n, f(n), f^{-1}(n)\}$; in all cases $\{n,k\} \in [b]^2 \cap c_0$). Therefore

$$(*) \qquad (\forall p \geq 3)(\forall a \in [\omega]^{R(p)})(\exists b \in [a]^p)\ b \cap f^{-1}(b) = \emptyset.$$

If $C \subseteq \omega$ and $e \in [\omega]^2$, then $C = \bigcup_{n \in e} C \setminus f^{-1}(\{n\})$. Since every $\mathcal{H}_i$ is a filter,

$$(**) \qquad (\forall C \in \mathcal{H}_i^{>0})(\forall e \in [\omega]^2)(\exists n \in e)\ C \setminus f^{-1}(\{n\}) \in \mathcal{H}_i^{>0}.$$

Let $C \in \mathcal{H}_i^{>0}$. Either $(\forall b \in [\omega]^{<\omega})\ C \setminus f^{-1}(b) \in \mathcal{H}_i^{>0}$ or let $k \in \omega$ be minimal such that $C \setminus f^{-1}(k \cup \{k\}) \in \mathcal{H}_i^{=0}$. In the latter case denote $C_m = C \setminus f^{-1}(m \setminus \{k\})$. By induction we show that $C_m \in \mathcal{H}_i^{>0}$ for all $m \in \omega$. Clearly $C_m \in \mathcal{H}_i^{>0}$ for $m \leq k+1$. Assume $C_m \in \mathcal{H}_i^{>0}$ for some $m \geq k+1$ $(= k \cup \{k\})$. By $(**)$ there is $n \in \{m, k\}$ such that $C_m \setminus f^{-1}(\{n\}) = C \setminus f^{-1}(m \cup \{n\} \setminus \{k\}) \in \mathcal{H}_i^{>0}$. Since $n \neq k$ we obtain $C_{m+1} \in \mathcal{H}_i^{>0}$. It follows that for every $i < s$,

$$(***) \qquad (\forall C \in \mathcal{H}_i^{>0})(\exists k \in \omega)(\forall b \in [\omega]^{<\omega})\ C \setminus f^{-1}(b \setminus \{k\}) \in \mathcal{H}_i^{>0}.$$

Let $r_m = R(m+s)$ and let $a \in [\omega]^{r_m}$ and $A \in \mathcal{F}$ be given; $r_m > m+s > m$ and $A \cap X_i \in \mathcal{H}_i^{>0}$ for all $i < s$. By $(*)$ there exists $d \in [a]^{m+s}$ such that $d \cap f^{-1}(d) = \emptyset$. By $(***)$ for every $i < s$ there exists $k_i \in \omega$ such that $(A \cap X_i) \setminus f^{-1}(d \setminus \{k_i\}) \in \mathcal{H}_i^{>0}$. Let $b \in [d \setminus \{k_i : i < s\}]^m$, $B' = \bigcup_{i<s}(A \cap X_i) \setminus f^{-1}(b)$, and $B = B' \setminus f[b]$. Then $B \in \mathcal{F}$ because $B' \in \mathcal{F}$ and $f[b]$ is finite, $b \cap f^{-1}(b) = \emptyset$ because $b \subseteq d$, by definition, $B \cap f^{-1}(b) \subseteq B' \cap f^{-1}(b) = \emptyset$, and $b \cap f^{-1}(B) = \emptyset$ because $B \cap f[b] = \emptyset$. □

Theorem 6.6. *Let $\mathcal{F} = \sum_{i<s_0}[\omega]^\omega \oplus \sum_{i<s_1} \mathcal{H}_{\boldsymbol{a}_i}^{>0}$ with $s_0, s_1 \in \omega$ and $\boldsymbol{a}_i \in \mathcal{P}_\infty$ for all $i < s_1$.* (a) *The semifilter $\mathcal{F}$ has no local diagonal.* (b) *For every $A \in \mathcal{F}$, $\mathcal{F} \cap \mathcal{P}(A)$ is a Rosenthal semifilter on A.*

Proof. We sketch the proof of (a). Assume that $A \in \mathcal{F}$ and $\{m_i\}_{i\in\omega}$ is a sequence of natural numbers ≥ 2. Lemma 6.5 can be used for a recursive construction of $a_i, b_i \in [\omega]^{<\omega}$ and $B_i \in \mathcal{F}^{>0}$ for $i \in \omega$ starting with $B_0 = A$ and $b_0 = \emptyset$ so that for all $i \in \omega$ the following conditions are satisfied:

(i) $a_i \in [B_i \setminus \bigcup_{j \leq i} b_j]^{r_{m_i}}$, $b_{i+1} \in [a_i]^{m_i}$, and $B_{i+1} \in \mathcal{F} \cap \mathcal{P}(B_i \setminus \bigcup_{j \leq i+1} b_j)$,
(ii) $b_i \cap f^{-1}(b_i) = \emptyset$,
(iii) $B_i \cap f^{-1}(b_i) = \emptyset$,
(iv) $b_i \cap f^{-1}(B_i) = \emptyset$.

Let $B = \bigcup_{i\in\omega} b_i$. One can observe that for the semifilter $\mathcal{F}$ in the theorem one can make choices of m_i and a_i for $i \in \omega$ so that $B \in \mathcal{F}$. We prove that $B \cap f^{-1}(B) = \emptyset$. Let $k \in B$ be arbitrary, i.e., $k \in b_i$ for some $i > 0$. Then $k \notin f^{-1}(b_i)$ by (ii); $k \notin f^{-1}(b_j)$ for all $j < i$ because $b_i \subseteq B_j$ and $B_j \cap f^{-1}(b_j) = \emptyset$ by (iii); and $k \notin f^{-1}(b_j)$ for all $j > i$ because $b_j \subseteq B_i$ and $b_i \cap f^{-1}(B_i) = \emptyset$ by (iv).

(b) As we have already mentioned before Lemma 6.5, $\mathcal{F}$ is an r-semifilter. By (a), $\mathcal{F}$ has no local diagonal and therefore by Theorem 6.1, $\mathcal{F} \cap \mathcal{P}(A)$ is a Rosenthal semifilter for every $A \in \mathcal{F}$. □

By Corollary 4.6, every Rosenthal filter $\mathcal{F}$ is the intersection of a finite family of ultrafilters pairwise incomparable in the Rudin-Keisler ordering. Due to the correspondence between filters and closed subsets of $\beta\omega$ this finite family of ultrafilters equals to the set $\{p \in \beta\omega : \mathcal{F} \subseteq p\}$. By the next theorem it is consistent to have an uncountable collection $\mathcal{C}$ of ultrafilters such that the intersection of every nonempty finite subfamily of $\mathcal{C}$ is a Rosenthal filter.

Theorem 6.7. *It is consistent with ZFC that there is an uncountable collection $\mathcal{C}$ of r-ultrafilters on ω with the following properties:*

(i) *$\mathcal{C}$ contains uncountably many selective ultrafilters and uncountably many non-selective ultrafilters.*
(ii) *If $\mathcal{C}_0 \subseteq \mathcal{C}$ is countable nonempty, then $\bigcap \mathcal{C}_0$ is an r-filter*
(iii) *If $\mathcal{C}_0 \subseteq \mathcal{C}$ is countable nonempty and all ultrafilters in $\mathcal{C}_0$ are selective, then $\bigcap \mathcal{C}_0$ is an s-filter.*
(iv) *If $\emptyset \neq \mathcal{C}_0 \subseteq \mathcal{C}$, then $\bigcap \mathcal{C}_0$ is a Rosenthal filter if and only if $\mathcal{C}_0$ is finite.*

Proof. Fix a partition $\boldsymbol{a} \in \mathcal{P}_\infty$ of ω consisting of finite sets and let $\boldsymbol{a}[x] = \bigcup_{n \in x} a_n$ for $x \subseteq \omega$. Let κ be an uncountable cardinal and for every $\alpha \in \kappa$ let $\mathcal{F}_\alpha = \mathcal{H}^{>0}_{\boldsymbol{a}}$ if α is even and $\mathcal{F}_\alpha = [\omega]^\omega$ if α is odd (where 2ξ is even and $2\xi + 1$ is odd). For every $\alpha \in \kappa$ fix $\pi_\alpha : \mathcal{F}_\alpha \to [\omega]^\omega$ such that

$$(\forall A \in \mathcal{F}_\alpha)(\forall x \in [\pi_\alpha(A)]^\omega)\ A \cap \boldsymbol{a}[x] \in \mathcal{F}_\alpha.$$

For example, if $\mathcal{F}_\alpha = \mathcal{H}^{>0}_{\boldsymbol{a}}$, then for $A \in \mathcal{F}_\alpha$ let $\pi_\alpha(A) \in [\omega]^\omega$ be such that $\lim_{n \in \pi_\alpha(A)} |a_n \cap A| = \infty$; if $\mathcal{F}_\alpha = [\omega]^\omega$, then for $A \in \mathcal{F}_\alpha$ let $\pi_\alpha(A) = \{n \in \omega : a_n \cap A \neq \emptyset\}$ (this works because $\boldsymbol{a}$ contains only finite sets).

Let $\mathbb{P} = \prod_{\alpha<\kappa} \mathcal{F}_\alpha$ be ordered by $p \leq q$ iff $(\forall \alpha \in \kappa)\ p(\alpha) \subseteq^* q(\alpha)$. Let $\mathcal{G} \subseteq \mathbb{P}$ be a V-generic filter on $\mathbb{P}$. The forcing $\mathbb{P}$ is ω-closed because $(\mathcal{F}_\alpha, \subseteq^*)$ is ω-closed for every $\alpha \in \kappa$. Therefore κ remains uncountable in $V[\mathcal{G}]$ and $V[\mathcal{G}]$ has no new reals and no new countable subsets of κ. For every $\alpha \in \kappa$, $\mathcal{G}_\alpha = \{p(\alpha) : p \in \mathcal{G}\}$ is a generic filter on $\mathcal{F}_\alpha$, consequently, an r-ultrafilter on ω which is selective if and only if α is odd. Let $\mathcal{C} = \{\mathcal{G}_\alpha : \alpha \in \kappa\}$. Condition (i) is obviously satisfied.

If $S \subseteq \kappa$ is countable, then for every $q \in \mathbb{P}$ there is $p \leq q$ such that $p(\alpha)$ for $\alpha \in S$ are pairwise disjoint sets. To prove this let $q \in \mathbb{P}$ be given. Find a system $\{x_\alpha : \alpha \in S\} \subseteq [\omega]^\omega$ of pairwise disjoint sets such that $x_\alpha \subseteq \pi_\alpha(q(\alpha))$ and define $p(\alpha) = q(\alpha)$, if $\alpha \in \kappa \setminus S$, and $p(\alpha) = q(\alpha) \cap \boldsymbol{a}[x_\alpha]$, if $\alpha \in S$.

This density argument shows that $\{\mathcal{G}_\alpha : \alpha \in S\}$ is a separated system of ultrafilters for every countable set $S \subseteq \kappa$. Then $\bigcap_{\alpha \in S} \mathcal{G}_\alpha =_{\mathrm{RK}} \sum_{\alpha \in S} \mathcal{G}_\alpha$ by Lemma 4.1 (3). Now, condition (ii) is a consequence of Lemma 5.5 (4) and condition (iii) is proved in Lemma 5.4 (2).

We verify (iv). Let $\emptyset \neq \mathcal{C}_0 \subseteq \mathcal{C}$. If $\mathcal{C}_0$ is infinite, then $\bigcap \mathcal{C}_0$ is decomposable because $\mathcal{C}_0$ contains a countable infinite separated family of ultrafilters. Then by Proposition 2.4, $\bigcap \mathcal{C}_0$ is not a Rosenthal family. Therefore, if $\bigcap \mathcal{C}_0$ is a Rosenthal filter, then $\mathcal{C}_0$ is finite. To prove the converse let $\mathcal{C}_0 = \{\mathcal{G}_\alpha : \alpha \in S\}$ where $S \in [\kappa]^{<\omega} \setminus \{\emptyset\}$ and let $\mathcal{G}' = \bigcap_{\alpha \in S} \mathcal{G}_\alpha$. Since S is finite there is a partition $\{A_\alpha : \alpha \in S\}$ of ω such that $A_\alpha \in \mathcal{G}_\alpha$ for $\alpha \in S$. Then by Lemma 5.5 (1, 4), the filter $\mathcal{G}'$ is an r-filter because $\mathcal{G}' = \{A \subseteq \omega : (\forall \alpha \in S)\ A \cap A_\alpha \in \mathcal{G}_\alpha\} =_{\mathrm{RK}} \sum_{\alpha \in S} \mathcal{G}_\alpha \cap \mathcal{P}(A_\alpha)$. To prove that $\mathcal{G}'$ is a Rosenthal filter, by Theorem 6.1 it is enough to prove that $\mathcal{G}'$ has no diagonal (since then, by Lemma 2.3, $\mathcal{G}'$ has no local diagonal).

Denote $\mathbb{P} \restriction S = \{p \restriction S : p \in \mathbb{P}\}$, $\mathcal{G} \restriction S = \{p \restriction S : p \in \mathcal{G}\}$, and $\mathbb{Q} = \{q \in \mathbb{P} \restriction S : q \leq \langle A_\alpha : \alpha \in S\rangle\}$. The semifilter $\mathcal{F} = \{A \subseteq \omega : (\forall \alpha \in S)\ A \cap A_\alpha \in \mathcal{F}_\alpha\}$ (partially ordered by $\subseteq$) is forcing equivalent to $\mathbb{Q}$ because the injection $\varphi : A \mapsto \langle A \cap A_\alpha : \alpha \in S\rangle$ for $A \in \mathcal{F}$ is a complete dense embedding from $\mathcal{F}$ into $\mathbb{Q}$. The filter $\mathcal{G}'$ is mapped by φ onto $(\mathcal{G} \restriction S) \cap \mathbb{Q}$ which is a generic subset of $\mathbb{Q}$. Therefore $\mathcal{G}'$ is a generic subset of $\mathcal{F}$. Let $f \in {}^\omega\omega$ be a non-diagonal function. Since $\mathcal{F} =_{\mathrm{RK}} \sum_{\alpha \in S} \mathcal{F}_\alpha \cap \mathcal{P}(A_\alpha)$, by

Theorem 6.6, f is not a local diagonal of $\mathcal{F}$ and hence, for every $A \in \mathcal{F}$ there exists $B \in \mathcal{F} \cap \mathcal{P}(A)$ such that $B \cap f^{-1}(B) = \emptyset$. By genericity of $\mathcal{G}'$ there is $B \in \mathcal{G}'$ such that $B \cap f^{-1}(B) = \emptyset$. This proves that $\mathcal{G}'$ has no diagonal. □

Recently, Piotr Koszmider and Arturo Martínez-Celis proved in [7] that every ultrafilter is a Rosenthal family and that $\mathfrak{ros} = \mathfrak{r}$. By Proposition 3.3 and Proposition 3.4 the equality $\mathfrak{ros} = \mathfrak{r}$ has this consequence:

Corollary 6.8. $\overline{\mathfrak{diag}} \leq \underline{\mathfrak{diag}} = \mathfrak{diag} = \mathfrak{dcmp} = \mathfrak{ros} = \mathfrak{r}$. □

Based on the previous results, it is natural to ask the following question:

Question 6.9. Is there in ZFC an infinite (or uncountable) family $\mathcal{C}$ of ultrafilters such that $\bigcap \mathcal{C}_0$ is a Rosenthal filter for every nonempty finite family $\mathcal{C}_0 \subseteq \mathcal{C}$?

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