IDEAL GENERALIZATIONS OF EGOROFF'S THEOREM

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Abstract. We investigate the classes of ideals for which the Egoroff's theorem or the generalized Egoroff's theorem holds between ideal versions of pointwise and uniform convergences. The paper is motivated by considerations of Michał Korch in [3].

1. Introduction

Egoroff's theorem states that for every sequence of measurable functions $f_n : [0,1] \to [0,1]$ for $n \in \omega$ which is pointwise convergent on $[0,1]$ and for every $\varepsilon > 0$ there is such a measurable set $A \subseteq [0,1]$ with $\mu(A) > 1-\varepsilon$ that the sequence of functions converges uniformly on A (here μ denotes the Lebesgue measure and μ^* denotes the Lebesgue outer measure). The generalized Egoroff's theorem states that for every sequence of real functions (possibly non-measurable) which is pointwise convergent on $[0,1]$ and for every $\varepsilon > 0$ there is such a set $A \subseteq [0,1]$ with $\mu^*(A) > 1-\varepsilon$ that the sequence of functions converges uniformly on A.

The independence of the generalized Egoroff's theorem was obtained by Weiss [8] and then Pinciroli [7] showed its relevance to cardinal invariants $\mathfrak{b}$ and $\mathrm{non}(\mathcal{N})$. Michał Korch [3] considered variants of these theorems for ideal generalizations of the pointwise convergence and of the uniform convergence. Analyzing the Pinciroli's approach to the generalized Egoroff's theorem in [7], Korch isolated two properties for a pair of convergences, denoted below by (H) and $(\bar{H})$, which ensure that all Pinciroli's arguments work for this pair of convergences. In fact, (H) implies the consistency of the generalized Egoroff's theorem and $(\bar{H})$ implies the consistency of the negation the generalized Egoroff's theorem for a pair of convergences, see Theorem 1.3. Following are the notation and terminology that we will use.

In the sets of functions of the form ${}^S\omega$ and ${}^T({}^S\omega)$ we consider the partial orderings, denoted in all cases by the same symbol as for the ordering of natural numbers $\leq$, given by $x \leq y$, if $x(s) \leq y(s)$ for all $s \in S$ where $x, y \in {}^S\omega$; and $\varphi \leq \psi$, if $\varphi(t) \leq \varphi(t)$ for all $t \in T$ where $\varphi, \psi \in {}^T({}^S\omega)$. We also consider the eventual quasi-ordering $\leq^*$ on ${}^\omega\omega$ defined by $x \leq^* y$, if $x(n) \leq y(n)$ for all but finitely many $n \in \omega$. The quasi-ordering $\leq^*$ on ${}^{[0,1]}({}^\omega\omega)$ is defined by $\varphi \leq^* \psi$, if $\varphi(t) \leq^* \psi(t)$ for all $t \in [0,1]$ where $\varphi, \psi \in {}^{[0,1]}({}^\omega\omega)$.

We say that a function $o : X \to P$ from a set X into a partially ordered set P is cofinal if for every $p \in P$ there exists $x \in X$ such that $p \leq o(x)$.

By a sequence of real functions $f_n : [0,1] \to [0,1]$ for $n \in \omega$ we understand a mapping $f : \omega \to {}^{[0,1]}[0,1]$ where we let $f(n) = f_n$ for $n \in \omega$. It will be useful to

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consider sequences $f : S \to {}^{[0,1]}[0,1]$ for arbitrary infinite countable sets S. We say that f is measurable, if $f_s : [0,1] \to [0,1]$ is measurable for all $s \in S$.

A mapping $o : {}^{S}({}^{[0,1]}[0,1]) \to {}^{[0,1]}X$, where X is a topological space, is said to be a measurability preserving mapping, if $o(f) : [0,1] \to X$ is measurable for every measurable f.

Assume that there are given two notions of convergence of sequences of functions $f \rightsquigarrow 0$ and $f \looparrowright 0$ on sets $A \subseteq [0,1]$. We assume that the convergence on a set implies the convergence on a subset. Usually we assume that $f \looparrowright 0$ on A implies $f \rightsquigarrow 0$ on A, i.e., $\looparrowright$ is stronger than $\rightsquigarrow$ (however, this is not necessary because we can replace $\looparrowright$ by conjuncion of $\looparrowright$ and $\rightsquigarrow$). We are not especially interested in the results of the convergences and term 0 in the formulas $f \rightsquigarrow 0$ and $f \looparrowright 0$ can represent anything or nothing to which the sequence f converges. Denote

$$\mathcal{F}_{\rightsquigarrow} = \{f \in {}^{\omega}({}^{[0,1]}[0,1]) : f \rightsquigarrow 0\}$$

and consider the following hypotheses between $\rightsquigarrow$ and $\looparrowright$:

- (M) There exists a measurability preserving mapping $o : {}^{S}({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^{T}\omega)$ with $|S| = |T| = \omega$ such that for every measurable $f \rightsquigarrow 0$ and measurable $A \subseteq [0,1]$, if $o(f)[A]$ is bounded in $({}^{T}\omega, \leq)$, then $f \looparrowright 0$ on A.
- (H) There exists $o : {}^{S}({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^{T}\omega)$ with $|S| = |T| = \omega$ such that for every $f \rightsquigarrow 0$ and $A \subseteq [0,1]$, if $o(f)[A]$ is bounded in $({}^{T}\omega, \leq)$, then $f \looparrowright 0$ on A.
- $(\bar{H})$ There exists cofinal $o : \mathcal{F}_{\rightsquigarrow} \to ({}^{[0,1]}({}^{\omega}\omega), \leq^*)$ such that for every $f \in \mathcal{F}_{\rightsquigarrow}$ and $A \subseteq [0,1]$, if $f \looparrowright 0$ on A, then $o(f)[A]$ is bounded in $({}^{\omega}\omega, \leq^*)$.

Obviously we can put $S = T = \omega$ in the conditions but sometimes definitions of functions o are easier to read if they do not contain enumerations of countable sets. Like in condition $(\bar{H})$, to verify (M) and (H) it is sufficient to define the restriction $o : \mathcal{F}_{\rightsquigarrow} \to {}^{[0,1]}({}^{\omega}\omega)$ because $o(f)$ can be arbitrary for $f \in {}^{\omega}({}^{[0,1]}[0,1)) \setminus \mathcal{F}_{\rightsquigarrow}$.

In Korch's notation (H) means $(H^{\Rightarrow}(\mathcal{F}, \looparrowright))$ for $\mathcal{F} = \mathcal{F}_{\rightsquigarrow}$ (where $\mathcal{F}$ identifies $\rightsquigarrow$). On the other hand, $(\bar{H})$ is a weakening of Korch's condition $(H^{\Leftarrow}(\mathcal{F}, \looparrowright))$ for $\mathcal{F} = \mathcal{F}_{\rightsquigarrow}$ using the structures $({}^{\omega}\omega, \leq^*)$ and $({}^{[0,1]}({}^{\omega}\omega), \leq^*)$ instead of $({}^{\omega}\omega, \leq)$ and $({}^{[0,1]}({}^{\omega}\omega), \leq)$.

Lemma 1.1 (Korch)**.** *Conditions (M), (H), $(\bar{H})$ hold between the pointwise convergence $\to$ and the uniform convergence $\rightrightarrows$.*

Proof. Take the measure preserving mapping $o : {}^{\omega}({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^{\omega}\omega)$ defined so that $o(f)(x)(n) = \min\{m \in \omega : (\forall k \geq m)\ f_k(x) \leq 2^{-n}\}$, if $f \to 0$, $x \in [0,1]$ and $n \in \omega$. □

Recall that $\mathcal{N}$ is the σ-ideal of sets of reals of measure zero, $\mathrm{non}(\mathcal{N})$ is the least cardinality of a set of reals not of measure zero, and $\mathfrak{b}$ is the least cardinality of a $\leq^*$-unbounded subset of ${}^{\omega}\omega$.

The main motivation for [3] and also for the present paper relies on Korch's extraction of Theorem 1.2 and Theorem 1.3 from Pinciroli's arguments applied with the pair of the pointwise and the uniform convergences.

Theorem 1.2 (Korch)**.** *Let $\Phi : [0,1] \to {}^{\omega}\omega$.*

(1) *If Φ is measurable, then for every $\varepsilon > 0$ there exists $A \subseteq [0,1]$ such that $\mu(A) \geq 1 - \varepsilon$ and $\Phi[A]$ is bounded in $({}^{\omega}\omega, \leq)$.*

(2) *If* $\mathrm{non}(\mathcal{N}) < \mathfrak{b}$*, then for every* $\varepsilon > 0$ *there exists* $A \subseteq [0,1]$ *such that* $\mu^*(A) \geq 1-\varepsilon$ *and* $\Phi[A]$ *is bounded in* $({}^\omega\omega, \leq)$*.*

Proof. (1) Since Φ is measurable we have $[0,1] = \bigcap_{n\in\omega}\bigcup_{k\in\omega} A_{n,k}$ with measurable sets $A_{n,k} = \{x \in [0,1] : \Phi(x)(n) = k\}$. For every $n \in \omega$ let $\varphi(n) \in \omega$ be minimal such that $\mu(\bigcup_{k<\varphi(n)} A_{n,k}) \geq 1 - \varepsilon 2^{-(n+1)}$ and let $A = \bigcap_{n\in\omega}\bigcup_{k<\varphi(n)} A_{n,k}$. Then $\Phi[A]$ is bounded by φ and $\mu(A) \geq 1 - \varepsilon\sum_{n\in\omega} 2^{-(n+1)} = 1-\varepsilon$.

(2) Since $\mathrm{non}(\mathcal{N}) < \mathfrak{b}$ there is a set $Y \subseteq [0,1]$ such that $|Y| < \mathfrak{b}$ and $\mu^*(Y) = 1$. Since $|\Phi[Y]| < \mathfrak{b}$ there is a sequence of compact sets $B_n \subseteq {}^\omega\omega$ for $n \in \omega$ such that $\Phi[Y] \subseteq \bigcup_{n\in\omega} B_n$. Denote $A_n = \Phi^{-1}(\bigcup_{i<n} B_i)$ and find $n \in \omega$ such that $\mu^*(A_n) \geq 1-\varepsilon$. Let $A = A_n$. Then $\Phi[A]$ is bounded because it is included in a compact set and $\mu^*(A) \geq 1-\varepsilon$. □

Theorem 1.3 (Korch)**.** *Let* $\rightsquigarrow$ *and* $\looparrowright$ *be arbitrary convergences.*

(1) *Assume that* (M) *holds between* $\rightsquigarrow$ *and* $\looparrowright$*. Then for every measurable* $f \rightsquigarrow 0$ *on* $[0,1]$ *and* $\varepsilon > 0$ *there exists a measurable set* $A \subseteq [0,1]$ *such that* $\mu(A) \geq 1-\varepsilon$ *and* $f \looparrowright 0$ *on* A*.*
(2) *Assume that* (H) *holds between* $\rightsquigarrow$ *and* $\looparrowright$*. If* $\mathrm{non}(\mathcal{N}) < \mathfrak{b}$*, then for every* $f \rightsquigarrow 0$ *on* $[0,1]$ *and* $\varepsilon > 0$ *there exists* $A \subseteq [0,1]$ *such that* $\mu^*(A) \geq 1-\varepsilon$ *and* $f \looparrowright 0$ *on* A*.*
(3) *Assume that* $(\bar{H})$ *holds between* $\rightsquigarrow$ *and* $\looparrowright$, $\mathrm{non}(\mathcal{N}) = \mathfrak{c}$ *and there exists a* $\mathfrak{c}$*-Lusin set. Then there exists* $f \rightsquigarrow 0$ *on* $[0,1]$ *such that for all sets* $A \subseteq [0,1]$ *with* $\mu^*(A) > 0$*,* $f \not\looparrowright 0$ *on* A*.*

Proof. (1)–(2) Take $\Phi = o(f)$ in Theorem 1.2.

(3) Let $Z \subseteq {}^\omega\omega$ be a $\mathfrak{c}$-Lusin set of cardinality $\mathfrak{c}$, i.e., $|B \cap Z| < \mathfrak{c}$ for every meager set $B \subseteq {}^\omega\omega$. Choose a bijection $\varphi : [0,1] \to Z$. Applying $(\bar{H})$ let $f \in \mathcal{F}_\rightsquigarrow$ be such that $\varphi \leq^* o(f)$. Let $A \subseteq [0,1]$ be arbitrary such that $f \looparrowright 0$ on A. By $(\bar{H})$, $o(f)[A]$ is $\leq^*$-bounded and then also $\varphi[A]$ is $\leq^*$-bounded because $\varphi \leq^* o(f)$. Therefore $\varphi[A]$ is meager and $|A| = |\varphi[A]| < \mathfrak{c}$ because $\varphi[A] \subseteq Z$. Then $\mu(A) = 0$ because $\mathrm{non}(\mathcal{N}) = \mathfrak{c}$. □

Note that the classical Egoroff's theorem and the generalized Egoroff's theorem are respectively the conclusions of Theorem 1.3 (1) and (2) between the pointwise and the uniform convergences. The ideal generalizations of these theorems were studied before Korch [3] by Das, Dutta, and Pal [1] and Mrożek [5, 6].

In the present paper we extend the Korch's results for pairs of convergences of the form $(\to_I, \rightrightarrows_I)$ and $(\to_{I^*}, \rightrightarrows_{I^*})$ which are defined in Section 2. By Lemma 2.1, $(\bar{H})$ holds for many pairs of ideal convergences, and in particular, $(\bar{H})$ holds for all pairs of the form $(\to_I, \rightrightarrows_I)$ and $(\to_{I^*}, \rightrightarrows_{I^*})$. In many cases Korch's conditions $(H^\Rightarrow)$ and $(H^\Leftarrow)$ and therefore also (H) and $(\bar{H})$ can be witnessed by the same function o (see the proofs of the results of [3] which include, e.g., the proof of Lemma 1.1). By Lemma 2.1 it follows that this regularity is limited in some way.

In Section 3 we prove closure properties of the systems of ideals I such that (M) or (H) holds between $\to_I$ and $\rightrightarrows_I$ or between $\to_{I^*}$ and $\rightrightarrows_{I^*}$ (Theorems 3.2–3.5).

In Section 4 we give several examples of ideals I such that that the Egoroff's theorem and the generalized Egoroff's theorem (hence also (M) and (H)) do not hold for the above mentioned pairs of ideal convergences.

2. Pairs of ideal convergences satisfying ($\bar{H}$)

By an ideal I on an infinite set S we mean a family of subsets of S such $\emptyset \in I$, $S \notin I$, $[S]^{<\omega} \subseteq I$, and I is closed under subsets and finite unions. Hence $\mathrm{Fin} = [S]^{<\omega}$ is the smallest ideal on S. For $f \in {}^{S}({}^{[0,1]}[0,1])$ we denote $f(s)(x)$ by $f_s(x)$; hence if $S = \omega$, $f = \langle f_m : m \in \omega \rangle$ is a sequence of real functions. For a sequence $\varepsilon \in {}^{S}[0,1]$ we write $\varepsilon \to_I 0$, if for every $\delta > 0$, $\{s \in S : \varepsilon_s \geq \delta\} \in I$. We recall definitions of *pointwise I-convergence*, *quasinormal I-convergence*, and *uniform I-convergence* of a sequence of real functions on a set $A \subseteq [0,1]$:

(i) $f \to_I 0$ on A if $(\forall x \in A)(\forall \varepsilon > 0)\ \{s \in S : f_s(x) \geq \varepsilon\} \in I$;
(ii) $f \xrightarrow{\mathrm{QN}}_I 0$ on A if $(\exists \varepsilon \in {}^{S}[0,1])\ [\varepsilon \to_I 0$ and $(\forall x \in A)\ \{s \in S : f_s(x) \geq \varepsilon_s\} \in I]$;
(iii) $f \rightrightarrows_I 0$ on A if $(\forall \varepsilon > 0)\ \{s \in S : \sup_{x \in A} f_s(x) \geq \varepsilon\} \in I$.

We recall another three kinds of convergences called *I^*-pointwise convergence*, *I^*-quasinormal convergence*, and *I^*-uniform convergence*, respectively:

(i) $f \to_{I^*} 0$ on A if $(\forall x \in A)(\exists M \in I)(\forall \varepsilon > 0)\ |\{s \in S : f_s(x) \geq \varepsilon\} \setminus M| < \omega$;
(ii) $f \xrightarrow{\mathrm{QN}}_{I^*} 0$ on A if $(\exists M \in I)(\exists \varepsilon \in {}^{S}[0,1])\ [\varepsilon \to_{\mathrm{Fin}} 0$ and $(\forall x \in A)\ |\{s \in S : f_s(x) \geq \varepsilon_s\} \setminus M| < \omega]$;
(iii) $f \rightrightarrows_{I^*} 0$ on A if $(\exists M \in I)(\forall \varepsilon > 0)\ |\{s \in S : \sup_{x \in A} f_s(x) \geq \varepsilon\} \setminus M| < \omega$.

The above convergences were all studied in [1] but the notation used here coincides with [3]. Let $K \subseteq I$ be two ideals on S. The above convergences are the limiting cases of the following two-ideal convergences (with $K = I$ and $K = \mathrm{Fin}$):

(i) $f \to_{K,I} 0$ on A if $(\forall x \in A)(\exists M \in I)(\forall \varepsilon > 0)\ \{s \in S : f_s(x) \geq \varepsilon\} \setminus M \in K$;
(ii) $f \xrightarrow{\mathrm{QN}}_{K,I} 0$ on A if $(\exists M \in I)(\exists \varepsilon \in {}^{S}[0,1])\ [\varepsilon \to_K 0$ and $(\forall x \in A)\ \{s \in S : f_s(x) \geq \varepsilon_s\} \setminus M \in K]$;
(iii) $f \rightrightarrows_{K,I} 0$ on A if $(\exists M \in I)(\forall \varepsilon > 0)\ \{s \in S : \sup_{x \in A} f_s(x) \geq \varepsilon\} \setminus M \in K$.

Recall that an ideal I on ω is a P-ideal, if for every sequence C_n for $n \in \omega$ of sets from I there is a set $C \in I$ such that $C_n \setminus C$ is finite for all $n \in \omega$. If I is a P-ideal, then $\to_{I^*} = \to_I$ and $\rightrightarrows_{I^*} = \rightrightarrows_I$. Moreover, if $K \subseteq I$ is a P-ideal (or contained in a P-ideal that is a subideal of I), then $\to_{I^*} = \to_{K,I}$ and $\rightrightarrows_{I^*} = \rightrightarrows_{K,I}$.

The implications between the convergences are summarized in the following diagram where "stronger" implies "weaker":

$$
\begin{array}{ccccccc}
\to_{\mathrm{Fin}} & \longrightarrow & \to_{I^*} & \longrightarrow & \to_{K,I} & \longrightarrow & \to_I \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\xrightarrow{\mathrm{QN}}_{\mathrm{Fin}} & \longrightarrow & \xrightarrow{\mathrm{QN}}_{I^*} & \longrightarrow & \xrightarrow{\mathrm{QN}}_{K,I} & \longrightarrow & \xrightarrow{\mathrm{QN}}_I \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\rightrightarrows_{\mathrm{Fin}} & \longrightarrow & \rightrightarrows_{I^*} & \longrightarrow & \rightrightarrows_{K,I} & \longrightarrow & \rightrightarrows_I
\end{array}
$$

By $\rightsquigarrow \cup \looparrowright$ we mean the convergence defined by $f \rightsquigarrow \cup \looparrowright 0$, if $f \rightsquigarrow 0$ or $f \looparrowright 0$. By next lemma many pairs of ideal convergences satisfy ($\bar{H}$):

Lemma 2.1. *Let $\rightsquigarrow$ and $\looparrowright$ be any convergences such that $\rightsquigarrow$ is weaker than $\to_{\mathrm{Fin}}$ and $\looparrowright$ is stronger than $\rightrightarrows_I \cup \xrightarrow{QN}_{I^*}$ for an ideal I on ω. Then ($\bar{H}$) holds between $\rightsquigarrow$ and $\looparrowright$. In particular, ($\bar{H}$) holds between $\to_I$ and $\rightrightarrows_I$ and ($\bar{H}$) holds between $\to_{I^*}$ and $\rightrightarrows_{I^*}$ for every ideal I on ω.*

Proof. For $f \in \mathcal{F}_{\rightsquigarrow}$, $x \in [0,1]$, and $n \in \omega$ let $C_{f,x,n} = \{m \in \omega : f_m(x) < 2^{-n}\}$ and define $o : \mathcal{F}_{\rightsquigarrow} \to {}^{[0,1]}({}^{\omega}\omega)$ by $o(f)(x)(n) = \min(C_{f,x,n})$, if $C_{f,x,n} \neq \emptyset$, and $o(f)(x)(n) = 0$, otherwise.

We prove that o is cofinal. Given $\varphi : [0,1] \to {}^{\omega}\omega$ define $f^{\varphi} \in \mathcal{F}_{\mathrm{Fin}} \subseteq \mathcal{F}_{\rightsquigarrow}$ by $f^{\varphi}_m(x) = \max\{2^{-n} : m \leq \varphi(x)(n) + n\}$. For all $m \leq \varphi(x)(n)$, $f^{\varphi}_m(x) \geq 2^{-n}$ and therefore $o(f^{\varphi})(x)(n) > \varphi(x)(n)$ because $C_{f^{\varphi},x,n} \neq \emptyset$.

Let $f \in \mathcal{F}_{\rightsquigarrow}$ and $f \looparrowright 0$ on $A \subseteq [0,1]$. Then $f \rightrightarrows_I 0$ on A or $f \xrightarrow{\mathrm{QN}}_{I^*} 0$ on A.

If $f \rightrightarrows_I 0$ on A, then the set $C_{f,n} = \{m \in \omega : \sup_{x \in A} f_m(x) < 2^{-n}\}$ is in the dual filter for every $n \in \omega$. Since $\emptyset \neq C_{f,n} \subseteq C_{f,x,n}$ for all $x \in A$, the function $\varphi(n) = \min(C_{f,n})$ is an upper bound of $o(f)[A]$ in $(\omega, \leq)$. Therefore $(\bar{H})$ holds between $\rightsquigarrow$ and $\looparrowright$.

Assume that $f \xrightarrow{\mathrm{QN}}_{I^*} 0$ on A. Then $A = \bigcup_{k \in \omega} A_k$ with $f \rightrightarrows_{I^*} 0$ on A_k for all $k \in \omega$ (this is an observation of Remark 3.2 in [1]). Since $\rightrightarrows_{I^*}$ is stronger than $\rightrightarrows_I$, by previous case there is an upper bound $\varphi_k \in {}^{\omega}\omega$ of $o(f)[A_k]$ in $({}^{\omega}\omega, \leq)$ for all $k \in \omega$. Then the function $\varphi \in {}^{\omega}\omega$ defined by $\varphi(n) = \max\{\varphi_k(n) : k \leq n\}$ is an upper bound of $o(f)[A]$ in $({}^{\omega}\omega, \leq^*)$. □

3. Pairs of ideal convergences satisfying (M) and (H)

Let I be an ideal on a set S (support of I), i.e., $S = \bigcup I$ and $[S]^{<\omega} \subseteq I$. The restriction of the ideal I onto a set $T \in \mathcal{P}(S) \setminus I$ is the ideal $I \restriction T = I \cap \mathcal{P}(T)$. We denote $\mathrm{Fin} = \mathrm{Fin}_S = [S]^{<\omega}$ and $\langle B \rangle = \langle B \rangle_S = \{E \subseteq S : |E \setminus B| < \omega\}$ for $B \subseteq S$; we omit the subscript S if the support S of the ideals is known from the context.

The join of a family of ideals $\mathcal{I}$, denoted by $\bigvee \mathcal{I}$, is the least ideal containing $\bigcup \mathcal{I}$. The intersection $\bigcap \mathcal{I}$ is an ideal, provided that the intersection of supports of ideals in $\mathcal{I}$ is not a member of $\bigcap \mathcal{I}$.

Let T be a finite or infinite set and for every $t \in T$ let I_t be an ideal on some set S_t. The set $S = \{(t,s) : t \in T \text{ and } s \in S_t\}$ is the disjoint sum of the family of sets $\{S_t : t \in T\}$. The ideal $\sum_{t \in T} I_t = \{A \subseteq S : (\forall t \in T)\ A_t \in I_t\}$ is the direct sum of the system of ideals $\{I_t : t \in T\}$, where $A_t = \{s \in S_t : (t,s) \in A\}$; $I_0 \oplus I_1 = \sum_{i \in \{0,1\}} I_i$. If T is infinite and J is an ideal on T, then the ideal $\sum^J_{t \in T} I_t = \{A \subseteq S : \{t \in T : A_t \notin I_t\} \in J\}$ is the sum of the system of ideals $\{I_t : t \in T\}$ with respect to the ideal J. The ideal $J \times I = \sum^J_{t \in T} I$ is the product of J and I. These definitions correspond with definitions of sums and products of filters in [2].

Countable sums of Borel ideals on ω are Borel and countable sums of analytic ideals on ω are analytic.

Let J be an ideal on T and for every $t \in T$ let I_t be an ideal on S_t. Define $\underline{\lim}^J_{t \in T} S_t = \{s : \{t \in T : s \notin S_t\} \in J\}$ and $\underline{\lim}^J_{t \in T} I_t = \{E : \{t \in T : E \notin I_t\} \in J\}$. Since $T \notin J$, $\underline{\lim}^J_{t \in T} S_t \subseteq \bigcup_{t \in T} S_t$ and $\underline{\lim}^J_{t \in T} I_t$ is a family of subsets of $\underline{\lim}^J_{t \in T} S_t$. Moreover, if $\underline{\lim}^J_{t \in T} S_t \notin \underline{\lim}^J_{t \in T} I_t$, then $\underline{\lim}^J_{t \in T} I_t$ is an ideal on $\underline{\lim}^J_{t \in T} S_t$. For $J = \mathrm{Fin}$ this definition gives $\underline{\lim}_{n \in \omega} I_n = \bigcup_{m \in \omega} \bigcap_{n > m} I_n$.

We consider the following partial orderings of ideals on ω:

(1) Rudin-Keisler partial ordering: $I \leq_{\mathrm{RK}} J$ if there is $g : \omega \to \omega$ such that $I = \{E \subseteq \omega : g^{-1}(E) \in J\}$.
(2) Rudin-Blass partial ordering: $I \leq_{\mathrm{RB}} J$ if there is a finite-to-one $g : \omega \to \omega$ such that $I = \{E \subseteq \omega : g^{-1}(E) \in J\}$.

Remark 3.1. The partial orderings $\leq_{\mathrm{RK}}$ and $\leq_{\mathrm{RB}}$ are used also for ideals with different supports. Let I, J and I_n for $n \in \omega$ be ideals on ω.

(1) $J \leq_{\mathrm{RK}} \sum^J_{n\in\omega} I_n$ is witnessed by the function $g : \omega \times \omega \to \omega$ defined by $g(m,n) = m$. In particular, $J \leq_{\mathrm{RK}} J \times I$.

(2) If $\bigcap_{n\in\omega} I_n \neq \mathrm{Fin}$, then $\bigcap_{n\in\omega} I_n \leq_{\mathrm{RB}} \sum_{n\in\omega} I_n$; if $\underline{\lim}^J_{n\in\omega} I_n \neq \mathrm{Fin}$, then $\underline{\lim}^J_{n\in\omega} I_n \leq_{\mathrm{RB}} \sum^J_{n\in\omega} I_n$. Hence, if $I \neq \mathrm{Fin}$, then $I \leq_{\mathrm{RB}} \sum_{n\in\omega} I$ and $I \leq_{\mathrm{RB}} J \times I$.

Proof of (2). The set $Z = \{(m,n) \in \omega \times \omega : m \leq n\}$ belongs to dual filters to both ideals $\sum_{n\in\omega} I_n \subseteq \sum^J_{n\in\omega} I_n$. Define a finite-to-one function $g : \omega \times \omega \to \omega$ so that $g(m,n) = n$ for $(m,n) \in Z$ and g maps $(\omega \times \omega) \setminus Z$ injectively onto an infinite set from the ideal $\bigcap_{n\in\omega} I_n$ or from the ideal $\underline{\lim}^J_{n\in\omega} I_n$, respectively. In this way we get the relations $\bigcap_{n\in\omega} I_n \leq_{\mathrm{RB}} \sum_{n\in\omega} I_n$ and $\underline{\lim}^J_{n\in\omega} I_n \leq_{\mathrm{RB}} \sum^J_{n\in\omega} I_n$. $\square$

Theorem 3.2. *Let $\mathcal{I}$ be the class of ideals I on countable sets such that (M) holds between $\to_I$ and $\rightrightarrows_I$.*

(1) $\mathrm{Fin} \in \mathcal{I}$ *and* $\langle B \rangle \in \mathcal{I}$ *for coinfinite* $B \subseteq \omega$.
(2) $\mathcal{I}$ *is closed under restrictions of ideals.*
(3) $\mathcal{I}$ *is downward* $\leq_{\mathrm{RK}}$*-closed.*
(4) $\mathcal{I}$ *is closed under direct sums* $\sum_{n\in\omega} I_n$, *where* $I_n \in \mathcal{I}$ *for all* $n \in \omega$.
(5) $\mathcal{I}$ *is closed under ideals that are intersections of countable subfamilies of* $\mathcal{I}$.
(6) $\mathcal{I}$ *is closed under increasing countable unions of analytic ideals from* $\mathcal{I}$.
(7) $\mathcal{I}$ *is closed under sums* $\sum^J_{n\in\omega} I_n$, *where* $J, I_n \in \mathcal{I}$ *and* I_n *are analytic for all* $n \in \omega$.
(8) $\mathcal{I}$ *is closed under ideals of the form* $\underline{\lim}^J_{n\in\omega} I_n$, *where* $J, I_n \in \mathcal{I}$ *and* I_n *are analytic for all* $n \in \omega$.

Theorem 3.3. *Let $\mathcal{I}$ be the class of ideals I on countable sets such that (H) holds between $\to_I$ and $\rightrightarrows_I$.*

(1) $\mathrm{Fin} \in \mathcal{I}$ *and* $\langle B \rangle \in \mathcal{I}$ *for coinfinite* $B \subseteq \omega$.
(2) $\mathcal{I}$ *is closed under restrictions of ideals.*
(3) $\mathcal{I}$ *is downward* $\leq_{\mathrm{RK}}$*-closed.*
(4) $\mathcal{I}$ *is closed under direct sums* $\sum_{n\in\omega} I_n$, *where* $I_n \in \mathcal{I}$ *for all* $n \in \omega$.
(5) $\mathcal{I}$ *is closed under ideals that are intersections of countable subfamilies of* $\mathcal{I}$.
(6) (a) *If* $I \in \mathcal{I}$ *and* $K \in \mathcal{I}$ *is a* P*-ideal, then* $I \vee K \in \mathcal{I}$.
    (b) $\mathcal{I}$ *is closed under increasing countable unions of ideals from* $\mathcal{I}$.
(7) $\mathcal{I}$ *is closed under sums* $\sum^J_{n\in\omega} I_n$ *where* $J, I_n \in \mathcal{I}$ *for all* $n \in \omega$.
(8) $\mathcal{I}$ *is closed under ideals of the form* $\underline{\lim}^J_{n\in\omega} I_n$, *where* $J, I_n \in \mathcal{I}$ *for all* $n \in \omega$.

Theorem 3.4. *Let $\mathcal{I}$ be the class of ideals I on countable sets such that (M) holds between $\to_{I^*}$ and $\rightrightarrows_{I^*}$.*

(1) $\mathrm{Fin} \in \mathcal{I}$ *and* $\langle B \rangle \in \mathcal{I}$ *for coinfinite* $B \subseteq \omega$.
(2) $\mathcal{I}$ *is closed under restrictions of ideals.*
(3) $\mathcal{I}$ *is downward* $\leq_{\mathrm{RB}}$*-closed.*
(4) $\mathcal{I}$ *is closed under direct sums* $\sum_{n\in\omega} I_n$, *where* $I_n \in \mathcal{I}$ *for all* $n \in \omega$.
(5) $\mathcal{I}$ *is closed under ideals that are intersections of countable subfamilies of* $\mathcal{I}$.
(6) $\mathcal{I}$ *is closed under increasing countable unions of analytic ideals from* $\mathcal{I}$.
(7) $\mathcal{I}$ *is closed under sums* $\sum^J_{n\in\omega} I_n$ *where* $J, I_n \in \mathcal{I}$ *and* I_n *are analytic for all* $n \in \omega$.

(8) *$\mathcal{I}$ is closed under ideals of the form $\underline{\lim}^J_{n\in\omega} I_n$, where $J, I_n \in \mathcal{I}$ and I_n are analytic for all $n \in \omega$.*

Theorem 3.5. *Let $\mathcal{I}$ be the class of ideals I on countable sets such that (H) holds between $\to_{I^*}$ and $\rightrightarrows_{I^*}$.*

(1) *$\mathrm{Fin} \in \mathcal{I}$ and $\langle B \rangle \in \mathcal{I}$ for coinfinite $B \subseteq \omega$.*
(2) *$\mathcal{I}$ is closed under restrictions of ideals.*
(3) *$\mathcal{I}$ is downward $\leq_{\mathrm{RB}}$-closed.*
(4) *$\mathcal{I}$ is closed under direct sums $\sum_{n\in\omega} I_n$, where $I_n \in \mathcal{I}$ for all $n \in \omega$.*
(5) *$\mathcal{I}$ is closed under ideals that are intersections of countable subfamilies of $\mathcal{I}$.*
(6) *$\mathcal{I}$ is closed under ideals that are joins of countable subfamilies of $\mathcal{I}$.*
(7) *$\mathcal{I}$ is closed under sums $\sum^J_{n\in\omega} I_n$ where $J, I_n \in \mathcal{I}$ for all $n \in \omega$.*
(8) *$\mathcal{I}$ is closed under ideals of the form $\underline{\lim}^J_{n\in\omega} I_n$, where $J, I_n \in \mathcal{I}$ for all $n \in \omega$.*

Proof of the theorems. The assertions of theorems follow by the following lemmas: (1) Lemma 3.6; (2) Lemma 3.7; (3) Lemma 3.9; (4) Lemma 3.10; (5) Lemma 3.11; (6) Lemma 3.12 and Lemma 3.13; (7) Lemma 3.15; (8) Lemma 3.16. □

To simplify speaking we use the following phrases:

"I satisfies (M')", if "(M) holds between $\to_I$ and $\rightrightarrows_I$",
"I satisfies (H')", if "(H) holds between $\to_I$ and $\rightrightarrows_I$",
"I satisfies (M'')", if "(M) holds between $\to_{I^*}$ and $\rightrightarrows_{I^*}$",
"I satisfies (H'')", if "(H) holds between $\to_{I^*}$ and $\rightrightarrows_{I^*}$".

The following lemma consists of special cases of paper [3] results.

Lemma 3.6. *The ideals* Fin *and $\langle B \rangle$ on ω satisfy (M'), (H'), (M''), (H'') for all coinfinite sets $B \subseteq \omega$.*

Proof. Since the ideal $I = \langle B \rangle$ is a P-ideal, it is enough to verify (M') and (H') for I because $\to_I = \to_{I^*}$ and $\rightrightarrows_I = \rightrightarrows_{I^*}$. Define $o : {}^{\omega}({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^{\omega}\omega)$ for $f \to_I 0$ by $o(f)(x)(n) = \min\{k \in \omega : \{m \geq k : f_m(x) \geq 2^{-n}\} \subseteq B\}$.

Obviously, o is measurability preserving. If $\varphi \in {}^{\omega}\omega$ is a bound of $o(f)[A]$ where $f \to_I 0$ and $A \subseteq [0,1]$, then $\{m \geq \varphi(n) : \sup_{x\in A} f_m(x) \geq 2^{-n}\} \subseteq B$ for every $n \in \omega$, i.e., $f \rightrightarrows_I 0$ on A. □

Lemma 3.7. *Let I be an ideal on ω and $S \in \mathcal{P}(\omega) \setminus I$. The restriction $I{\restriction}S$ has any of the properties (M'), (H'), (M''), (H''), whenever I has the same property.*

Proof. Let z denote the constant real function with value 0. For $f \in {}^{S}({}^{[0,1]}[0,1])$ define $\bar{f} \in {}^{\omega}({}^{[0,1]}[0,1])$ by $\bar{f}_m = f_m$ for $m \in S$ and $\bar{f}_m = z$ for $m \in \omega \setminus S$. For both ideal convergences, $\bar{f}$ converges with respect to I if and only if f converges with respect to $I{\restriction}S$. It is easy to verify that whenever $o_I : {}^{\omega}({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^{\omega}\omega)$ is a witness for a particular property concerning to I required by the lemma, then $o : {}^{S}({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^{\omega}\omega)$ defined by $o(f) = o_I(\bar{f})$ is a witness for the same property concerning to $I{\restriction}S$. □

Remark 3.8. Let $I \leq_{\mathrm{RK}} J$ be ideals on ω and let $I = \{E \subseteq \omega : g^{-1}(E) \in J\}$ for some $g : \omega \to \omega$. If $\mathrm{Fin} \subsetneq J$, then there is a surjection $h : \omega \to \omega$ such that $I = \{E \subseteq \omega : h^{-1}(E) \in J\}$, and moreover, h is finite-to-one, if g is finite-to-one, and h is injective, if g is injective. To see this we find $B \in J$ either finite or infinite such that $|B| = |\omega \setminus g(\omega \setminus B)|$. Then for any bijection $\pi : B \to \omega \setminus g(\omega \setminus B)$ the function $h = \pi \cup g{\restriction}(\omega \setminus B)$ has the required properties.

Choose an infinite set $B_0 \in J$ and let $A_0 = \{x \in g(B_0) : g^{-1}(\{x\}) \subseteq B_0\}$ and $A_1 = \{x \in g(B_0) : g^{-1}(\{x\}) \setminus B_0 \neq \emptyset\}$. Then $g(B_0) = A_0 \cup A_1$. If $|A_0| = \omega$, then let $B = g^{-1}(A_0)$. If $|A_1| = \omega$, then choose $B \subseteq B_0$ such that $|B| = |\omega \setminus \mathrm{rng}(g)|$ and $g(B) \subseteq A_1$. If $|A_0 \cup A_1| < \omega$, then choose an $x \in g(B_0)$ such that $|g^{-1}(\{x\}) \cap B_0| = \omega$ and choose $B \subsetneq g^{-1}(\{x\}) \cap B_0$ such that $|B| = |\omega \setminus \mathrm{rng}(g)|$.

Lemma 3.9. *Let I and J be ideals on ω.*

1. *If $I \leq_{\mathrm{RK}} J$, then*
   (a) *I satisfies (M'), whenever J satisfies (M');*
   (b) *I satisfies (H'), whenever J satisfies (H').*
2. *If $I \leq_{\mathrm{RB}} J$, then*
   (a) *I satisfies (M''), whenever J satisfies (M'');*
   (b) *I satisfies (H''), whenever J satisfies (H'').*

Proof. Let $I = \{E \subseteq \omega : g^{-1}(E) \in J\}$ for a function $g : \omega \to \omega$ that is finite-to-one in case (2). If $J = \mathrm{Fin}$, then the set $E = \omega \setminus \mathrm{rng}(g)$ is in I because $g^{-1}(E) = \emptyset$, and for every $E \in I$, $|E \cap \mathrm{rng}(g)| < \omega$ because $|g^{-1}(E)| < \omega$. Hence $I = \langle \omega \setminus \mathrm{rng}(g) \rangle$ and both assertions of the lemma follow by Lemma 3.6. Therefore we can assume that $J \neq \mathrm{Fin}$ and by Remark 3.8 we can moreover assume that g is a surjection. For $f \in {}^{\omega}({}^{[0,1]}[0,1])$ define $\bar{f} \in {}^{\omega}({}^{[0,1]}[0,1])$ by $\bar{f}_m = f_{g(m)}$ for $m \in \omega$.

If $f \to_I 0$, then $\bar{f} \to_J 0$ because for $x \in [0,1]$ and $\varepsilon > 0$, $\{m \in \omega : \bar{f}_m(x) \geq \varepsilon\} = g^{-1}(\{k \in \omega : f_k(x) \geq \varepsilon\}) \in J$.

If $f \to_{I^*} 0$ and g is finite-to-one, then $\bar{f} \to_{J^*} 0$. To see this assume that for every $x \in [0,1]$ there is $M_x \in I$ such that $\{k \in \omega : f_k(x) \geq \varepsilon\} \setminus M_x \in \mathrm{Fin}$ for all $\varepsilon > 0$. Then $\{m \in \omega : \bar{f}_m(x) \geq \varepsilon\} \setminus g^{-1}(M_x) = g^{-1}(\{k \in \omega : f_k(x) \geq \varepsilon\} \setminus M_x) \in \mathrm{Fin}$ for all $\varepsilon > 0$ because g is finite-to-one.

Assume that $o_J : {}^{\omega}({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^{\omega}\omega)$ witnesses any of the properties (M'), (H'), (M''), (H'') for J and define $o : {}^{\omega}({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^{\omega}\omega)$ by $o(f) = o_J(\bar{f})$; o is measurability preserving, whenever o_J is such.

(1) If $f \to_I 0$ and $A \subseteq [0,1]$ are such that $o(f)[A] = o_J(\bar{f})[A]$ is bounded in ${}^{\omega}\omega$, then (in cases of (M') and (H') for J) $\bar{f} \rightrightarrows_J 0$ on A and hence for $\varepsilon > 0$, $g^{-1}(\{k \in \omega : \sup_{x\in A} f_k(x) \geq \varepsilon\}) = \{m \in \omega : \sup_{x \in A} \bar{f}_m(x) \geq \varepsilon\} \in J$. Therefore $f \rightrightarrows_I 0$ on A.

(2) Assume that $f \to_{I^*} 0$ and $A \subseteq [0,1]$ are such that $o(f)[A] = o_J(\bar{f})[A]$ is bounded in ${}^{\omega}\omega$. Let $M_\varepsilon = \{k \in \omega : \sup_{x\in A} f_k(x) \geq \varepsilon\}$ and $N_\varepsilon = \{m \in \omega : \sup_{x\in A} \bar{f}_m(x) \geq \varepsilon\} = g^{-1}(M_\varepsilon)$ for $\varepsilon > 0$. Since (in cases of (M'') and (H'') for J) $\bar{f} \rightrightarrows_{J^*} 0$ on A, there is $N \in J$ such that $N_\varepsilon \setminus N \in \mathrm{Fin}$ for all $\varepsilon > 0$. Let $M = \{k \in \omega : g^{-1}(\{k\}) \subseteq N\}$. Then $M \in I$ and $g^{-1}(\{k\}) \cap (N_\varepsilon \setminus N) \neq \emptyset$ for all $k \in M_\varepsilon \setminus M$. Therefore $M_\varepsilon \setminus M \in \mathrm{Fin}$ for all $\varepsilon > 0$ and $f \rightrightarrows_{I^*} 0$ on A. □

Lemma 3.10. *The direct sum $\sum_{t\in T} I_t$ of a countable family of ideals on countable sets satisfies any of the properties (M'), (H'), (M''), (H''), whenever all ideals I_t for $t \in T$ satisfy the same property.*

Proof. Let $I = \sum_{t\in T} I_t$; I is an ideal on the set $S = \bigcup_{t\in T}\{t\} \times S_t$. For $f : S \to {}^{[0,1]}[0,1]$ define $\hat{f}^t : S_t \to {}^{[0,1]}[0,1]$ by $(\hat{f}^t)_s(x) = f_{t,s}(x)$. For every $t \in T$, if $f \to_I 0$, then $\hat{f}^t \to_{I_t} 0$; if $f \to_{I^*} 0$, then $\hat{f}^t \to_{I^*} 0$; if f is measurable, then $\hat{f}^t$ is measurable. For any of the required properties it is easy to verify that whenever $o_t : {}^{S_t}({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^{S_t}\omega)$ witnesses this property for I_t for all $t \in T$, then the function $o : {}^S({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^S\omega)$ defined by $o(f)(x)(t,s) = o_t(\hat{f}^t)(x)(s)$ witnesses the same property for I. $\square$

Lemma 3.11. *If an intersection $\bigcap_{n\in\omega} I_n$ of a sequence of ideals on countable sets is an ideal, then it has any of the properties (M'), (H'), (M''), (M''), whenever all ideals I_n have the same property.*

Proof. The ideal $I = \bigcap_{n\in\omega} I_n$ is an ideal on an infinite countable set S and since $I = \bigcap_{n\in\omega} I_n{\restriction}S$, by Remark 3.1, $I = \mathrm{Fin}$ or $I \leq_{\mathrm{RB}} \sum_{n\in\omega} I_n{\restriction}S$. Therefore the lemma is a consequence of Lemma 3.6, Lemma 3.9, and Lemma 3.10. $\square$

For example, the P-ideal of pseudo-intersections of a countable sequence of infinite subsets of ω satisfies (M'), (H'), (M''), (H'').

Let I and J be ideals on ω and $g : [0,1] \to I \vee J$ be measurable. In general we can say nothing about the existence of a measurable function $h : [0,1] \to J$ such that $g(x) \setminus h(x) \in I$ and $h(x) \subseteq g(x)$. This is one of the reasons why measurable variants are omitted in next lemma (they hold for analytic ideals due to $\mathbf{\Sigma}^1_2$-uniformization provided that $\mathbf{\Sigma}^1_2$ sets are Lebesgue measurable).

Lemma 3.12. *Let I and J be ideals on countable sets.*

1. *$I \vee J$ satisfies (H'), whenever I and J satisfy (H') and J is a P-ideal.*
2. *$I \vee J$ satisfies (H''), whenever I and J satisfy (H'').*

Proof. Let I, J, and $K = I \vee J$ be ideals on sets S, T, and $U = S \cup T$, respectively. For functions $f : U \to {}^{[0,1]}[0,1]$ and $\varepsilon > 0$ denote $U_{f,x,\varepsilon} = \{m \in U : f_m(x) \geq \varepsilon\}$.

(1) If $f \to_K 0$, then $U_{f,x,\varepsilon} \in K$. Since J is a P-ideal, for every $x \in [0,1]$ we can find $M_{f,x} \in J$ (hence $M_{f,x} \subseteq T$) such that $U_{f,x,\varepsilon} \setminus M_{f,x} \in I \vee [T \setminus S]^{<\omega}$ for all $\varepsilon > 0$. Let us define $f^I : S \to {}^{[0,1]}[0,1]$ and $f^J : T \to {}^{[0,1]}[0,1]$ by

$$f^I_m(x) = \begin{cases} f_m(x), & \text{if } m \in S \setminus M_{f,x}, \\ 0, & \text{if } m \in S \cap M_{f,x}, \end{cases}$$

$$f^J_m(x) = \begin{cases} f_m(x), & \text{if } m \in T \setminus (S \setminus M_{f,x}), \\ 0, & \text{if } m \in T \cap (S \setminus M_{f,x}). \end{cases}$$

Then $f^I \to_I 0$ and $f^J \to_J 0$ because for all $\varepsilon > 0$,

$$\{m \in S : f^I_m(x) \geq \varepsilon\} = U_{f,x,\varepsilon} \cap (S \setminus M_{f,x}) = (U_{f,x,\varepsilon} \setminus M_{f,x}) \cap S \in I,$$

$$\{m \in T : f^J_m(x) \geq \varepsilon\} \subseteq U_{f,x,\varepsilon} \setminus (S \setminus M_{f,x}) \subseteq (U_{f,x,\varepsilon} \setminus S) \cup M_{f,x} \in J.$$

Let $o_I : {}^S({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^\omega\omega)$ and $o_J : {}^T({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^\omega\omega)$ witness (H') for I and J. Define $o : {}^U({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^\omega\omega)$ for $f \to_K 0$ by

$$o(f)(x)(n) = \max\{o_I(f^I)(x)(n), o_J(f^J)(x)(n)\}.$$

Assume that $f \to_K 0$ and $A \subseteq [0,1]$ are such that $o(f)[A]$ is bounded in ${}^\omega\omega$. Then $o_I(f^I)[A]$ and $o_J(f^J)[A]$ are bounded and applying (H') we have $f^I \rightrightarrows_I 0$ on A and $f^J \rightrightarrows_J 0$ on A. Denote $E_\varepsilon = \{m \in S : \sup_{x\in A} f^I_m(x) \geq \varepsilon\}$ and $F_\varepsilon = \{m \in T : \sup_{x\in A} f^J_m(x) \geq \varepsilon\}$ for $\varepsilon > 0$; hence $E_\varepsilon \in I$ and $F_\varepsilon \in J$. Then for every $\varepsilon > 0$, $\{m \in U : \sup_{x\in A} f_m(x) \geq \varepsilon\} = E_\varepsilon \cup F_\varepsilon \in K$ because $\sup_{x\in A} f_m(x) = \max\{\sup_{x\in A} f^I_m(x), \sup_{x\in A} f^J_m(x)\}$, where every missing term $f^I_m(x)$ for $m \in T \setminus S$ and $f^J_m(x)$ for $m \in S \setminus T$ is replaced by 0. Hence $f \rightrightarrows_K 0$.

(2) If $f \to_{K^*} 0$, then for every $x \in [0,1]$ fix $N_{f,x} \in I$ and $M_{f,x} \in J$ such that $N_{f,x} \cap M_{f,x} = \emptyset$ and $U_{f,x,\varepsilon} \setminus (N_{f,x} \cup M_{f,x}) \in \mathrm{Fin}$ for all $\varepsilon > 0$. Let us define

$f^I : S \to {}^{[0,1]}[0,1]$ and $f^J : T \to {}^{[0,1]}[0,1]$ by the same formula as in case (1). Then $f^I \to_{I^*} 0$ and $f^J \to_{J^*} 0$ because for all $\varepsilon > 0$ the following sets are finite:

$$\{m \in S : f^I_m(x) \geq \varepsilon\} \setminus N_{f,x} \subseteq U_{f,x,\varepsilon} \setminus (M_{f,x} \cup N_{f,x}),$$

$$\{m \in T : f^J_m(x) \geq \varepsilon\} \setminus M_{f,x} \subseteq (U_{f,x,\varepsilon} \setminus (S \setminus M_{f,x})) \setminus M_{f,x} = U_{f,x,\varepsilon} \setminus (S \cup M_{f,x}).$$

Let $o_I : {}^S({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^\omega\omega)$ and $o_J : {}^T({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^\omega\omega)$ witness (H'') for I and J. Define $o : {}^U({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^\omega\omega)$ by

$$o(f)(x)(n) = \max\{o_I(f^I)(x)(n), o_J(f^J)(x)(n)\}.$$

Assume that $f \to_{K^*} 0$ and $A \subseteq [0,1]$ are such that $o(f)[A]$ is bounded in ${}^\omega\omega$. Then $o_I(f^I)[A]$ and $o_J(f^J)[A]$ are bounded and applying (H'') we get $f^I \rightrightarrows_{I^*} 0$ on A and $f^J \rightrightarrows_{J^*} 0$ on A. Hence, if we denote $E_\varepsilon = \{m \in S : \sup_{x\in A} f^I_m(x) \geq \varepsilon\}$ and $F_\varepsilon = \{m \in T : \sup_{x\in A} f^J_m(x) \geq \varepsilon\}$, then there are $N \in I$ and $M \in J$ such that $E_\varepsilon \setminus N$ and $F_\varepsilon \setminus M$ are finite for all $\varepsilon > 0$. Then $\{m \in U : \sup_{x\in A} f_m(x) \geq \varepsilon\} \setminus (N \cup M) = (E_\varepsilon \cup F_\varepsilon) \setminus (N \cup M)$ is finite for all $\varepsilon > 0$ because $\sup_{x\in A} f_m(x) = \max\{\sup_{x\in A} f^I_m(x), \sup_{x\in A} f^J_m(x)\}$, where every missing term $f^I_m(x)$ for $m \in T \setminus S$ and $f^J_m(x)$ for $m \in S \setminus T$ is replaced by 0. □

Korch [3] proved that countably generated ideals satisfy (H') and (H''). The following lemma together with Lemma 3.6 and Lemma 3.12 gives much more. A proof of assertion (1a) for analytic ideals and proof of assertions (1b), (2a), (2b) can be obtained also by Lemma 3.16 below.

We will use this simple fact: If $g : [0,1] \to X$ is a measurable function into a Polish space X, then there is a Borel set $B \subseteq [0,1]$ of full measure such that $g \restriction B$ is Borel.

Lemma 3.13. *Let the ideal $I = \bigcup_{k\in\omega} I_k$ be the union of an increasing sequence of ideals on countable sets.*

(1) (a) *I satisfies (M'), whenever the ideals I_k for $k \in \omega$ are analytic and satisfy (M').*
    (b) *I satisfies (H'), whenever the ideals I_k for $k \in \omega$ satisfy (H').*
(2) (a) *I satisfies (M''), whenever the ideals I_k for $k \in \omega$ are analytic and satisfy (M'').*
    (b) *I satisfies (H''), whenever the ideals I_k for $k \in \omega$ satisfy (H'').*

Proof. For every $k \in \omega$, I_k is an ideal on some countable set S_k and $S_k \subseteq S_{k+1}$. Let $S = \bigcup_{k\in\omega} S_k$ and let $I'_k = \{A \subseteq S : A \cap S_k \in I_k \text{ and } |A \cap (S \setminus S_k)| < \omega\}$. Then $I = \bigcup_{k\in\omega} I'_k$. If $|S \setminus S_k| = \omega$, then I'_k is an isomorphic copy of $I_k \oplus \mathrm{Fin}$; if $|S \setminus S_k| < \omega$, then I'_k is the restriction of a copy of $I_k \oplus \mathrm{Fin}$ onto S. If I_k has any of the properties (M'), (H'), (M''), (H''), then I'_k has the same property by Lemma 3.7, Lemma 3.9, and Lemma 3.10. Therefore without loss of generality we can assume that all ideals I and I_k are ideals on ω, excluding the trivial case we can assume that $I \neq I_k$ for all $k \in \omega$, and passing to a subsequence, if necessary, we can assume that $I_k \neq I_{k+1}$ for all $k \in \omega$.

(1) If $f \to_I 0$, denote $B_{f,x,n} = \{m \in \omega : f_m(x) > 2^{-n}\}$ for $x \in [0,1]$ and $n \in \omega$, and define $\psi(f) : [0,1] \to {}^\omega\omega$ and $\bar{f}^k : \omega \to {}^{[0,1]}[0,1]$ for $k \in \omega$ by

$$\psi(f)(x)(n) = \min\{k \in \omega : B_{f,x,n} \in I_k\},$$

$$(\bar{f}^k)_m(x) = \begin{cases} f_m(x), & \text{if } (\exists n \in \omega)\ 2^{-n} < f_m(x) \leq 2^{-(n-1)} \text{ and } \psi(f)(x)(n) \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\bar{f}^k \to_{I_k} 0$ because for every $x \in [0,1]$ and $n \in \omega$, $\{m \in \omega : (\bar{f}^k)_m(x) > 2^{-n}\} = \{m \in \omega : (\exists i \leq n)\ 2^{-i} < f_m(x) \leq 2^{-(i-1)}$ and $\psi(f)(x)(i) \leq k\} \subseteq \bigcup\{B_{f,x,i} : i \leq n$ and $B_{f,x,i} \in I_k\} \in I_k$.

Assume that for every $k \in \omega$, $o_k : {}^{\omega}({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^{\omega}\omega)$ witnesses (M') or (H') for I_k and define $o : {}^{\omega}({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^{\omega\times\omega}\omega)$ for $f \to_I 0$ by

$$o(f)(x)(k,n) = \max\{o_k(\bar{f}^k)(x)(n), \psi(f)(x)(n)\}.$$

If f is measurable and I_k are analytic ideals for all $k \in \omega$, then all $\bar{f}^k$ are measurable (there is a Borel set $B \subseteq [0,1]$ of full measure such that $f_m \restriction B$ are Borel as well as the restriction of the measurable mapping $x \mapsto \langle B_{f,x,n} : n \in \omega\rangle$ onto B, then $\psi(f)\restriction B$ and $(\bar{f}^k)_m \restriction B$ are analytic, and then $\bar{f}^k$ is measurable). Hence o is measurability preserving, whenever all o_k are such.

Let $f \to_I 0$ and $A \subseteq [0,1]$ be such that $o(f)[A]$ is bounded by a function $\varphi : \omega \times \omega \to \omega$. Then $o_k(\bar{f}^k)[A]$ is bounded by $\varphi_k \in {}^{\omega}\omega$ defined by $\varphi_k(n) = \varphi(k,n)$ and therefore $\bar{f}^k \rightrightarrows_{I_k} 0$ on A for all $k \in \omega$. We apply this fact for all k of the form $k = \varphi_0(n)$ for $n \in \omega$. For every $x \in A$ and $n \in \omega$, $\{m \in \omega : f_m(x) > 2^{-n}\} = \{m \in \omega : (\exists i \leq n)\ 2^{-i} < f_m(x) \leq 2^{-(i-1)}\} \subseteq \{m \in \omega : (\bar{f}^{\varphi_0(n)})_m(x) > 2^{-n}\}$ because $\psi(f)(x)(i) \leq \psi(f)(x)(n) \leq \varphi_0(n)$ for all $i \leq n$. Therefore $\{m \in \omega : \sup_{x\in A} f_m(x) > 2^{-n}\} \subseteq \{m \in \omega : \sup_{x\in A}(\bar{f}^{\varphi_0(n)})_m(x) > 2^{-n}\} \in I_{\varphi_0(n)} \subseteq I$ for all $n \in \omega$.

(2) If $f \to_{I^*} 0$, denote $B_{f,x,n} = \{m \in \omega : f_m(x) > 2^{-n}\}$ for $x \in [0,1]$ and $n \in \omega$, and define $\psi(f) : [0,1] \to {}^{\omega}\omega$ and $\bar{f}^k : \omega \to {}^{[0,1]}[0,1]$ for $k \in \omega$ by

$$\psi(f)(x) = \min\{k \in \omega : (\exists M \in I_k)(\forall n \in \omega)\ |B_{f,x,n} \setminus M| < \omega\},$$

$$(\bar{f}^k)_m(x) = \begin{cases} f_m(x), & \text{if } \psi(f)(x) \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

We show that if $f \to_{I^*} 0$, then $\bar{f}^k \to_{I_k^*} 0$. For every $x \in [0,1]$ choose $M_{f,x} \in I_{\psi(f)(x)}$ such that $|B_{f,x,n} \setminus M_{f,x}| < \omega$ for all $n \in \omega$. Either $\psi(f)(x) \leq k$ and then $M_{f,x} \in I_k$ and $\{m \in \omega : (\bar{f}^k)_m(x) > 2^{-n}\} \setminus M_{f,x} = B_{f,x,n} \setminus M_{f,x} \in \mathrm{Fin}$, or $\psi(f)(x) > k$ and $\{m \in \omega : (\bar{f}^k)_m(x) > 2^{-n}\} = \emptyset$.

Assume that for every $k \in \omega$, $o_k : {}^{\omega}({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^{\omega}\omega)$ witnesses (M'') or (H'') for I_k and define $o : {}^{\omega}({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^{\omega\times\omega}\omega)$ for $f \to_{I^*} 0$ by

$$o(f)(x)(k,n) = \max\{o_k(\bar{f}^k)(x)(n), \psi(f)(x)\}.$$

If f is measurable and I_k are analytic ideals for all $k \in \omega$, then all $\bar{f}^k$ are measurable (same as in the previous case). Hence o is measurability preserving, whenever all o_k are such.

Let $f \to_{I^*} 0$ and $A \subseteq [0,1]$ be such that $o(f)[A]$ is bounded by a function $\varphi : \omega \times \omega \to \omega$. Then $o_k(\bar{f}^k)[A]$ is bounded by $\varphi_k \in {}^{\omega}\omega$ defined by $\varphi_k(n) = \varphi(k,n)$ and therefore $\bar{f}^k \rightrightarrows_{I_k^*} 0$ on A. Let $k = \varphi(0,0)$ and let $M \in I_k$ be such that $|\{m \in \omega : \sup_{x\in A}(\bar{f}^k)_m(x) > 2^{-n}\} \setminus M| < \omega$ for all $n \in \omega$. For $x \in A$ and $n \in \omega$, $\psi(f)(x) \leq k$ and hence, $\{m \in \omega : f_m(x) > 2^{-n}\} = \{m \in \omega : (\bar{f}^k)_m(x) > 2^{-n}\}$. Therefore $\{m \in \omega : \sup_{x\in A} f_m(x) > 2^{-n}\} \setminus M \in \mathrm{Fin}$ for all $n \in \omega$. □

Lemma 3.14. *Let J be an ideal on ω, let I_t be ideals on S_t for $t \in \omega$, let $I = \sum_{t\in\omega}^{J} I_t$ be the ideal on $S = \bigcup_{t\in\omega}\{t\} \times S_t$, and let $a \in {}^{S}[0,1]$, $f \in {}^{S}({}^{[0,1]}[0,1])$, $A \subseteq [0,1]$.*

(1) *$a \to_{I^*} 0$ if and only if $\{t \in \omega : \langle a_{t,s}\rangle_{s\in S_t} \not\to_{I_t^*} 0\} \in J$.*

(2) *$f \to_{I^*} 0$ on A if and only if $(\forall x \in A)$ $\{t \in \omega : \langle f_{t,s}(x)\rangle_{s\in S_t} \not\to_{I_t^*} 0\} \in J$.*

(3) $f \rightrightarrows_{I^*} 0$ *on* A *if and only if* $\{t \in \omega : \langle f_{t,s}\rangle_{s\in S_t} \not\rightrightarrows_{I_t^*} 0 \text{ on } A\} \in J$.

Proof. (1) Denote $E = \{t \in \omega : \langle a_{t,s}\rangle_{s\in S_t} \not\to_{I_t^*} 0\}$.

Assume $a \to_{I^*} 0$. Let $M \in I$ be such that $|\{(t,s) \in S : a_{t,s} \geq \varepsilon\} \setminus M| < \omega$ for all $\varepsilon > 0$. Denote $M_t = \{s \in S_t : (t,s) \in M\}$ and $N = \{t \in \omega : M_t \notin I_t\}$. Then $N \in J$. If $t \in \omega \setminus N$, then $M_t \in I_t$ and for every $\varepsilon > 0$, $|\{s \in S_t : a_{t,s} \geq \varepsilon\} \setminus M_t| \leq |\{(u,s) \in S : a_{u,s} \geq \varepsilon\} \setminus M| < \omega$ and hence $t \notin E$. Therefore $E \subseteq N \in J$.

Assume $E \in J$. For each $t \in \omega \setminus E$ let $M_t \in I_t$ be such that $F_{t,n} = \{s \in S_t : a_{t,s} \geq 2^{-n}\} \setminus M_t$ is finite for all $n \in \omega$. The set $M = \{(t,s) : t \in \omega \setminus E$ and $s \in M_t \cup F_{t,t}\}$ belongs to I and $\{(t,s) \in S : a_{t,s} \geq 2^{-n}\} \setminus M \subseteq \{(t,s) \in S : t < n$ and $s \in F_{t,n}\}$ are finite sets for all $n \in \omega$. Therefore $a \to_{I^*} 0$.

(2) Use (1) with $a_{t,s} = f_{t,s}(x)$ for $x \in A$.

(3) Use (1) with $a_{t,s} = \sup_{x\in A} f_{t,s}(x)$; $a \to_{I^*} 0$ if and only if $f \rightrightarrows_{I^*} 0$ on A. □

Mrożek [6] and Korch [3] showed that for iterations Fin^α, $\alpha < \omega_1$ of the ideal Fin obtained by repeating the operation $\sum^{\mathrm{Fin}}$, the Egoroff's theorem and the generalized Egoroff's theorem hold between $\to_{\mathrm{Fin}^\alpha}$ and $\rightrightarrows_{\mathrm{Fin}^\alpha}$. The following lemma generalizes both these results.

Lemma 3.15. *Let* J *be an ideal on* T*, let* I_t *be ideals on* S_t *for* $t \in T$*, where* T *and all sets* S_t *are infinite countable, and let* $I = \sum_{t\in T}^J I_t$ *be the ideal on the set* $S = \bigcup_{t\in T}\{t\} \times S_t$.

(1) (a) I *satisfies* (M')*, whenever* J *and* I_t *for* $t \in T$ *satisfy* (M') *and the ideals* I_t *for* $t \in T$ *are analytic.*
(b) I *satisfies* (H')*, whenever* J *and* I_t *for* $t \in T$ *satisfy* (H').

(2) (a) I *satisfies* (M'')*, whenever* J *and* I_t *for* $t \in T$ *satisfy* (M'') *and the ideals* I_t *for* $t \in T$ *are analytic.*
(b) I *satisfies* (H'')*, whenever* J *and* I_t *for* $t \in T$ *satisfy* (H'').

Proof. (1) For $f : S \to {}^{[0,1]}[0,1]$ we define $\bar{f} : T \to {}^{[0,1]}[0,1]$ and $\hat{f}^t : S_t \to {}^{[0,1]}[0,1]$ for $t \in T$ in the following way: Let $t \in T$ and $x \in [0,1]$. If $\langle f_{t,s}(x)\rangle_{s\in S_t} \to_{I_t} 0$, then for all $s \in S_t$ let

$$\bar{f}_t(x) = 0 \quad \text{and} \quad (\hat{f}^t)_s(x) = f_{t,s}(x).$$

Otherwise, let $n(f,t,x)$ be the least $n \in \omega$ such that $\{s \in S_t : f_{t,s}(x) \geq 2^{-n}\} \notin I_t$ and for all $s \in S_t$ let

$$\bar{f}_t(x) = 2^{-n(f,t,x)},$$

$$(\hat{f}^t)_s(x) = \begin{cases} f_{t,s}(x), & \text{if } f_{t,s}(x) \geq 2^{-(n(f,t,x)-1)}, \\ 0, & \text{if } f_{t,s}(x) < 2^{-(n(f,t,x)-1)}. \end{cases}$$

If $f \to_I 0$, then $\bar{f} \to_J 0$ because for all $x \in [0,1]$ and $n \in \omega$, $\{t \in T : \bar{f}_t(x) \geq 2^{-n}\} = \{t \in T : n \geq n(f,t,x)\} = \{t \in T : \{s \in S_t : f_{t,s}(x) \geq 2^{-n}\} \notin I_t\} \in J$; and for every $t \in T$, $\hat{f}^t \to_{I_t} 0$ because either $\langle(\hat{f}^t)_s(x)\rangle_{s\in S_t} = \langle f_{t,s}(x)\rangle_{s\in S_t} \to_{I_t} 0$, or otherwise $\{s \in S_t : (\hat{f}^t)_s(x) > 0\} \in I_t$.

One can verify that if f is measurable, then $\bar{f}$ and $\hat{f}^t$ for $t \in T$ are measurable: If f is Borel, the functions $\bar{f}$ and $\hat{f}^t$ are obtained by changing f in a simple way on sets $X_t = \{x \in [0,1] : \langle f_{t,s}(x)\rangle_{s\in S_t} \to_{I_t} 0\}$ and $Z_{t,n} = \{x \in [0,1] \setminus X_t : n(f,t,x) = n\}$ that are measurable because they belong to the algebra generated by analytic sets since the ideals I_t are analytic. If f is measurable, then there is a Borel measure zero set $B \subseteq [0,1]$ such that $f\restriction([0,1]\setminus B)$ is Borel and then the sets X_t and $Z_{t,n}$ are measurable because $X_t \setminus B$ and $Z_{t,n} \setminus B$ are measurable by previous argument.

Assume that $o_J : {}^T({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^T\omega)$ and $o_t : {}^{S_t}({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^{S_t}\omega)$ are witnesses that the ideals J and I_t for $t \in T$ satisfy (M') or (H'). We define $o : {}^S({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^S\omega)$ by

$$o(f)(x)(t,s) = \max\{o_J(\bar{f})(x)(t), o_t(\hat{f}^t)(x)(s)\}$$

for $f \to_I 0$, $x \in [0,1]$, $t \in T$ and $s \in S_t$. If o_J and o_t are all measurability preserving, then also o is such.

Let $f \to_I 0$ and $A \subseteq [0,1]$ be such that $o(f)[A]$ is bounded in $({}^S\omega, \leq)$ by a function $\nu : S \to \omega$. Then $o_J(\bar{f})[A]$ is bounded in $({}^T\omega, \leq)$ by the function $\bar{\nu} : T \to \omega$ defined by $\bar{\nu}(t) = \nu(t,0)$ and $o_t(\hat{f}^t)[A]$ is bounded in $({}^{S_t}\omega, \leq)$ by the function $\hat{\nu}_t : S_t \to \omega$ defined by $\hat{\nu}_t(s) = \nu(t,s)$. It follows that $\bar{f} \rightrightarrows_J 0$ on A and $\hat{f}^t \rightrightarrows_{I_t} 0$ on A for all $t \in T$. We show that the set $X_n = \{(t,s) \in S : \sup_{x\in A} f_{t,s}(x) \geq 2^{-n}\}$ is in I for all $n \in \omega$ and hence $f \rightrightarrows_I 0$ on A.

For $n \in \omega$ and $t \in T$ let $T_n = \{v \in T : \sup_{x\in A} \bar{f}_v(x) \geq 2^{-n}\}$ and $S_{t,n} = \{s \in S_t : \sup_{x\in A} (\hat{f}^t)_s(x) \geq 2^{-n}\}$; hence $T_n \in J$ and $S_{t,n} \in I_t$. Fix $t \in T \setminus T_n$ and $s \in S_t \setminus S_{t,n}$. Then $\sup_{x\in A} \bar{f}_t(x) < 2^{-n}$ and $\sup_{x\in A}(\hat{f}^t)_s(x) < 2^{-n}$. Let $x \in A$ be arbitrary. If $\langle f_{t,u}(x)\rangle_{u\in S_t} \to_{I_t} 0$, then $f_{t,s}(x) = (\hat{f}^t)_s(x) < 2^{-n}$. Otherwise, $2^{-n(f,t,x)} = \bar{f}_t(x) < 2^{-n}$. Hence $n(f,t,x) > n$ and, either $f_{t,s}(x) \geq 2^{-(n(f,t,x)-1)}$ and then $f_{t,s}(x) = (\hat{f}^t)_s(x) < 2^{-n}$, or $f_{t,s}(x) < 2^{-(n(f,t,x)-1)} \leq 2^{-n}$. Therefore $(t,s) \notin X_n$ for all $t \in T \setminus T_n$ and $s \in S_t \setminus S_{t,n}$. It follows that $X_n \in I$.

(2) For $f : S \to {}^{[0,1]}[0,1]$ we define $\bar{f} : T \to {}^{[0,1]}[0,1]$ and $\hat{f}^t : S_t \to {}^{[0,1]}[0,1]$ for $t \in T$ by the formulas

$$\begin{aligned} &\bar{f}_t(x) = 0 \quad \text{and} \quad (\hat{f}^t)_s(x) = f_{t,s}(x), && \text{if } \langle f_{t,u}(x)\rangle_{u\in S_t} \to_{I_t^*} 0, \\ &\bar{f}_t(x) = 1 \quad \text{and} \quad (\hat{f}^t)_s(x) = 0, && \text{otherwise.} \end{aligned}$$

If $f \to_{I^*} 0$, then $\hat{f}^t \to_{I_t^*} 0$ for all $t \in T$, and by Lemma 3.14 (2), $\bar{f} \to_{J^*} 0$. If f is measurable, then also $\bar{f}$ and $\hat{f}^t$ are measurable because the ideals I_t are analytic.

Assume that $o_J : {}^T({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^T\omega)$ and $o_t : {}^{S_t}({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^{S_t}\omega)$ are witnesses that the ideals J and I_t for $t \in T$ satisfy (M'') or (H'') and define $o : {}^S({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^S\omega)$ by

$$o(f)(x)(t,s) = \max\{o_J(\bar{f})(x)(t), o_t(\hat{f}^t)(x)(s)\}$$

for $f \to_{I^*} 0$, $x \in [0,1]$, $t \in T$ and $s \in S_t$. If o_J and o_t are all measurability preserving, then also o is such.

Let $f \to_{I^*} 0$ and $A \subseteq [0,1]$ be such that $o(f)[A]$ is bounded in $({}^S\omega, \leq)$ by a function $\nu : S \to \omega$. Then $o_J(\bar{f})[A]$ is bounded in $({}^T\omega, \leq)$ by the function $\bar{\nu} : T \to \omega$ defined by $\bar{\nu}(t) = \nu(t,0)$ and $o_t(\hat{f}^t)[A]$ is bounded in $({}^{S_t}\omega, \leq)$ by the function $\hat{\nu}_t : S_t \to \omega$ defined by $\hat{\nu}_t(s) = \nu(t,s)$. It follows that $\bar{f} \rightrightarrows_{J^*} 0$ on A and $\hat{f}^t \rightrightarrows_{I_t^*} 0$ on A for all $t \in T$. Let $M = \{t \in T : \sup_{x\in A} \bar{f}_t(x) \geq 1\}$. Then $M \in J$, $M = \{t \in T : \sup_{x\in A} \bar{f}_t(x) > 0\}$, and $(\hat{f}^t)_s(x) = f_{t,s}(x)$ for all $t \in T \setminus M$, $s \in S_t$ and $x \in [0,1]$. Since $\hat{f}^t \rightrightarrows_{I_t^*} 0$ on A for all $t \in T \setminus M$, by Lemma 3.14 (3) it follows that $f \rightrightarrows_{I^*} 0$ on A. $\square$

Lemma 3.16. *Let J be an ideal on T, let I_t be ideals on S_t for $t \in T$, where all sets T and S_t are infinite countable, let $S = \underline{\lim}^J_{t\in T} S_t$, and let $I = \underline{\lim}^J_{t\in T} I_t$. If $S \notin I$, then for the ideal I all conclusions of Lemma 3.15 hold.*

Proof. For $t \in T$ let $I'_t = \{A \subseteq S : A\cap S_t \in I_t\restriction(S\cap S_t)$ and $|A\cap(S\setminus S_t)| < \omega\}$. Then I'_t is an ideal on S and I'_t is isomorphic to an ideal of the form $(I_t\oplus \mathrm{Fin})\restriction E$ for some set E. By Lemma 3.7 and Lemma 3.10, I'_t satisfies any of the properties (M'), (H'), (M''), (H'') whenever I_t satisfies it. Since $I = \underline{\lim}^J_{t\in T} I'_t$, by Remark 3.1, $I = \mathrm{Fin}$ or $I \leq_{\mathrm{RB}} \sum^J_{t\in T} I'_t$. Therefore the lemma is a consequence of Lemma 3.6, Lemma 3.9, and Lemma 3.15. □

For an ideal I on ω and a family of functions $G \subseteq {}^\omega\omega$ let $G^\leftarrow(I) = \{E \subseteq \omega : (\forall g \in G)\ g(E) \in I\}$. Then $G^\leftarrow(I)$ is an ideal if and only if $\omega \notin G^\leftarrow(I)$ if and only if $(\exists g \in G)\ \mathrm{rng}(g) \notin I$. If $G^\leftarrow(I)$ is an ideal, then $G^\leftarrow(I) = G_0^\leftarrow(I)$ where $G_0 = \{g \in G : \mathrm{rng}(g) \notin I\}$. If $G = \{g\}$, we write $g^\leftarrow(I)$ instead of $G^\leftarrow(I)$. Hence $G^\leftarrow(I) = \bigcap_{g\in G} g^\leftarrow(I)$.

Lemma 3.17. *Let I and J be ideals on ω such that $J = G^\leftarrow(I)$ for a countable family of functions $G \subseteq {}^\omega\omega$.*

1. *J satisfies (M') or (H'), whenever I satisfies the same property.*
2. *If G consists of finite-to-one functions, then J satisfies (M'') or (H''), whenever I satisfies the same property.*

Proof. Since $G^\leftarrow(I) = \bigcap_{g\in G} g^\leftarrow(I)$, by Lemma 3.11 it is enough to prove the lemma in the case $|G| = 1$. So let $g \in {}^\omega\omega$ be such that $\mathrm{rng}(g) \notin I$, g is finite-to-one in case (2), and let $J = g^\leftarrow(I) = \{E \subseteq \omega : g(E) \in I\}$. For $f : \omega \to {}^{[0,1]}[0,1]$ define $\bar f : \omega \to {}^{[0,1]}[0,1]$ by $\bar f_k(x) = \sup\{f_m(x) : m \in g^{-1}(\{k\})\}$, if $k \in \mathrm{rng}(g)$, and $\bar f_k$ is the constant real function with value 0, if $k \notin \mathrm{rng}(g)$.

If $f \to_J 0$, then $\bar f \to_I 0$ because for $x \in [0,1]$ and $\varepsilon > 0$, $\{k \in \omega : \bar f_k(x) > \varepsilon\} = \{k \in \omega : (\exists m \in g^{-1}(\{k\}))\ f_m(x) > \varepsilon\} = g(\{m \in \omega : f_m(x) > \varepsilon\}) \in I$.

If $f \to_{J^*} 0$, then $\bar f \to_{I^*} 0$. To see this, for every $x \in [0,1]$ choose $M_x \in J$ such that $\{m \in \omega : f_m(x) > \varepsilon\} \setminus M_x \in \mathrm{Fin}$ for all $\varepsilon > 0$. Then $g(M_x) \in I$ and $\{k \in \omega : \bar f_k(x) > \varepsilon\} \setminus g(M_x) \subseteq \{k \in \omega : (\exists m \in g^{-1}(\{k\}) \setminus M_x)\ f_m(x) > \varepsilon\} = g(\{m \in \omega : f_m(x) > \varepsilon\} \setminus M_x) \in \mathrm{Fin}$.

Assume that $o_I : {}^\omega({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^\omega\omega)$ witnesses any of the properties for I mentioned in the lemma. Define $o : {}^\omega({}^{[0,1]}[0,1]) \to {}^{[0,1]}({}^\omega\omega)$ by $o(f) = o_I(\bar f)$. Clearly, if o_I is measurability preserving, then also o is measurability preserving.

Let $f \to_J 0$ and $A \subseteq [0,1]$ be such that $o(f)[A] = o_I(\bar f)[A]$ is bounded in ${}^\omega\omega$. By (H') for I (or by (M') in the measurability case) we obtain $\bar f \rightrightarrows_I 0$ on A and hence for every $\varepsilon > 0$, $\{m \in \omega : \sup_{x\in A} f_m(x) > \varepsilon\} \subseteq g^{-1}(\{k \in \omega : \sup_{x\in A} \bar f_k(x) > \varepsilon\}) \in J$. Therefore $f \rightrightarrows_J 0$ on A and (H') (or (M')) holds for J.

Let $f \to_{J^*} 0$ and $A \subseteq [0,1]$ be such that $o(f)[A] = o_I(\bar f)[A]$ is bounded in ${}^\omega\omega$. By (H'') for I (or by (M'')) we obtain $\bar f \rightrightarrows_{I^*} 0$ on A and hence there is $M \in I$ such that $\{k \in \omega : \sup_{x\in A} \bar f_k(x) > \varepsilon\} \setminus M \in \mathrm{Fin}$ for all $\varepsilon > 0$. Then $g^{-1}(M) \in J$ and $\{m \in \omega : \sup_{x\in A} f_m(x) > \varepsilon\}\setminus g^{-1}(M) = g^{-1}(\{k \in \omega : \sup_{x\in A} \bar f_k(x) > \varepsilon\}\setminus M) \in \mathrm{Fin}$ because g is finite-to-one. Therefore $f \rightrightarrows_{J^*} 0$ on A and (H'') (or (M'')) holds for J. □

Lemma 3.9 concerns to preservation of properties of ideals $\leq_{\mathrm{RK}}$-downward and $\leq_{\mathrm{RB}}$-downward. In the upward direction we have the following lemma:

Lemma 3.18. *For ideals I and J on ω there exists an ideal $J_0 \subseteq J$ such that:*

(1) *If $I \leq_{\mathrm{RK}} J$, then $I \leq_{\mathrm{RK}} J_0$ and J_0 satisfies (M') or (H') if and only if I satisfies the same property.*
(2) *If $I \leq_{\mathrm{RB}} J$, then $I \leq_{\mathrm{RB}} J_0$ and J_0 satisfies any of the properties (M'), (H'), (M''), (H'') if and only if I satisfies the same property.*

Proof. This is a consequence of Lemma 3.9 and Lemma 3.17 because if $g : \omega \to \omega$ is such that $I = \{E \subseteq \omega : g^{-1}(E) \in J\}$ and $J_0 = g^{\leftarrow}(I) = \{E \subseteq \omega : g(E) \in I\}$, then $J_0 \subseteq J$ and $I = \{E \subseteq \omega : g^{-1}(E) \in J_0\}$. □

4. Negative results on (M) and (H)

An ideal I on ω is said to be thick, if there is a measurable function $g : [0,1] \to I$ such that $\mu(\{x \in [0,1] : g(x) \subseteq E\}) = 0$ for all $E \in I$. In this definition the interval $[0,1]$ can be replaced by ${}^{\omega}2$ since there exists a measure preserving Borel isomorphism between $[0,1]$ and ${}^{\omega}2$. Recall that functions $g : [0,1] \to \mathcal{P}(\omega)$ and $h : [0,1] \to {}^{\omega}\mathcal{P}(\omega)$ are measurable, if the sets $\{x \in [0,1] : m \in g(x)\}$ and $\{x \in [0,1] : m \in h_n(x)\}$ are measurable for all $m, n \in \omega$.

Lemma 4.1. *If I is a thick ideal on ω, then neither the classical Egoroff's theorem nor the generalized Egoroff's theorem between $\to_I$ and $\rightrightarrows_I$ and between $\to_{I^*}$ and $\rightrightarrows_{I^*}$ holds.*

Proof. Let $g : [0,1] \to I$ be measurable such that $\mu(\{x \in [0,1] : g(x) \subseteq E\}) = 0$ for all $E \in I$. Define a sequence of measurable functions $f : \omega \to {}^{[0,1]}[0,1]$ by

$$f_m(x) = \begin{cases} 1, & \text{if } m \in g(x), \\ 0, & \text{otherwise.} \end{cases} \qquad (*)$$

Then $f \to_I 0$ and $f \to_{I^*} 0$ on $[0,1]$ because $\{m \in \omega : f_m(x) \geq \varepsilon\} = g(x)$ for all $\varepsilon > 0$. If $A \subseteq [0,1]$ is arbitrary such that $f \rightrightarrows_I 0$ on A, then $\mu(A) = 0$ because the set $E = \{m \in \omega : \sup_{z \in A} f_m(z) \geq 1\}$ is in I and $g(x) \subseteq E$ for every $x \in A$. The same argument works for $\rightrightarrows_{I^*}$ because it implies $\rightrightarrows_I$. □

Due to Lemma 4.1 the following is a consequence of Theorem 1.3:

Corollary 4.2. *Let I be a thick ideal on ω.*

(1) *(M) holds neither between $\to_I$ and $\rightrightarrows_I$, nor between $\to_{I^*}$ and $\rightrightarrows_{I^*}$.*
(2) *If $\mathrm{non}(\mathcal{N}) < \mathfrak{b}$, then (H) holds neither between $\to_I$ and $\rightrightarrows_I$, nor between $\to_{I^*}$ and $\rightrightarrows_{I^*}$.* □

Lemma 4.3. *Let I, J, and I_n for $n \in \omega$ be ideals on ω and let $B \in \mathcal{P}(\omega) \setminus I$.*

(1) *If $I \restriction B$ is thick, then also I is thick.*
(2) *If $J \leq_{\mathrm{RK}} I$ and J is thick, then also I is thick.*
(3) *If $J \leq_{\mathrm{RK}} I \restriction B$ and J is thick, then also I is thick.*
(4) *If J is thick, then also $\sum^{J}_{n \in \omega} I_n$, $J \times I$, $I \times J$ are thick.*

Proof. (1) is easy; (3) is a consequence of (1) and (2); and by Remark 3.1, (4) is a consequence of (2). We prove (2).

Let $g : \omega \to \omega$ be such that $J = \{E \subseteq \omega : g^{-1}(E) \in I\}$. Assume that $h : [0,1] \to J$ is measurable and $\mu(\{x \in [0,1] : h(x) \subseteq E\}) = 0$ for all $E \in J$. The function $g^{-1} : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is continuous and therefore the function $\bar{h} : [0,1] \to I$ defined by $\bar{h}(x) = g^{-1}(h(x))$ is measurable. Given set $F \in I$ let $E = \{n \in \omega :$

$g^{-1}(\{n\}) \subseteq F\}$, i.e., $E \in J$ and E is the largest subset of ω such that $g^{-1}(E) \subseteq F$. Then $\{x \in [0,1] : \bar{h}(x) \subseteq F\} = \{x \in [0,1] : g^{-1}(h(x)) \subseteq F\} = \{x \in [0,1] : g^{-1}(h(x)) \subseteq g^{-1}(E)\} = \{x \in [0,1] : h(x) \subseteq E\}$ because $\omega \setminus \operatorname{rng}(g) \subseteq E$. Therefore $\mu(\{x \in [0,1] : \bar{h}(x) \subseteq F\}) = 0$. □

Now we introduce several ideals and we prove that they are thick.

Example 4.4. (a) Let $\varphi : {}^{<\omega}2 \to \omega$ be a bijection such that $\varphi(s) < \varphi(t)$ whenever $|s| < |t|$. Then $2^{|s|} \le \varphi(s)+1 < 2^{|s|+1}$ for $s \in {}^{<\omega}2$, i.e., φ maps ${}^{n}2$ onto the interval $K_n = [2^n - 1, 2^{n+1} - 1)$ and $2^{-(n+1)} < 1/(k+1) \le 2^{-n}$ for $k \in K_n$. For $x \in {}^{\omega}2$ let $E_x = \{\varphi(x{\restriction}n) : n \in \omega\}$. The mapping $x \mapsto E_x$ is a homeomorphism of ${}^{\omega}2$ onto the family $C = \{E_x : x \in {}^{\omega}2\} \subseteq [\omega]^{\omega}$, $\omega = \bigcup C$, and C is a perfect compact almost disjoint family on ω (shortly, a. d. family, i.e., $|E_x \cap E_y| < \omega$ for $x \ne y$). For $A \subseteq {}^{\omega}2$ denote $E(A) = \bigcup_{x \in A} E_x$. Consider the ideals

$$I_0 = \{E \subseteq \omega : (\exists A \in [{}^{\omega}2]^{<\omega})\ E \subseteq E(A)\},$$
$$I_1 = \{E \subseteq \omega : (\exists A \in [{}^{\omega}2]^{\omega})\ \{E_x : x \in {}^{\omega}2 \setminus A\} \cup \{E\} \text{ is an a. d. family}\},$$
$$I_2 = \{E \subseteq \omega : (\exists A \in \mathcal{N})\ \{E_x : x \in {}^{\omega}2 \setminus A\} \cup \{E\} \text{ is an a. d. family}\},$$
$$I_3 = \{E \subseteq \omega : \textstyle\sum_{k \in E} 1/(k+1) < \infty\},$$
$$I_4 = \{E \subseteq \omega : \lim_{n\to\infty} |E \cap n|/n = 0\}$$
$$= \{E \subseteq \omega : \lim_{n\to\infty} |E \cap [2^n, 2^{n+1})|/2^n = 0\}.$$

(b) A family $D \subseteq [\omega]^{\omega}$ is an independent family, if $|\bigcap D_0 \cap \bigcap_{F \in D_1} (\omega \setminus F)| = \omega$ for any two disjoint finite subfamilies $D_0, D_1 \subseteq D$. Let $D = \{F_x : x \in {}^{\omega}2\}$ be a perfect compact independent family on ω such that $\omega = \bigcup D$ and the mapping $x \mapsto F_x$ is a homeomorphism. (For example, on the countable set $S = \{(n,u) : n \in \omega$ and $u \subseteq {}^{n}2\}$ the family of sets $F'_x = \{(n,u) \in S : n \in \omega$ and $x{\restriction}n \in u\}$ for $x \in {}^{\omega}2$ is a perfect compact independent family. This is a modification of the Hausdorff's construction, see [4].) For $A, B_1, \dots, B_k \subseteq {}^{\omega}2$ denote $F(A) = \bigcup_{x \in A} F_x$ and $F(A; B_1, \dots, B_k) = F(A) \cup \bigcup_{i=1}^{k} (\omega \setminus F(B_i))$. Let $\mathcal{N}^+$ denote the family of sets of positive measure in ${}^{\omega}2$. Consider the ideals

$$I'_0 = \{E \subseteq \omega : (\exists A \in [{}^{\omega}2]^{<\omega})\ E \subseteq F(A)\},$$
$$I'_1 = \{E \subseteq \omega : (\exists A \in [{}^{\omega}2]^{<\omega})(\exists B_1, \dots, B_k \in \mathcal{N}^+)\ E \subseteq F(A; B_1, \dots, B_k)\},$$
$$I'_2 = \{E \subseteq \omega : (\exists A \in [{}^{\omega}2]^{<\omega})(\exists B_1, \dots, B_k \in [{}^{\omega}2]^{\omega})\ E \subseteq F(A; B_1, \dots, B_k)\}.$$

Note that $\mathcal{N}^+$ can be replaced by the family of closed sets of positive measure in definition of I'_1. In fact, I_0 and I'_i are $\mathbf{\Sigma}^1_1$, I_1 and I_2 are $\mathbf{\Sigma}^1_2$, I_3 is $\mathbf{\Sigma}^0_2$, and I_4 is $\mathbf{\Pi}^0_3$. It follows that I_0, I_3, I_4, I'_0, I'_1, I'_2 are meager and have measure zero.

For $E \subseteq \omega$ let $X_E = \{x \in {}^{\omega}2 : E_x \subseteq E\}$ and $Z_E = \{x \in {}^{\omega}2 : F_x \subseteq E\}$. Then X_E is closed, $X_\emptyset = \emptyset$, $X_\omega = {}^{\omega}2$, and $X_{\bigcap_i E_i} = \bigcap_i X_{E_i}$. The same holds for Z_E.

Lemma 4.5. (1) *I_0, I_1, I_2, I_3, I_4, I'_0, I'_1, I'_2 are ideals and I_2, I_3, I_4 are P-ideals.*
(2) *$I_0 \subseteq I_1 \subseteq I_2$, $I_0 \subseteq I_3 \subseteq I_4$, and $I'_0 \subseteq I'_1 \subseteq I'_2$.*

Proof. (1) We show that I'_1 and I'_2 are ideals and that I_2 is a P-ideal. The other facts are obvious or known.

By definition I'_1 and I'_2 are closed under finite unions and subsets. If $A \in [{}^{\omega}2]^{<\omega}$ and $B_1, \dots, B_k$ are infinite, then there are distinct reals $x_1 \in B_1 \setminus A, \dots, x_k \in$

$B_k \setminus A$. Since D is independent, the set $\bigcap_{i=1}^k F_{x_i} \cap \bigcap_{x\in A}(\omega \setminus F_x)$ is infinite and disjoint from $F(A; B_1, \dots, B_k)$. Therefore $\omega \notin I'_1$ and $\omega \notin I'_2$.

The sets X_E are closed and therefore one can easily verify that

$$I_2 = \{E \subseteq \omega : \lim_{n\in\omega} \mu(X_{(\omega\setminus(E\setminus n))}) = 1\}.$$

Assume that $E_n \subseteq E_{n+1} \in I_2$ for all $n \in \omega$. Find an increasing sequence $\langle k_n\rangle_{n\in\omega}$ in ω such that $\mu(X_{(\omega\setminus(E_n\setminus k_n))}) \geq 1 - 2^{-(n+1)}$ and let $E = \bigcup_{n\in\omega}(E_n \setminus k_n)$. Then $\mu(X_{(\omega\setminus(E\setminus k_n))}) \geq \mu(X_{(\omega\setminus\bigcup_{i\geq n}(E_i\setminus k_i))}) = \mu(\bigcap_{i\geq n} X_{(\omega\setminus(E_i\setminus k_i))}) \geq 1 - \sum_{i\geq n} 2^{-(i+1)} = 1 - 2^{-n}$. This proves that I_2 is a P-ideal.

(2) For every $x \in {}^\omega 2$ we have $1 = \sum_{n\in\omega} 2^{-(n+1)} < \sum_{n\in\omega} 1/(\varphi(x\restriction n) + 1) = \sum_{k\in E_x} 1/(k + 1) < \sum_{n\in\omega} 2^{-n} = 2$ and hence $E_x \in I_3$. Therefore $I_0 \subseteq I_3$. The inclusion $I_3 \subseteq I_4$ follows by the fact that for $E \subseteq \omega$ and $n \in \omega$, $\sum_{k\in E\cap K_n} 1/(k+1) \geq \sum_{k\in E\cap K_n} 2^{-n} = |E \cap K_n|/2^n$. The remaining inclusions are obvious. □

An ideal I on ω is said to be a summable ideal, if there is a function $a : B \to [0, 1]$ such that $\sum_{k\in B} a(k) = \infty$ and $I = S(a)$, where

$$S(a) = \{E \subseteq \mathrm{dom}(a) : \textstyle\sum_{k\in E} a(k) < \infty\}.$$

An ideal I on ω is said to be simply summable in a set $B \subseteq \omega$, if there is a function $a : B \to [0, 1]$ such that $\sum_{k\in B} a(k) = \infty$, $\lim_{k\in B} a(k) = 0$, and $I\restriction B = S(a)$.

If I is a summable ideal on ω, then there is $B \subseteq \omega$ such that either $I = \langle B\rangle$ (this includes $I = \mathrm{Fin}$), or I is simply summable in B. Really, if $I = S(a)$ for a divergent series $a : \omega \to [0, 1]$, then either there is $\delta > 0$ such that $\sum\{a(k) : a(k) < \delta\} < \infty$ and then $I = \langle B\rangle$ for $B = \{k \in \omega : a(k) < \delta\}$, or otherwise by induction find a decreasing sequence of positive reals $\delta_n < 2^{-n}$ and pairwise disjoint finite sets B_n such that $\sum_{k\in B_n} a_k \geq 1$ and $\delta_{n+1} \leq a_k < \delta_n$ for $k \in B_n$. Then I is simply summable in $B = \bigcup_{n\in\omega} B_n$.

By next theorem simply summable ideals are thick and therefore by Lemma 3.6, a summable ideal on ω is thick if and only if it is simply summable in a subset of ω.

Theorem 4.6. *Let I be an ideal on ω.*

1. *If $I_0 \subseteq I \subseteq I_2$ or $I_0 \subseteq I \subseteq I_4$, then I is thick.*
2. *If I is simply summable in a subset of ω, then I is thick.*
3. *If $I'_0 \subseteq I \subseteq I'_2$ or $I'_1 \subseteq I$, then I is thick. In particular, there are prime thick ideals on ω.*

Proof. (1) Since $I_0 \subseteq I$ we can define $g : {}^\omega 2 \to I$ by $g(x) = E_x$. The function g is continuous and $X_E = \{x \in {}^\omega 2 : g(x) \subseteq E\}$ for $E \subseteq \omega$.

If $E \in I_2$, then $X_E \in \mathcal{N}$, i.e., $\mu(X_E) = 0$, by definition of I_2.

If $E \in I_4$, then $\mu(X_E) = 0$ because $X_E \subseteq \bigcup\{[s] : \varphi(s) \in E \cap K_n\}$ for all $n \in \omega$, $\mu([s]) = 1/2^n$ for $\varphi(s) \in K_n$, and $\lim_{n\in\omega} |E \cap K_n|/2^n = 0$.

(2) Assume that an ideal I on ω is simply summable in a set $B' \subseteq \omega$ by a series $a : B' \to [0, 1]$. We find an infinite set $B \subseteq B'$ and an ideal $J \leq_{\mathrm{RB}} I\restriction B$ on ω such that $I_0 \subseteq J \subseteq I_4$. Then I is thick by (1) and by Lemma 4.3.

Since a is converging to 0 and $\sum_{k\in B'} a_k$ is divergent, by induction we can construct a system of pairwise disjoint nonempty finite sets $B_{n,s} \subseteq B'$ for $n \in \omega$ and $s \in {}^n 2$ such that $2^{-(n+1)} \leq \sum_{k\in B_{n,s}} a(k) < 2^{-n}$. Let $B = \bigcup_{n\in\omega} \bigcup_{s\in {}^n 2} B_{n,s}$ and let $\varphi : {}^{<\omega}2 \to \omega$ be the bijection from Example 4.4. Define $g : B \to \omega$ by $g(k) = \varphi(s)$ for $k \in B_{|s|,s}$ and $s \in {}^{<\omega}2$ and let $J = \{E \subseteq \omega : g^{-1}(E) \in I\restriction B\}$ where $I\restriction B = S(a\restriction B)$. Then $J \leq_{\mathrm{RB}} I\restriction B$ because g is finite-to-one.

We show that $I_0 \subseteq J \subseteq I_4$. For every $x \in {}^\omega 2$, $E_x \in J$ because $g^{-1}(E_x) = \bigcup_{n\in\omega} B_{n,x\restriction n}$ and $\sum_{n\in\omega}\sum_{k\in B_{n,x\restriction n}} a_k < \sum_{n\in\omega} 2^{-n} < \infty$. It follows that $I_0 \subseteq J$. Let $E \in J$ be arbitrary, i.e., $g^{-1}(E) = \bigcup_{\varphi(s)\in E} B_{|s|,s} = \bigcup_{n\in\omega}\bigcup_{\varphi(s)\in E\cap K_n} B_{n,s} \in I$. Then

$$\sum_{n\in\omega} |E\cap K_n|/2^{n+1} = \sum_{n\in\omega}\sum_{\varphi(s)\in E\cap K_n} 2^{-(n+1)} \leq \sum_{n\in\omega}\sum_{\varphi(s)\in E\cap K_n}\sum_{k\in B_{n,s}} a_k < \infty$$

and hence $\lim_{n\to\infty} |E\cap K_n|/2^n = 0$. Therefore $J \subseteq I_4$.

(3) The function $g : {}^\omega 2 \to I_0'$ defined by $g(x) = F_x$ is continuous. Then $Z_E = \{x \in {}^\omega 2 : g(x) \subseteq E\}$ for $E \subseteq \omega$.

Let $E \in I_2'$ be arbitrary, i.e., $E \subseteq F(A; B_1, \dots, B_k)$ for some $A \in [{}^\omega 2]^{<\omega}$ and $B_1, \dots, B_k \in [{}^\omega 2]^\omega$. We show that $Z_E \subseteq A$. Let $x \in {}^\omega 2 \setminus A$. Choose distinct $x_i \in B_i \setminus (A \cup \{x\})$ for $i = 1, \dots, k$ and denote $G = \bigcap_{y\in A}(\omega \setminus F_y) \cap \bigcap_{i=1}^k F_{x_i}$. Then $G \cap F_x \neq \emptyset$ because D is independent. Since $G \cap F(A; B_1, \dots, B_k) = \emptyset$ it follows that $F_x \nsubseteq F(A; B_1, \dots, B_k)$ and hence $x \in {}^\omega 2 \setminus Z_E$. Therefore Z_E is finite for every $E \in I_2'$. It follows that every ideal I with $I_0' \subseteq I \subseteq I_2'$ is thick.

Let $I \supseteq I_1'$ be an arbitrary ideal. Then $\mu(Z_E) = 0$ for every $E \in I$ since otherwise for a closed set $B = Z_E$ of positive measure we have $F(B) \subseteq E \in I$ and $\omega \setminus F(B) = F(\emptyset; B) \in I_1' \subseteq I$. Therefore I is thick. □

Remark 4.7. The conclusion of Corollary 4.2 in case of the thick ideals $I = I_0$ and $I = I_0'$ does not require the assumption $\mathrm{non}(\mathcal{N}) < \mathfrak{b}$ and does not need Theorem 1.2. This is due to the fact that for every function $\Phi : [0,1] \to {}^\omega\omega$ there is an infinite set $A \subseteq [0,1]$ such that $\Phi[A]$ is bounded (but the sequences of functions $(*)$ from the proof of Lemma 4.1 witnessing the thickness of I, I-uniformly converges on a set of the form $A = X_E$ and $A = Z_E$, respectively, only if A is finite): If $Z = \Phi([0,1])$ is an uncountable subset of ${}^\omega\omega$, then Z has an accumulation point z_0 which together with a sequence converging to z_0 gives an infinite countable compact set $Z_0 \subseteq Z$. Then for the infinite set $A = \Phi^{-1}(Z_0)$, $\Phi[A]$ is bounded. If Z is countable, then for some $z_0 \in Z$, the set $A = \Phi^{-1}(\{z_0\})$ is infinite and $\Phi[A]$ is bounded.

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