COFINALITY OF THE LAVER IDEAL

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Abstract. Yurii Khomskii observed that $\mathrm{cof}(l^0) > \mathfrak{c}$ assuming $\mathfrak{b} = \mathfrak{c}$ and he asked whether the inequality $\mathrm{cof}(l^0) > \mathfrak{c}$ is provable in ZFC. We find several conditions that imply some variants of this inequality for tree ideals. Applying a recent result of J. Brendle, Y. Khomskii, and W. Wohofsky we show that l^0 satisfies some of these conditions and consequently, $\mathrm{cof}(l^0) = \mathfrak{d}({}^{\mathfrak{c}}l^0) \geq \mathfrak{d}({}^{\mathfrak{c}}\mathfrak{c}) > \mathfrak{c}$. We also prove that if the cellularity of a Boolean algebra B is hereditarily $\geq \kappa$, then every κ-sequence in B^+ has a κ-subsequence with a disjoint refinement.

1. Introduction

Yurii Khomskii [5] observed that assuming $\mathfrak{b} = \mathfrak{c}$, the cofinality of the Laver ideal l^0 is $> \mathfrak{c}$, i.e., $\mathrm{cof}(l^0) > \mathfrak{c}$. He asked whether the inequality $\mathrm{cof}(l^0) > \mathfrak{c}$ is provable in ZFC. After writing a previous version of this paper Yurii Khomskii kindly informed us about the proof of $\mathrm{cof}(l^0) > \mathfrak{c}$. It is contained in the paper of J. Brendle, Y. Khomskii, and W. Wohofsky [2]. We find several conditions that imply this and also a bit stronger inequalities for tree ideals. Applying a result of [2] we prove that $\mathrm{cof}(l^0) = \mathfrak{d}({}^{\mathfrak{c}}l^0) \geq \mathfrak{d}({}^{\mathfrak{c}}\mathfrak{c}) > \mathfrak{c}$. We deal also with disjoint refinements in Boolean algebras. We prove that if the cellularity of a Boolean algebra B is hereditarily $\geq \kappa$, then every κ-sequence in B^+ has a κ-subsequence with a disjoint refinement. This result helps to classify the considered conditions.

Throughout this paper $\mathbb{P}_{\mathbb{L}}$ is the system of all Laver perfect sets in ${}^{\omega}\omega$ (i.e., the sets of the form $[T] = \{x \in {}^{\omega}\omega : (\forall n \in \omega)\ x{\restriction}n \in p\}$ where $T \subseteq {}^{<\omega}\omega$ is a Laver tree) and $\mathbb{P}_{\mathbb{S}}$ is the system of all perfect sets in ${}^{\omega}\omega$. We are primarily interested in the Laver ideal l^0 but most of the assertions hold also for other tree ideals and also in a more general context: For a family $\mathbb{P} \subseteq \mathcal{P}({}^{\omega}\omega)$ let $s(\mathbb{P}) = \{X \subseteq {}^{\omega}\omega : (\forall p \in \mathbb{P})$ $(\exists q \in \mathbb{P})\ q \subseteq p$ and $q \cap X = \emptyset\}$ and $s^+(\mathbb{P}) = \mathcal{P}({}^{\omega}\omega) \setminus s(\mathbb{P})$. Hence $s(\mathbb{P})$ is an ideal associated to the poset $(\mathbb{P}, \subseteq)$. In particular, $l^0 = s(\mathbb{P}_{\mathbb{L}})$ is the Laver ideal and $s^0 = s(\mathbb{P}_{\mathbb{S}})$ is the Marczewski ideal. Let $l^+ = s^+(\mathbb{P}_{\mathbb{L}})$ and $s^+ = s^+(\mathbb{P}_{\mathbb{S}})$. We assume that $\mathbb{P}$ is a family with the following properties:

(a) $\mathbb{P}$ is a separable family of sets (see [8, 10]), i.e., $\mathbb{P} \subseteq \mathrm{dec}(\mathbb{P})$ where $\mathrm{dec}(\mathbb{P}) = \{X \subseteq {}^{\omega}\omega : (\forall p \in \mathbb{P})(\exists q \in \mathbb{P})\ q \subseteq p$ and either $q \subseteq X$ or $q \cap X = \emptyset\}$.
(b) Every $p \in \mathbb{P}$ has $\mathfrak{c}$ pairwise disjoint subsets in $\mathbb{P}$.
(c) $s(\mathbb{P}){\restriction}p = s(\mathbb{P}) \cap \mathcal{P}(p)$ is isomorphic to $s(\mathbb{P})$ for every $p \in \mathbb{P}$.

Note that $\mathrm{dec}(\mathbb{P})$ is an algebra of sets, $s(\mathbb{P}) \subseteq \mathrm{dec}(\mathbb{P})$, and by (a), $p, q \in \mathbb{P}$ are incompatible if and only if $p \cap q \in s(\mathbb{P})$. By (b), $\mathrm{non}(s(\mathbb{P})) = \mathfrak{c}$; (c) is necessary only for $\mathrm{cof}(s(\mathbb{P})) = \mathfrak{d}({}^{\mathfrak{c}}s(\mathbb{P}))$.

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Recall that for an ideal I on a set X, $\mathrm{non}(I) = \min\{|A| : A \subseteq X \text{ and } A \notin I\}$, $\mathrm{cov}(I) = \min\{|I_0| : I_0 \subseteq I \text{ and } \bigcup I_0 = X\}$, $\mathrm{cof}(I) = \min\{|I_0| : I_0 \subseteq I \text{ and } (\forall A \in I)$ $(\exists B \in I_0)\ A \subseteq B\}$. If P is a partially ordered set then $\mathfrak{d}(P)$ denotes the least cardinality of a dominating subset of P and $\mathfrak{b}(P)$ is the least cardinality of an unbounded subset of P.

If X is a family of sets and $p, q \in {}^{\kappa}X$, then $p \le q$ means $(\forall \alpha < \kappa)\ p_\alpha \subseteq q_\alpha$ and we say that p refines q (we identify $p \in {}^{\kappa}X$ with $\langle p_\alpha : \alpha < \kappa \rangle$). We say that p quasi-refines q if there is a one-to-one function $f : \kappa \to \kappa$ such that $(\forall \alpha < \kappa)$ $p_\alpha \subseteq q_{f(\alpha)}$.

Lemma 1.1. *Let κ be an infinite cardinal and let $I \subseteq \mathcal{P}(\kappa) \setminus \{\kappa\}$ be a family of sets, e.g., a proper ideal on κ. The following holds:*

(i) $\kappa < \mathfrak{d}({}^{\kappa}\kappa)$.
(ii) *If $\kappa \subseteq I$, then $\mathfrak{d}({}^{\kappa}\kappa) \le \mathfrak{d}({}^{\kappa}I)$.*
(iii) *If $[\kappa]^{<\kappa} \subseteq I$, then $\kappa < \mathfrak{d}({}^{\mathrm{cf}(\kappa)}I) \le \mathfrak{d}({}^{\kappa}I)$.*

Proof. If $F = \{f_\alpha : \alpha < \kappa\} \subseteq {}^{\kappa}\kappa$, then F is not cofinal in ${}^{\kappa}\kappa$ because the function $f(\alpha) = f_\alpha(\alpha) + 1$ is not dominated by any member of F. Therefore $\kappa < \mathfrak{d}({}^{\kappa}\kappa)$.

If $\kappa \subseteq I$, then there is a pair of functions (Galois–Tukey embedding) $\varphi : {}^{\kappa}\kappa \to {}^{\kappa}I$ and $\psi : {}^{\kappa}I \to {}^{\kappa}\kappa$ such that $\varphi(f) \le g$ implies $f \le \psi(g)$ (define $\varphi(f) = f$ and $\psi(g)(\alpha) = \sup\{\xi < \kappa : \xi \subseteq g(\alpha)\}$). It follows that $\mathfrak{d}({}^{\kappa}\kappa) \le \mathfrak{d}({}^{\kappa}I)$.

Let $\{\xi_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ be a cofinal sequence of ordinals in κ and assume that $[\kappa]^{<\kappa} \subseteq I$. If $F = \{f_\alpha : \alpha < \kappa\} \subseteq {}^{\mathrm{cf}(\kappa)}I$, then F is not cofinal in ${}^{\mathrm{cf}(\kappa)}I$ because there is $f : \mathrm{cf}(\kappa) \to [\kappa]^{<\kappa} \subseteq I$ such that $(\forall \alpha < \mathrm{cf}(\kappa))(\forall \xi < \xi_\alpha)\ f(\alpha) \setminus f_\xi(\alpha) \ne \emptyset$; f is not dominated by any member of F. □

In this section as well as in the next section the letter μ denotes the cofinality of the continuum, i.e., $\mu = \mathrm{cf}(\mathfrak{c})$. By previous lemma the following inequalities hold:

$$\mathfrak{c} < \mathfrak{d}({}^{\mathfrak{c}}\mathfrak{c}) \le \mathfrak{d}({}^{\mathfrak{c}}s(\mathbb{P})), \qquad \mathfrak{c} < \mathfrak{d}({}^{\mu}s(\mathbb{P})) \le \mathfrak{d}({}^{\mathfrak{c}}s(\mathbb{P})),$$
$$\mathfrak{c} < \mathfrak{d}({}^{\mathfrak{c}}\mathfrak{c}) \le \mathfrak{d}({}^{\mathfrak{c}}([\mathfrak{c}]^{<\mathfrak{c}})), \qquad \mathfrak{c} < \mathfrak{d}({}^{\mu}([\mathfrak{c}]^{<\mathfrak{c}})) \le \mathfrak{d}({}^{\mathfrak{c}}([\mathfrak{c}]^{<\mathfrak{c}})).$$

Below we list several conditions and later we show that each of them imply that $\mathrm{cof}(s(\mathbb{P}))$ is above some of the cardinals $\mathfrak{c}^+$, $\mathfrak{d}({}^{\mathfrak{c}}\mathfrak{c})$, $\mathfrak{d}({}^{\mathfrak{c}}s(\mathbb{P}))$, $\mathfrak{d}({}^{\mu}s(\mathbb{P}))$ that are all bigger than $\mathfrak{c}$ (on the other hand we have no comparison between $\mathrm{cof}(s(\mathbb{P}))$ and the cardinals $\mathfrak{d}({}^{\mu}([\mathfrak{c}]^{<\mathfrak{c}})) \le \mathfrak{d}({}^{\mathfrak{c}}([\mathfrak{c}]^{<\mathfrak{c}}))$ provided that $\mathfrak{c}$ is singular).

(1) There exists a maximal antichain $A \subseteq \mathbb{P}$ of cardinality $\mathfrak{c}$ with pairwise disjoint sets.
(2) There exists a maximal antichain $\{p_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathbb{P}$ such that $(\forall \alpha < \mathfrak{c})$ $(\exists q \in \mathbb{P})\ q \cap \bigcup_{\beta<\alpha} p_\beta = \emptyset$.
(3) There exists a maximal antichain $\{p_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathbb{P}$ such that $(\forall \alpha < \mathfrak{c})$ ${}^{\omega}\omega \setminus \bigcup_{\beta<\alpha} p_\beta \in s^+(\mathbb{P})$.
(4) $(\forall p \in {}^{\mathfrak{c}}\mathbb{P})[p \text{ is an antichain} \to (\exists A \in s(\mathbb{P}))(\forall \alpha < \mathfrak{c})\ A \cap p_\alpha \ne \emptyset]$.
(5) $(\exists p \in {}^{\mathfrak{c}}\mathbb{P})(\forall q \in \{p' \in {}^{\mathfrak{c}}\mathbb{P} : p' \le p\})(\exists A \in s(\mathbb{P}))(\forall \alpha < \mathfrak{c})\ A \cap q_\alpha \ne \emptyset$.

The strengthening of condition (4) by removing the requirement that a $\mathfrak{c}$-sequence $p \in {}^{\mathfrak{c}}\mathbb{P}$ is an antichain is false (consider an enumeration of $\mathbb{P}$).

Let us consider also the following selection and refinement properties for $\mathbb{P}$:

(σ_2) There is $p \in {}^{\mathfrak{c}}\mathbb{P}$ such that $\bigcup_{\alpha<\mathfrak{c}} A_\alpha \in s(\mathbb{P})$ whenever $(\forall \alpha < \mathfrak{c})\ A_\alpha \subseteq p_\alpha$ and $A_\alpha \in s(\mathbb{P})$.

(σ_1) There is $p \in {}^{\mu}\mathbb{P}$ such that $\bigcup_{\alpha<\mu} A_\alpha \in s(\mathbb{P})$ whenever $(\forall \alpha < \mu)\ A_\alpha \subseteq p_\alpha$ and $A_\alpha \in s(\mathbb{P})$.

(σ_0^*) There is $p \in {}^{\mathfrak{c}}\mathbb{P}$ such that $\bigcup_{\alpha<\mathfrak{c}} A_\alpha \in s(\mathbb{P})$ whenever $(\forall \alpha < \mathfrak{c})\ A_\alpha \in [p_\alpha]^{<\mathfrak{c}}$.

(σ_0) There is $p \in {}^{\mathfrak{c}}\mathbb{P}$ such that $\bigcup_{\alpha<\mathfrak{c}} A_\alpha \in s(\mathbb{P})$ whenever $(\forall \alpha < \mathfrak{c})\ A_\alpha \in [p_\alpha]^{\leq 1}$.

(ρ_2) $(\exists p \in {}^{\mathfrak{c}}\mathbb{P})(\forall q \in \mathbb{P})(\exists q' \in \mathbb{P})\ q' \subseteq q$ and $|\{\alpha < \mathfrak{c} : q' \cap p_\alpha \neq \emptyset\}| \leq 1$.

(ρ_1) $(\exists p \in {}^{\mu}\mathbb{P})(\forall q \in \mathbb{P})(\exists q' \in \mathbb{P})\ q' \subseteq q$ and $|\{\alpha < \mu : q' \cap p_\alpha \neq \emptyset\}| \leq 1$.

(ρ_0) $(\exists p \in {}^{\mathfrak{c}}\mathbb{P})(\forall q \in \mathbb{P})(\exists q' \in \mathbb{P})\ q' \subseteq q$ and $|\{\alpha < \mathfrak{c} : q' \cap p_\alpha \neq \emptyset\}| < \mathfrak{c}$.

If $\mu = \mathfrak{c}$, then $\sigma_1 \leftrightarrow \sigma_2$ and $\rho_1 \leftrightarrow \rho_2$.

By Theorem 3.3 below for every $p \in {}^{\mathfrak{c}}\mathbb{P}$ there is an antichain $q \in {}^{\mathfrak{c}}\mathbb{P}$ and a one-to-one function $f : \mathfrak{c} \to \mathfrak{c}$ such that $q_\alpha \subseteq p_{f(\alpha)}$ for $\alpha < \mathfrak{c}$ (it is said that the antichain q pseudo-refines p). Hence, the witnessing sequences for σ_i and ρ_i can be pseudo-refined by antichains of $\mathbb{P}$. Moreover, the witnessing sequences for σ_2, σ_1, ρ_2, ρ_1 can be pseudo-refined so that the sets p_α will be pairwise disjoint.

If $p \in {}^{\mathfrak{c}}\mathbb{P}$ is a witnessing sequence for σ_0, then $\{x \in {}^{\omega}\omega : |\{\alpha < \mathfrak{c} : x \in p_\alpha\}| = \mathfrak{c}\} \in s(\mathbb{P})$. To see this let $a_x = \{\alpha < \mathfrak{c} : x \in p_\alpha\}$ for $x \in {}^{\omega}\omega$ and let $A = \{x \in {}^{\omega}\omega : |a_x| = \mathfrak{c}\}$. Let $f : A \to \mathfrak{c}$ be a one-to-one function such that $f(x) \in a_x$ for all $x \in A$. Define $A_{f(x)} = \{x\}$ for $x \in A$ and $A_\alpha = \emptyset$ for $\alpha \in \mathfrak{c} \setminus \operatorname{rng}(f)$. Then $A = \bigcup_{\alpha<\mathfrak{c}} A_\alpha \in s(\mathbb{P})$ by the assumption on p.

Theorem 1.2. *The implications of Figure* 1 *hold. If* $\mathfrak{c}$ *is regular, then Figure* 1 *"reduces" to Figure* 2.

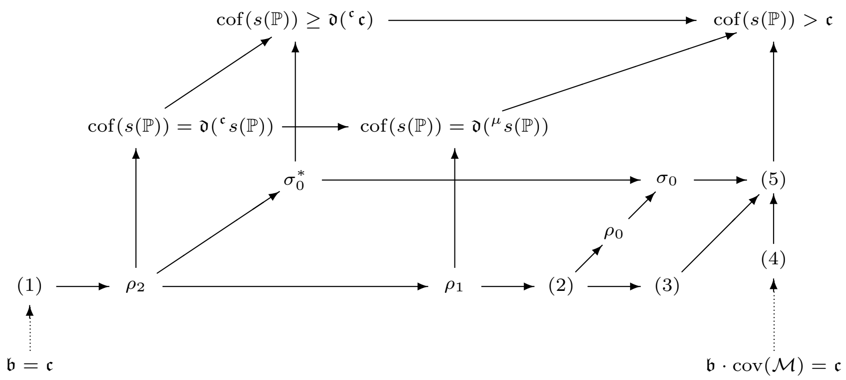

Figure 1. Implications between the assertions about $\mathbb{P}$ ($\sigma_2 \leftrightarrow \rho_2$ and $\sigma_1 \leftrightarrow \rho_1$). Dotted arrows are stated for $\mathbb{P}_{\mathbb{L}}$ only.

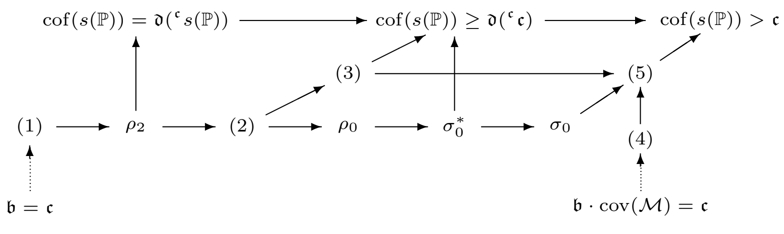

Figure 2. Implications between the assertions assuming $\mu = \mathfrak{c}$ ($\rho_2 \leftrightarrow \rho_1 \leftrightarrow \sigma_2 \leftrightarrow \sigma_1$). Dotted arrows are stated for $\mathbb{P}_{\mathbb{L}}$ only.

Corollary 1.3.

(i) $(\mathfrak{b} = \mathfrak{c}) \to (\mathrm{cof}(l^0) = \mathfrak{d}({}^{\mathfrak{c}}l^0))$.
(ii) $(\mathfrak{b} \cdot \mathrm{cov}(\mathcal{M}) = \mathfrak{c}) \to (\mathrm{cof}(l^0) > \mathfrak{c})$.

Proof. These are the implications of Figure 1 or Figure 2. □

Let $\mathcal{D} \subseteq l^0$ denote the σ-ideal on ${}^{\omega}\omega$ consisting of sets that are not strongly dominating. Recall that a set $A \subseteq {}^{\omega}\omega$ is strongly dominating if for every function $f : {}^{<\omega}\omega \to \omega$ there is $x \in A$ such that $x(k) > f(x{\restriction}k)$ for all but finitely many $k \in \omega$. Then $\mathrm{add}(\mathcal{D}) = \mathrm{cov}(\mathcal{D}) = \mathfrak{b}$ and $\mathrm{cof}(\mathcal{D}) = \mathrm{non}(\mathcal{D}) = \mathfrak{d}$ (see [4, Lemma 2.4]) and $\mathfrak{b} \leq \mathfrak{b} \cdot \mathrm{cov}(\mathcal{M}) \leq \mathfrak{d}$.

Corollary 1.4. $\mathrm{cov}(l^0) < \mathrm{cof}(l^0)$.

Proof. If $\mathrm{cov}(l^0) < \mathfrak{c}$, then $\mathrm{cov}(l^0) < \mathfrak{c} = \mathrm{non}(l^0) \leq \mathrm{cof}(l^0)$. If $\mathrm{cov}(l^0) = \mathfrak{c}$, then $\mathfrak{b} = \mathrm{cov}(\mathcal{D}) \geq \mathrm{cov}(l^0) = \mathfrak{c}$, and then by Corollary 1.3 (i), $\mathrm{cof}(l^0) = \mathfrak{d}({}^{\mathfrak{c}}l^0) > \mathfrak{c} = \mathrm{cov}(l^0)$. □

Example 1.5. Let W_κ be a generic extension of a transitive model V of ZFC by finite support iterations of κ many Cohen reals for an uncountable cardinal κ. Then $W_\kappa \models$ "$\mathrm{cov}(l^0) = \mathrm{cov}(s^0) = \mathfrak{b} = \omega_1$ and $\mathfrak{d} \geq \kappa$" (recall that s^0 is the Marczewski ideal). The equality $\mathrm{cov}(l^0) = \omega_1$ holds because whenever $u \in {}^{\omega}\omega$ is an unbounded real over W_α, then $\mathbb{R} \cap W_\alpha \in l^0$: Every Laver perfect set p has a Laver perfect subset $p' = \{x \in p : (\forall n \in \omega)\ x(n) \geq u(n)\}$ that contains no real from W_α. If C_α denotes the notion of forcing for adding α many Cohen reals, then $C_\kappa \simeq C_\kappa \times C_{\omega_1}$ for $\kappa \geq \omega_1$. Similarly the equality $\mathrm{cov}(s^0) = \omega_1$ holds in W_{ω_1}, and hence also in W_κ, because ${}^{\omega}2 \cap W_\alpha \in s^0$ in W_{ω_1} for every $\alpha < \omega_1$. We prove that ${}^{\omega}2 \cap W_\alpha \in s^0$ in W_{ω_1}. Let $p \subseteq {}^{\omega}2$ be arbitrary perfect set coded in W_{ω_1}. The Borel code of p belongs to W_β for some countable $\beta \geq \alpha$. Let $f : {}^{\omega}2 \times {}^{\omega}2 \to p$ be an homeomorphism coded in W_β and let $p' = f``({}^{\omega}2 \times \{r\})$ for an $r \in {}^{\omega}2 \setminus W_\beta$. Then $p' \cap ({}^{\omega}2 \cap W_\alpha) \subseteq p' \cap ({}^{\omega}2 \cap W_\beta) = \emptyset$ because $r \notin W_\beta$ can be defined from any element of p' by means of f.

Fact 1.6. Conditions (1) and (4) hold in ZFC for $\mathbb{P} = \mathbb{P}_{\mathbb{S}}$.

Proof. We prove that for every maximal antichain $A \subseteq \mathbb{P}_{\mathbb{S}}$ there exists a maximal antichain $\{p_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathbb{P}_{\mathbb{S}}$ refining A consisting of pairwise disjoint sets. (The term "refining family" is used here in the sense that every p_α is a subset of some $q \in A$. This differs from the already mentioned "refining sequence".) To prove this, let $\{r_\xi : \xi < \mathfrak{c}\}$ be an enumeration of $\mathbb{P}_{\mathbb{S}}$. By splitting one element of the partition if necessary we can assume without loss of generality that $|A| = \mathfrak{c}$. By induction on $\alpha < \mathfrak{c}$ define $\{p_\alpha : \alpha < \mathfrak{c}\}$ as follows: Let ξ_α be the least ξ such that r_ξ is incompatible with all p_β, $\beta < \alpha$, i.e., all $|r_\xi \cap p_\beta| \leq \omega$ for all $\beta < \alpha$ (such ξ exists because by the induction hypothesis $\{p_\beta : \beta < \alpha\}$ refines A and $|A| = \mathfrak{c}$). Chose $q \in A$ compatible with r_{ξ_α} and let $p_\alpha \in \mathbb{P}$ be a subset of $r_{\xi_\alpha} \cap q$ disjoint from the s^0-set $\bigcup_{\beta<\alpha}(r_{\xi_\alpha} \cap p_\beta)$. The sets p_α are pairwise disjoint by the construction. The sequence of ordinals ξ_α, $\alpha < \mathfrak{c}$ is strictly increasing, hence cofinal in $\mathfrak{c}$. Therefore every r_ξ is compatible with some p_α. Hence $\{p_\alpha : \alpha < \mathfrak{c}\}$ is a maximal antichain refining A with pairwise disjoint sets. This gives (1). One can easily verify that every selector for $\{p_\alpha : \alpha < \mathfrak{c}\}$ is an s^0-set that meets every $q \in A$. Then (4) follows, too, because every antichain can be extended to a maximal one. □

Therefore $\operatorname{cof}(s^0) = \mathfrak{d}({}^{\mathfrak{c}}s^0) \geq \mathfrak{d}({}^{\mathfrak{c}}\mathfrak{c}) > \mathfrak{c}$ holds in ZFC (in fact, the inequality $\operatorname{cof}(s^0) > \mathfrak{c}$ was observed by Khomskii [5]). For $\mathbb{P} = \mathbb{P}_{\mathbb{L}}$ in [2] the following was proved:

(ρ) There is a system $\{p_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathbb{P}$ of pairwise disjoint sets such that for every $q \in \mathbb{P}_{\mathbb{L}}$ there is $q' \subseteq q$ in $\mathbb{P}$ such that either there is $\alpha < \mathfrak{c}$ such that $q' \subseteq p_\alpha$ or $|q' \cap p_\alpha| \leq 1$ for all $\alpha < \mathfrak{c}$.

Lemma 1.7. $\rho \to \rho_2$.

Proof. Let $p = \langle p_\alpha : \alpha < \mathfrak{c}\rangle \in {}^{\mathfrak{c}}\mathbb{P}$ be a witness for (ρ). Let $\{q_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of the set $\{q \in \mathbb{P} : (\forall \alpha < \mathfrak{c})\ |q \cap p_\alpha| \leq 1\}$. For every $\alpha < \mathfrak{c}$, $|p_\alpha \cap \bigcup_{\beta\leq\alpha} q_\beta| < \mathfrak{c}$ and $|q_\alpha \cap \bigcup_{\beta<\alpha} p_\beta| < \mathfrak{c}$. Then for every $\alpha < \mathfrak{c}$ there are $r_\alpha, q'_\alpha \in \mathbb{P}$ such that $r_\alpha \subseteq p_\alpha \setminus \bigcup_{\beta\leq\alpha} q_\beta = \emptyset$ and $q'_\alpha \subseteq q_\alpha \setminus \bigcup_{\beta\leq\alpha} p_\beta = \emptyset$ because $\operatorname{cov}(s(\mathbb{P})) = \mathfrak{c}$. We claim that the sequence $\{r_\alpha : \alpha < \mathfrak{c}\}$ is a witness for (ρ_2). Let $q \in \mathbb{P}$ be given. If there is $q' \subseteq q$ in $\mathbb{P}$ such that $q' \subseteq p_\alpha$ for some α, then $\{\beta < \mathfrak{c} : q' \cap p_\beta \neq \emptyset\} = \{\alpha\}$. Otherwise by ($\rho$) there is $\alpha < \mathfrak{c}$ such that $q_\alpha \subseteq q$. Then $q'_\alpha \subseteq q$ and $q'_\alpha \cap \bigcup_{\beta<\mathfrak{c}} r_\beta \subseteq (q'_\alpha \cap \bigcup_{\beta\leq\alpha} p_\beta) \cup (q_\alpha \cap \bigcup_{\beta>\alpha} r_\beta) = \emptyset$, i.e., $\{\beta < \mathfrak{c} : q'_\alpha \cap p_\beta \neq \emptyset\} = \emptyset$. □

Corollary 1.8. $\operatorname{cof}(l^0) = \mathfrak{d}({}^{\mathfrak{c}}l^0) \geq \mathfrak{d}({}^{\mathfrak{c}}\mathfrak{c}) > \mathfrak{c}$. □

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2. Proofs

Lemma 2.1. $\rho_2 \to \rho_1 \to \rho_0$.

Proof. The implication $\rho_2 \to \rho_1$ is trivial. Assume that $p \in {}^{\mu}\mathbb{P}$ is a witness for ρ_1 and let $f : \mu \to \mathfrak{c}$ is a cofinal function. For every $\alpha < \mu$ choose a system $\{p_{\alpha,\xi} : \xi < f(\alpha)\} \subseteq \mathbb{P}$ of pairwise disjoint subsets of p_α and let $r = \langle r_\alpha : \alpha < \mathfrak{c}\rangle$ be a one-to-one enumeration of $\{p_{\alpha,\xi} : \alpha < \mu \text{ and } \xi < f(\alpha)\}$. Clearly r is a witness for ρ_0 and hence $\rho_1 \to \rho_0$ holds. □

Lemma 2.2. $\sigma_2 \to \sigma_1 \to \sigma_0$ *and* $\sigma_2 \to \sigma_0^*$.

Proof. If $p \in {}^{\mu}\mathbb{P}$ is a witness for σ_1, then any $p' \in {}^{\mathfrak{c}}\mathbb{P}$ obtained from p by splitting each member of p into $< \mathfrak{c}$ sets is a witness for σ_0. Therefore $\sigma_1 \to \sigma_0$ holds. The other implications are trivial. □

Lemma 2.3. $\sigma_2 \leftrightarrow \rho_2$, $\sigma_1 \leftrightarrow \rho_1$, $\rho_0 \to \sigma_0$.

Proof. We prove $\rho_2 \to \sigma_2$ and $\sigma_2 \to \rho_2$; proofs of other implications are similar.

Let $p \in {}^{\mathfrak{c}}\mathbb{P}$ be a witness for ρ_2 and let $A_\alpha \subseteq p_\alpha$ be arbitrary $s(\mathbb{P})$-sets for $\alpha < \mathfrak{c}$. We show that $A = \bigcup_{\alpha<\mathfrak{c}} A_\alpha$ is an $s(\mathbb{P})$-set and hence σ_2 holds. Let $q \in \mathbb{P}$ be arbitrary. There is $q' \subseteq q$ in $\mathbb{P}$ and $\alpha < \mathfrak{c}$ such that $q' \cap p_\beta = \emptyset$ for all $\beta \neq \alpha$. Then $A \cap q' = A_\alpha \cap q' \in s(\mathbb{P})$ and then there is $q'' \subseteq q'$ in $\mathbb{P}$ such that $q'' \cap A = \emptyset$. Therefore $A \in s(\mathbb{P})$.

Let $p \in {}^{\mathfrak{c}}\mathbb{P}$ be a witness for σ_2. By Theorem 3.4 there is a disjoint pseudo-refinement $r \in {}^{\mathfrak{c}}\mathbb{P}$ of p, i.e., for all $\alpha < \beta < \mathfrak{c}$, $r_\alpha \cap r_\beta \in s(\mathbb{P})$ and there is a one-to-one function $f : \mathfrak{c} \to \mathfrak{c}$ such that $r_\alpha \subseteq p_{f(\alpha)}$ for all $\alpha < \mathfrak{c}$. We verify that r is a witness for ρ_2. Let $q \in \mathbb{P}$. Since r is an antichain of $\mathbb{P}$ and $\mathbb{P}$ is separable, there are $q' \subseteq q$ in $\mathbb{P}$ and $\alpha < \mathfrak{c}$ such that $q' \cap r_\beta \in s(\mathbb{P})$ for all $\beta \neq \alpha$. Then

$q' \cap \bigcup_{\beta\neq\alpha} r_\beta \in s(\mathbb{P})$ by the choice of p. Therefore there is $q'' \subseteq q'$ in $\mathbb{P}$ such that $q'' \cap \bigcup_{\beta\neq\alpha} r_\beta = \emptyset$. □

The same proof as that of $\rho_0 \to \sigma_0$ gives the following:

Lemma 2.4. *If $\mathfrak{c}$ is regular, then $\rho_0 \to \sigma_0^*$.* □

Lemma 2.5. $(1) \to (\rho_2)$, $(\rho_1) \to (2) \to (\rho_0)$, $(2) \to (3) \to (5)$, $(4) \to (5)$, $(\sigma_0) \to (5)$.

Proof. The implications $(\rho_1) \to (2) \to (\rho_0)$ and $(3) \to (5)$ need some explanation; the other implications are trivial.

$(\rho_1) \to (2)$. Let $p \in {}^\mu\mathbb{P}$ be a witness for (ρ_1), i.e., the set $D = \{q \in \mathbb{P} : |\{\alpha < \mu : q \cap p_\alpha \neq \emptyset\}| \leq 1\}$ is dense in $\mathbb{P}$. By refining the sets p_α if necessary we can assume that $p_\alpha \in D$ and hence they are pairwise disjoint for $\alpha < \mu$. We can also assume that the antichain p is not maximal in $\mathbb{P}$. If $q \in D$ is incompatible with every p_α, then since q can meet at most one member of p and $\mathbb{P}$ is separable, there is $q' \subseteq q$ in $\mathbb{P}$ such that $q' \cap \bigcup_{\alpha<\mu} p_\alpha = \emptyset$. Hence there is a maximal antichain $A \supseteq \{p_\alpha : \alpha < \mu\}$ in D of cardinality $\mathfrak{c}$ such that $q \cap \bigcup_{\alpha<\mu} p_\alpha = \emptyset$ for all $q \in A \setminus \{p_\alpha : \alpha < \mu\}$. The enumeration of A in which $\{p_\alpha : \alpha < \mu\}$ is cofinal is a witness for (2).

$(2) \to (\rho_0)$. Let $\{r_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathbb{P}$ be a maximal antichain such that $(\forall \alpha < \mathfrak{c})(\exists r \in \mathbb{P})\ r \cap \bigcup_{\beta<\alpha} r_\beta = \emptyset$ and let $f \in {}^\mu\mathfrak{c}$ be an increasing cofinal function. By induction on $\alpha < \mu$ find $g(\alpha) < \mathfrak{c}$ and families $\{q_{\alpha,\xi} : \xi < f(\alpha)\} \subseteq \mathbb{P}$ of pairwise disjont sets such that $q_{\alpha,\xi} \subseteq r_{g(\alpha)} \setminus \bigcup_{\beta<f(\alpha)} r_\beta$. Let $p \in {}^{\mathfrak{c}}\mathbb{P}$ be a one-to-one enumeration of $\{q_{\alpha,\xi} : \alpha < \mu \text{ and } \xi < f(\alpha)\}$. Then p is a witness for ρ_0: If $q \in \mathbb{P}$ is given find $q' \subseteq q$ in $\mathbb{P}$ such that $q' \subseteq r_\beta$ for some $\beta < \mathfrak{c}$. Then $q' \cap q_{\alpha,\xi} = \emptyset$ whenever $f(\alpha) > \beta$.

$(3) \to (5)$. For $X \in s^+(\mathbb{P})$ let $\mathbb{P}(X) = \{p \in \mathbb{P} : (\forall q \in \mathbb{P})\ q \subseteq p \to q \cap X \neq \emptyset\}$. Assume that (3) holds and let $\{r_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathbb{P}$ be a maximal antichain such that the sets $X_\alpha = {}^\omega\omega \setminus \bigcup_{\beta<\alpha} r_\beta$ are in $s^+(\mathbb{P})$ for all $\alpha < \mathfrak{c}$. Using the maximality of the antichain, by induction construct $u \in {}^\mu\mathbb{P}$ and an increasing continuous cofinal function $f \in {}^\mu\mathfrak{c}$ with $f(0) = 0$ so that for every $\xi < \mu$, $u_\xi \in \mathbb{P}(X_{f(\xi)})$ and $u_\xi \subseteq r_\beta$ for some $\beta < f(\xi+1)$. Define $p \in {}^{\mathfrak{c}}\mathbb{P}$ by $p_\alpha = u_\xi$ for $\alpha \in [f(\xi), f(\xi+1))$ and $\xi < \mu$. Let $q \in {}^{\mathfrak{c}}\mathbb{P}$ be arbitrary such that $q \leq p$. Take any set $A = \{x_\alpha : \alpha < \mathfrak{c}\}$ with $x_\alpha \in q_\alpha \cap X_{f(\xi)}$ whenever $\alpha \in [f(\xi), f(\xi+1))$. Then $A \cap q_\alpha \neq \emptyset$ for all $\alpha < \mathfrak{c}$. We prove that $A \in s(\mathbb{P})$ and hence (5) holds. If $v \in \mathbb{P}$ there are β, ξ, and $v' \in \mathbb{P}$ such that $\beta < f(\xi)$ and $v' \subseteq v \cap r_\beta$. Then $A \cap v' \subseteq \{x_\alpha : \alpha < f(\xi)\}$ has cardinality $< \mathfrak{c}$ and hence there is $v'' \subseteq v'$ in $\mathbb{P}$ such that $A \cap v' = \emptyset$. □

Lemma 2.6. *Let $\mathbb{P} = \mathbb{P}_{\mathbb{L}}$.*

(i) $(\mathfrak{b} = \mathfrak{c}) \to (1)$.

(ii) $(\mathfrak{b} \cdot \mathrm{cov}(\mathcal{M}) = \mathfrak{c}) \to (4)$.

Proof. The proof of (1) and (4) for $\mathbb{P} = \mathbb{P}_{\mathbb{L}}$ under the assumption $\mathfrak{b} = \mathfrak{c}$ is same as that for the $\mathbb{P} = \mathbb{P}_{\mathbb{S}}$ in Fact 1.6. (Following the proof of Fact 1.6 with $\mathbb{P}_{\mathbb{L}}$ in the role of $\mathbb{P}_{\mathbb{S}}$, let ξ_α be the least ordinal such that $r_{\xi_\alpha} \cap p_\beta \in s(\mathbb{P}_{\mathbb{L}})$ for all $\beta < \alpha$. Then $\bigcup_{\beta<\alpha}(r_{\xi_\alpha} \cap p_\beta) \in s(\mathbb{P}_{\mathbb{L}})$ because $\mathrm{add}(\mathcal{D}) = \mathfrak{b}$ and we assume $\mathfrak{b} = \mathfrak{c}$. Therefore for some $q \in A$ we can find $p_\alpha \subseteq r_{\xi_\alpha} \cap q$ in $\mathbb{P}_{\mathbb{L}}$ disjoint from $\bigcup_{\beta<\alpha}(r_{\xi_\alpha} \cap p_\beta)$. This is the only distinct point of the proofs.)

Assume $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$. To verify (4) assume that $p \in {}^{\mathfrak{c}}\mathbb{P}_{\mathbb{L}}$ is an antichain and we find $A \in l^0$ such that $A \cap p_\alpha \neq \emptyset$ for all $\alpha < \mathfrak{c}$. Without loss of generality we can assume that p is a maximal antichain. For every $\alpha < \mathfrak{c}$ there is $x_\alpha \in p_\alpha \setminus \bigcup_{\beta<\alpha} p_\beta$

because $p_\alpha \cap p_\beta$ is nowhere dense in p_α whenever $\alpha \neq \beta$. Let $A = \{x_\alpha : \alpha < \mathfrak{c}\}$. We prove that $A \in l^0$. Let $q \in \mathbb{P}_\mathbb{L}$ be arbitrary. By maximality of the antichain p there is $\alpha < \mathfrak{c}$ and $q' \in \mathbb{P}_\mathbb{L}$ such that $q' \subseteq q \cap p_\alpha$. Then $q' \cap A \subseteq \{x_\beta : \beta \leq \alpha\}$, and since $\mathrm{non}(l^0) = \mathfrak{c}$, there is $q'' \in \mathbb{P}_\mathbb{L}$ such that $q'' \subseteq q'$ and $q'' \cap A = \emptyset$. □

The following lemma summarizes hypotheses under which $\mathrm{cof}(s(\mathbb{P})) > \mathfrak{c}$.

Lemma 2.7. *Denote* $\mu = \mathrm{cf}(\mathfrak{c})$.

(i) $\sigma_0^* \to \mathrm{cof}(s(\mathbb{P})) \geq \mathfrak{d}({}^\mathfrak{c}\mathfrak{c})$.
(ii) $\rho_2 \to \mathrm{cof}(s(\mathbb{P})) = \mathfrak{d}({}^\mathfrak{c}s(\mathbb{P}))$.
(iii) $\rho_1 \to \mathrm{cof}(s(\mathbb{P})) = \mathfrak{d}({}^\mu s(\mathbb{P}))$.
(iv) (5) $\to \mathrm{cof}(s(\mathbb{P})) > \mathfrak{c}$.
(v) *If* $\mathfrak{c} = \mu$, *then* (3) $\to \mathrm{cof}(s(\mathbb{P})) \geq \mathfrak{d}({}^\mathfrak{c}\mathfrak{c})$.

Proof. (i) Let $p \in {}^\mathfrak{c}\mathbb{P}$ be such that $\bigcup_{\alpha<\mathfrak{c}} A_\alpha \in s(\mathbb{P})$ whenever $(\forall \alpha < \mathfrak{c})\ A_\alpha \in [p_\alpha]^{<\mathfrak{c}}$. There is a pair of functions (Galois–Tukey embedding) $\varphi : {}^\mathfrak{c}\mathfrak{c} \to s(\mathbb{P})$ and $\psi : s(\mathbb{P}) \to {}^\mathfrak{c}\mathfrak{c}$ such that $\varphi(f) \subseteq A$ implies $f \leq \psi(A)$, and consequently, $\mathfrak{d}({}^\mathfrak{c}\mathfrak{c}) \leq \mathrm{cof}(s(\mathbb{P}))$. To see this fix an enumeration $\{x_\alpha : \alpha < \mathfrak{c}\}$ of ${}^\omega\omega$ and define $\varphi(f) = \bigcup_{\alpha<\mathfrak{c}}(p_\alpha \cap \{x_\beta : \beta < f(\alpha)\})$ and $\psi(A)(\alpha) = \min\{\beta < \mathfrak{c} : x_\beta \in p_\alpha \setminus A\}$.

(ii) Assume that $p \in {}^\mathfrak{c}\mathbb{P}$ is a witness for σ_2 (because $\rho_2 \leftrightarrow \sigma_2$ by Lemma 2.3). Then $\mathrm{cof}(s(\mathbb{P})) \geq \mathfrak{d}(\prod_{\alpha<\mathfrak{c}} s(\mathbb{P}){\restriction}p_\alpha) = \mathfrak{d}({}^\mathfrak{c}s(\mathbb{P})) \geq \mathrm{cof}(s(\mathbb{P}))$ because $s(\mathbb{P}){\restriction}p_\alpha = s(\mathbb{P}) \cap \mathcal{P}(p_\alpha)$ is isomorphic to $s(\mathbb{P})$ for every α. The proof of (iii) is same.

(iv) Let $p \in {}^\mathfrak{c}\mathbb{P}$ be a witness for (5). Let $\{A_\alpha : \alpha < \mathfrak{c}\}$ be arbitrary family of $s(\mathbb{P})$-sets of cardinality $\mathfrak{c}$. Find $q \in {}^\mathfrak{c}\mathbb{P}$ such that for every $\alpha < \mathfrak{c}$, $q_\alpha \subseteq p_\alpha$ and $q_\alpha \cap A_\alpha = \emptyset$. By (5) there is $A \in s(\mathbb{P})$ such that $(\forall \alpha < \mathfrak{c})\ A \cap q_\alpha \neq \emptyset$. Then A is not covered by any set from $\mathcal{A}$. This proves that $\mathrm{cof}(s(\mathbb{P})) > \mathfrak{c}$.

(v) Assume that (3) holds and let $\{p_\xi : \xi < \mathfrak{c}\} \subseteq \mathbb{P}$ be a maximal antichain in $\mathbb{P}$ such that all sets $Z_\beta = {}^\omega\omega \setminus \bigcup_{\xi<\beta} p_\xi$ for $\beta < \mathfrak{c}$ are in $s^+(\mathbb{P})$. By the maximality of the antichain for every $\beta < \mathfrak{c}$ there is $\xi \geq \beta$ such that $Z_\beta \cap p_\xi \in s^+(\mathbb{P})$. Hence, by induction we can define an increasing sequence of ordinals $\{\xi_\alpha : \alpha < \mu\}$ cofinal in $\mathfrak{c}$ such that the sets $X_\alpha = Z_{\xi_\alpha} \setminus Z_{\xi_{\alpha+1}}$ are in $s^+(\mathbb{P})$. Since $\mathrm{non}(s(\mathbb{P})) = \mathfrak{c}$, using the maximality of the antichain one can easily verify that $\bigcup_{\alpha<\mu} A_\alpha \in s(\mathbb{P})$ whenever $(\forall \alpha < \mu)\ A_\alpha \in [X_\alpha]^{<\mathfrak{c}}$. Then the same argument as in case (i) proves that $\mathrm{cof}(s(\mathbb{P})) \geq \mathfrak{d}({}^\mu\mathfrak{c})$. □

Question 2.8. Assumimng $\mathfrak{b} < \mathfrak{c}$, is $\mathrm{cof}(l^0) = \mathfrak{d}({}^\mathfrak{b}l^0)$? Notice that $\mathrm{cof}(l^0) = \mathfrak{d}({}^\kappa l^0)$ whenever $1 \leq \kappa < \mathfrak{b}$ and if $\mathfrak{b} = \mathfrak{c}$, then the equality holds also for $\kappa = \mathfrak{b}$.

3. Refinemens

Let B be a Boolean algebra. For $a \in B$ let $B{\restriction}a = \{x \in B : x \leq a\}$ be the relativization of B with respect to a and let $B^+ = B \setminus \{0\}$.

Let κ and λ be cardinal numbers and let $a = \langle a_\alpha : \alpha < \lambda\rangle$ and $b = \langle b_\alpha : \alpha < \lambda\rangle$ be any λ-sequences in B^+. We say that a is a disjoint sequence if $a_\alpha \wedge a_\beta = 0$ for $\beta < \alpha < \lambda$. We say that a is κ-disjoint if $(\forall \alpha < \lambda)\ |\{\beta < \lambda : a_\alpha \wedge a_\beta \neq \emptyset\}| < \kappa$ (in particular, 2-disjoint has the same meaning as disjoint and if $\lambda < \kappa$, then each λ-sequence is κ-disjoint). We say that b is a refinement of a, if $b_\alpha \leq a_\alpha$ for all $\alpha < \lambda$; we say that b is a pseudo-refinement of a, if there is a one-to-one function $f : \lambda \to \lambda$ such that $b_\alpha \leq a_{f(\alpha)}$ for all $\alpha < \lambda$.

Theorem 3.1 (Balcar–Vojtáš; [1, 6])**.** *Assume that κ is an infinite cardinal and for each $x \in B^+$ there is an antichain of $B{\restriction}x$ of cardinality κ^+. Then each κ-sequence in B^+ has a disjoint refinement.* □

This theorem can be expressed in the following form:

Theorem 3.2. *Assume that κ is a regular cardinal and for each $x \in B^+$ there is an antichain of $B{\restriction}x$ of cardinality κ. Then each κ-disjoint sequence in B^+ has a disjoint refinement.*

Proof. Let $\langle a_\alpha : \alpha < \lambda\rangle$ be a κ-disjoint sequence in B^+. The following are the lines from [6]. For $X \subseteq B$ and $a \in B$ let $X(a) = \{x \in X : x \wedge a \neq 0\}$. By induction on $\alpha < \lambda$ we construct a chain $\langle X_\alpha : \alpha < \lambda\rangle$ of antichains of B^+ increasing with respect to the inclusion such that for every $\alpha, \beta < \lambda$,

(i) $X_\alpha(a_\beta) = \emptyset$ or $|X_\alpha(a_\beta)| \geq \kappa$ and

(ii) $|X_{\alpha+1}(a_\alpha)| \geq \kappa$.

Put $X_\emptyset = \emptyset$ and let $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ for α limit. Assume that X_α has been constructed. If $|X_\alpha(a_\alpha)| = \kappa$, set $X_{\alpha+1} = X_\alpha$. Otherwise $X_\alpha(a_\alpha) = \emptyset$ and hence $a \wedge a_\alpha = 0$ for all $a \in X_\alpha$. Choose any antichain $Y \subseteq B{\restriction}a_\alpha$ of size κ and let $Y' = Y \setminus \bigcup\{Y(a_\beta) : a_\alpha \wedge a_\beta \neq 0 \text{ and } |Y(a_\beta)| < \kappa\}$. By regularity of κ and because a is κ-disjoint, $|Y'| = \kappa$. Define $X_{\alpha+1} = X_\alpha \cup Y'$. This finishes the construction.

Let $X = \bigcup_{\alpha<\lambda} X_\alpha$. Then $|X(a_\alpha)| = \kappa$ for all $\alpha < \lambda$. By induction on $\alpha < \kappa$ choose $x_\alpha \in X(a_\alpha) \setminus \{x_\beta : \beta < \alpha \text{ and } a_\alpha \wedge a_\beta \neq 0\}$ and define a disjoint refining sequence $\langle b_\alpha : \alpha < \lambda\rangle$ for $\langle a_\alpha : \alpha < \lambda\rangle$ by setting $b_\alpha = x_\alpha \wedge a_\alpha$. □

Theorem 3.3. *Assume that κ is an infinite cardinal and for each $x \in B^+$ there is an antichain of $B{\restriction}x$ of cardinality κ. Then each κ-sequence in B^+ has a disjoint pseudo-refinement.*

Proof. Let κ-sequence $a \in {}^\kappa(B^+)$ be fixed. There are several cases.

Case 1. κ is regular. We claim that there exists a disjoint sequence $d \in {}^\kappa(B^+)$ such that for every $S \in [\kappa]^{<\kappa}$ there are $\beta, \gamma \in \kappa \setminus S$ such that $d_\gamma \wedge a_\beta \neq 0$.

To prove the claim, for every $\alpha < \kappa$ choose an antichain $D_\alpha \subseteq (B{\restriction}a_\alpha)^+$ of size κ. Assuming that disjoint sequences $d \in {}^\kappa(B{\restriction}a_\alpha)^+$ for any $\alpha < \kappa$ have not the property in the claim (otherwise there is nothing to prove) we can find for every $\alpha < \kappa$ a set $D'_\alpha \in [D_\alpha]^{<\kappa}$ and an ordinal $g(\alpha) < \kappa$ such that $d \wedge a_\beta = 0$ for all $d \in D_\alpha \setminus D'_\alpha$ and $\beta \geq g(\alpha)$. Since κ is regular and $g(\alpha) > \alpha$ for all $\alpha < \kappa$ the function $f : \kappa \to \kappa$ inductively defined by $f(\alpha) = g(\sup f{\restriction}\alpha)$ for $\alpha < \kappa$ is strictly increasing. For $\alpha < \beta < \kappa$, because $f(\beta) \geq f(\alpha + 1) = g(f(\alpha))$, for all $d \in D_{f(\alpha)} \setminus D'_{f(\alpha)}$ and $e \in D_{f(\beta)} \setminus D'_{f(\beta)}$ we have $d \wedge e \leq d \wedge a_{f(\beta)} = 0$. It follows that $D = \bigcup_{\alpha<\kappa}(D_{f(\alpha)} \setminus D'_{f(\alpha)})$ is an antichain of size κ and each $a_{f(\alpha)}$ meets κ many elements of D. Therefore the antichain D (i.e., enumeration of D by κ) has the property in the claim and the proof of the claim is finished.

The claim allows inductively define one-to-one functions $f, g : \kappa \to \kappa$ by

$$f(\alpha) = \min\{\beta \in \kappa \setminus \operatorname{rng}(f{\restriction}\alpha) : (\exists \gamma \in \kappa \setminus \operatorname{rng}(g{\restriction}\alpha))\ d_\gamma \wedge a_\beta \neq 0\},$$
$$g(\alpha) = \min\{\gamma \in \kappa \setminus \operatorname{rng}(g{\restriction}\alpha) : d_\gamma \wedge a_{f(\alpha))} \neq 0\}.$$

Then the sequence $b \in {}^\kappa(B^+)$ defined by $b_\alpha = d_{g(\alpha)} \wedge a_{f(\alpha)}$ is disjoint and $b_\alpha \leq a_{f(\alpha)}$, i.e., b is a disjoint pseudo-refinement of a.

Case 2. κ is singular. Let $\mu = \operatorname{cf}(\kappa)$, and let $\langle \kappa_\xi : \xi < \mu\rangle$ be an increasing cofinal sequence of regular cardinals in κ bigger than μ. We consider two subcases.

Case 2a. There exists a disjoint sequence $d \in {}^{\lambda}(B^+)$ with $\lambda < \kappa$ such that for every $S' \in [\kappa]^{<\kappa}$ and every $T' \in [\lambda]^{<\mu}$ there are $\beta \in \kappa \setminus S'$ and $\gamma \in \lambda \setminus T'$ such that $d_\gamma \wedge a_\beta \neq 0$. For $\gamma < \lambda$ let $S_\gamma = \{\beta < \kappa : d_\gamma \wedge a_\beta \neq 0\}$, let $T = \{\gamma < \lambda : |S_\gamma| = \kappa\}$, and let $S = \bigcup_{\gamma \in \lambda \setminus T} S_\gamma$.

If $|T| \geq \mu$, let $\langle \gamma_\xi : \xi < \mu \rangle$ be any sequence of distinct elements of T. For $\xi < \mu$ we can choose $B_\xi \in [S_{\gamma_\xi}]^{\kappa_\xi}$ so that the sets B_ξ's are pairwise disjoint. This is possible because $\kappa_\xi > \sum_{\eta < \xi} \kappa_\eta$. By Balcar–Vojtáš theorem, for every $\xi < \mu$ there is a disjoint refinement $\langle b_\alpha : \alpha \in B_\xi \rangle$ of the sequence $\langle d_{\gamma_\xi} \wedge a_\alpha : \alpha \in B_\xi \rangle$. Then $\langle b_\alpha : \alpha \in \bigcup_{\xi < \mu} B_\xi \rangle$ is a disjoint pseudo-refinement of a.

If $|T| < \mu$, then $|S| = \kappa$ by the hypothesis for Case 2a. For $\gamma \in \lambda \setminus T$ let $B_\gamma = S_\gamma \setminus \bigcup \{S_\delta : \delta < \gamma \text{ and } \delta \in \lambda \setminus T\}$. Then $S = \bigcup_{\gamma \in \lambda \setminus T} B_\gamma$ is a disjoint union of sets of cardinalities $< \kappa$. By Balcar–Vojtáš theorem, for every $\gamma \in \lambda \setminus T$ there is a disjoint refinement $\langle b_\alpha : \alpha \in B_\gamma \rangle$ of the sequence $\langle d_\gamma \wedge a_\alpha : \alpha \in B_\gamma \rangle$. Then $\langle b_\alpha : \alpha \in S \rangle$ is a disjoint pseudo-refinement of a.

Case 2b. For every disjoint sequence $d \in {}^{\lambda}(B^+)$ with $\lambda < \kappa$ there exist $S \in [\kappa]^{<\kappa}$ and $T \in [\lambda]^{<\mu}$ such that for all $\beta \in \kappa \setminus S$ and $\gamma \in \lambda \setminus T$ we have $d_\gamma \wedge a_\beta = 0$.

By Balcar–Vojtáš theorem, for every $\xi < \mu$ let $\langle b_\alpha : \alpha \in \kappa_{\xi+1} \setminus \kappa_\xi \rangle$ be a disjoint refinement of $\langle a_\alpha : \alpha \in \kappa_{\xi+1} \setminus \kappa_\xi \rangle$. Find $S_\xi \in [\kappa]^{<\kappa}$ and $T_\xi \in [\kappa_{\xi+1} \setminus \kappa_\xi]^{<\mu}$ such that $b_\gamma \wedge a_\beta = 0$ for all $\beta \in \kappa \setminus S_\xi$ and $\gamma \in \kappa_{\xi+1} \setminus (\kappa_\xi \cup T_\xi)$. By induction construct a cofinal subset $X \subseteq \mu$ (the range of an increasing sequence) such that $|S_\eta| < \kappa_\xi$ whenever $\eta < \xi$ are both in X. For $\xi \in X$ let $B_\xi = \kappa_{\xi+1} \setminus (\kappa_\xi \cup T_\xi \cup \bigcup_{\eta < \xi,\, \eta \in X} S_\eta)$. If $\eta < \xi$ are both in X, then $B_\xi \subseteq \kappa \setminus S_\eta$ and $B_\eta \subseteq \kappa_{\eta+1} \setminus (\kappa_\eta \cup T_\eta)$, and therefore $b_\gamma \wedge b_\beta \leq b_\gamma \wedge a_\beta = 0$ for $\beta \in B_\xi$ and $\gamma \in B_\eta$. Then $b = \langle b_\alpha : \alpha \in \bigcup_{\xi \in X} B_\xi \rangle$ is a disjoint sequence in B^+ and $|b| = \sum_{\xi \in X} \kappa_{\xi+1} = \kappa$. Therefore b is a disjoint pseudo-refinement of a. □

A Boolean algebra B is said to be $(\nu, \cdot, \kappa)$-distributive if for every ν-sequence of antichains $A_\alpha \subseteq B^+$, $\alpha < \nu$ there exists a maximal antichain $A \subseteq B^+$ such that for every $x \in A$ and every $\alpha < \nu$, $|\{y \in A_\alpha : x \wedge y \neq 0\}| < \kappa$ (see [7]). We say that B is $(\nu, \cdot, \kappa)^*$-distributive if for every ν-sequence of antichains $A_\alpha \subseteq B^+$, $\alpha < \nu$ there exists a maximal antichain $A \subseteq B^+$ such that $|\{y \in \bigcup_{\alpha < \nu} A_\alpha : x \wedge y \neq 0\}| < \kappa$ for every $x \in A$.

Let S be an infinite set. For $f \in {}^{S}B$ let $\operatorname{supp}(f) = \{s \in S : f(s) > 0\}$. We say that $f \in {}^{S}B$ has a disjoint refinement if $f \restriction \operatorname{supp}(f)$ has a disjoint refinement. Assuming that $f \in {}^{S}B$ has no disjoint refinement we define

$$I(f) = \{X \subseteq S : f \restriction X \text{ has a disjoint refinement}\}.$$

Theorem 3.4. *Assume that κ is an infinite cardinal and for each $x \in B^+$ there is an antichain of $B \restriction x$ of cardinality κ. If $f \in {}^{S}B$ has no disjoint refinement, then $I(f)$ is an ideal on S such that $[S]^{<\kappa} \subseteq I(f)$ and $I(f) \cap [X]^\kappa \neq \emptyset$ for all $X \in [S]^\kappa$. If B is $(\nu, \cdot, \kappa)^*$-distributive, then $I(f)$ is a ν-complete ideal on S.*

Proof. Obviously $I(f)$ is closed for subsets. By Balcar–Vojtáš theorem $[S]^{<\kappa} \subseteq I(f)$ and by previous theorem $I(f) \cap [X]^\kappa \neq \emptyset$ for all $X \in [S]^\kappa$. Every Boolean algebra is $(\nu, \cdot, 2)$-distributive for $\nu < \omega$ and therefore the fact that $I(f)$ is an ideal is a special case of the claim about the additivity of $I(f)$.

Assume that B is $(\nu, \cdot, \kappa)^*$-distributive and let $X_i \in I(f)$ for $i < \nu$ be pairwise disjoint. We prove that the union $X = \bigcup_{i \in \omega} X_i$ belongs to $I(f)$. Without loss of generality we can assume that $X \subseteq \operatorname{supp}(f)$. Because the sets X_i's are pairwise disjoint there is $g \in {}^X(B^+)$ such that $g{\restriction}X_i$ is a disjoint refinement of $f{\restriction}X_i$ for $i < \nu$. By the distributivity property of B there is a maximal antichain $A \subseteq B^+$ such that for every $a \in A$ the set $Y_a = \{s \in X : g(s) \wedge a \neq 0\}$ has cardinality $< \kappa$. Let $\preceq$ be a well-ordering of A. The sets $Y'_a = Y_a \setminus \bigcup_{b \prec a} Y_b$ for $a \in A$ are pairwise disjoint and by maximality of A we have $X = \bigcup_{a\in A} Y_a = \bigcup_{a \in A} Y'_a$. Since $|Y'_a| < \kappa$, by Balcar–Vojtáš theorem there is a disjoint refinement $f_a : Y'_a \to B^+$ of $g{\restriction}Y'_a$ for every $a \in A$. It follows that $\bigcup_{a\in A} f_a \in {}^X(B^+)$ is a disjoint refinement of $f{\restriction}X$ and hence $X \in I(f)$. □

Example 3.5. Let $\mathcal{R}$ be the disjoint refinement ideal on ${}^\omega 2$, i.e., the ideal $I(f)$ for the Boolean algebra $B = \mathcal{P}(\omega)/\mathrm{fin}$ and $f : {}^\omega 2 \to B$ defined by $f(x) = \{n \in \omega : x(n) = 1\}/\mathrm{fin}$ for $x \in S$. Then $[{}^\omega 2]^{<\mathfrak{c}} \subseteq \mathcal{R}$, $\mathcal{R} \cap [X]^{\mathfrak{c}} \neq \emptyset$ for all $X \in [{}^\omega 2]^{\mathfrak{c}}$, and $\mathrm{add}(\mathcal{R}) \geq \mathfrak{h}$ (recall that $\mathfrak{h}$ is the least cardinal κ such that $\mathcal{P}(\omega)/\mathrm{fin}$ is not κ-distributive). Let us recall two other σ-ideals $\mathcal{I}_0 \subsetneq \mathfrak{P}_2$ on ${}^\omega 2$:

$$\mathfrak{P}_2 = \{A \subseteq {}^\omega 2 : (\forall a \in [\omega]^\omega)\ A{\restriction}a \neq {}^a 2\},$$
$$\mathcal{I}_0 = \{A \subseteq {}^\omega 2 : (\forall a \in [\omega]^\omega)(\exists b \in [a]^\omega)\ |A{\restriction}b| \leq \omega\},$$

where $A{\restriction}a = \{x{\restriction}a : x \in A\}$. The ideal $\mathfrak{P}_2$ was introduced by Rosłanowski [11] in connection to an infinite game of Mycielski. The ideals $\mathcal{I}_0$ and $\mathfrak{P}_2$ coincide on analytic sets and $\mathrm{add}(\mathcal{I}_0) \geq \mathfrak{h}$ (see [9]). By Balcar–Vojtáš theorem one can easily verify that

$$\mathcal{I}_0 \subseteq \{A \subseteq {}^\omega 2 : (\forall a \in [\omega]^\omega)(\exists b \in [a]^\omega)\ |A{\restriction}b| < \mathfrak{c}\} \subseteq \mathcal{R}.$$

On the other hand, $\mathcal{R} \setminus \mathfrak{P}_2 \neq \emptyset$, and moreover, there is a closed set $A \subseteq {}^\omega 2$ such that $A \in \mathcal{R}$ and $(\forall b \in [\omega]^\omega)(\exists c \in [b]^\omega)\ A{\restriction}c = {}^c 2$. To see this, let $P \subseteq [\omega]^\omega$ be a compact almost disjoint family. For $a \in P$ let $A_a = \{x \in {}^\omega 2 : (\forall n \in a)\ x(n) = 1\}$ and let $A = \bigcup_{a \in P} A_a$. Then the set A is a closed subset of ${}^\omega 2$ (A is a projection of the compact set $\{(a, x) : a \in P \text{ and } x \in A_a\}$) and $A \in \mathcal{R}$ (because for every $x \in A$ there is $a \in P$ such that $a \subseteq^* \{n \in \omega : x(n) = 1\}$ and for every $a \in A$ there is an almost disjoint family of subsets of a of cardinality $\mathfrak{c}$). Let $b \in [\omega]^\omega$ be given. Find $a \in P$ such that $c = b \setminus a$ is infinite. Then $A{\restriction}c \supseteq A_a{\restriction}c = {}^c 2$.

Example 3.6. Every $\mathfrak{c}$-sequence $B \in {}^{\mathfrak{c}}l^+$ has a disjoint refinement by l^+-sets. Assume that $B_\alpha \in l^+$ for all $\alpha < \mathfrak{c}$. For every $\alpha < \mathfrak{c}$ choose $p_\alpha \in \mathbb{P}(B_\alpha)$, i.e., $|q \cap B_\alpha| = \mathfrak{c}$ for all $q \in \mathbb{P}$ with $q \subseteq p_\alpha$. Let $\{p_{\alpha,\beta} : \beta < \mathfrak{c}\}$ be an enumeration of the set $\{q \in \mathbb{P} : q \subseteq p_\alpha\}$ and let $\pi : \mathfrak{c} \times \mathfrak{c} \to \mathfrak{c}$ be one-to-one. By induction on $\pi(\alpha, \beta)$ choose $x_{\alpha,\beta} \in B_\alpha \cap p_{\alpha,\beta} \setminus \{x_{\alpha',\beta'} : \pi(\alpha', \beta') < \pi(\alpha, \beta)\}$. The sets $B'_\alpha = \{x_{\alpha,\beta} : \beta < \mathfrak{c}\}$ are pairwise disjoint l^+-sets and $B'_\alpha \subseteq B_\alpha$.

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