SETS OF POINTS OF SYMMETRIC CONTINUITY

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Abstract. We study the sets of symmetric continuity of real functions in connection with the sets of continuity. We prove that sets of reals of cardinality $< \mathfrak{p}$ and subsets of weakly independent G_δ sets of reals are sets of symmetric continuity. The latter strengthens a similar result of Darji. We improve results of Fried and Belna saying that the set of points of symmetric continuity of a real function that are not continuity points does not contain a nonmeager set with Baire property and has inner measure zero by introducing another notion of smallness below meager and measure zero.

1. Introduction

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be symmetrically continuous at a point $x \in \mathbb{R}$, if $\lim_{h\to 0}(f(x+h) - f(x-h)) = 0$. The symmetric continuity was introduced by Hausdorff [6]. The reader can find an overview of results on symmetric continuity in [15]. Denote

$$\mathrm{SC}(f) = \{x \in \mathbb{R} : \lim_{h\to 0}(f(x+h) - f(x-h)) = 0\}.$$

There is an unsolved problem of Marcus [15, Problem 14] to find a characterization of sets of the form $\mathrm{SC}(f)$. Marcus [11] proved that every G_δ set has this form. Darji [2] proved that whenever $C \subseteq \mathbb{R}$ is a perfect linearly independent set over $\mathbb{Q}$, then for every set $A \subseteq C$ there is a function f such that $\mathrm{SC}(f) = A$. In particular this means that there is no restriction on projective complexity of sets of symmetric continuity.

If X is a separable metric space and $f : X \to \mathbb{R}$ then let

$$\mathrm{C}(f) = \{x \in \mathbb{R} : \lim_{y\to x} f(y) = f(x)\},$$
$$\mathrm{C}^+(f) = \{x \in \mathbb{R} : \lim_{y\to x} f(y) \in \mathbb{R}\}.$$

Clearly $\mathrm{C}(f) \subseteq \mathrm{C}^+(f)$ and for $f : \mathbb{R} \to \mathbb{R}$ also $\mathrm{C}^+(f) \subseteq \mathrm{SC}(f)$.

Example 1.1 (Hausdorff [6]). For every countable set $Q \subseteq \mathbb{R}$ there is $f : \mathbb{R} \to \mathbb{R}$ such that $\mathrm{SC}(f) = \mathrm{C}^+(f) = \mathbb{R}$ and $\mathrm{C}(f) = \mathbb{R} \setminus Q$. Let $\{r_n : n \in \omega\}$ be an enumeration of Q possibly with repetitions (if Q is finite). Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2^{-n}$ if $x = r_n$ and $x \notin \{r_i : i < n\}$ and $f(x) = 0$ for $x \in \mathbb{R} \setminus Q$.

Example 1.2. Let $Q \subseteq \mathbb{R}$ be an infinite countable set without isolated points. There is a function $f : \mathbb{R} \to \mathbb{R}$ such that $\mathrm{C}(f) = \mathrm{C}^+(f) = \mathrm{SC}(f) = \mathbb{R} \setminus Q$.

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Proof. Fix a one-to-one enumeration $\{r_n : n \in \omega\}$ of Q and choose a disjoint system $\{R_n : n \in \omega\} \subseteq [Q]^\omega$ of sets of order-type ω or ω^* such that r_n is the limit point of R_n and $r_n \notin R_n$ for all $n \in \omega$; hence $\mathrm{cl}(R_n) = R_n \cup \{r_n\}$ and either $r_n = \inf R_n$ or $r_n = \sup R_n$. Define $f(x) = 2^{-n}$ for $x \in R_n$ and $f(x) = 0$ for $x \in \mathbb{R} \setminus \bigcup_{n\in\omega} R_n$. It is easy to see that $\mathrm{SC}(f) \subseteq \mathbb{R} \setminus Q$. We show that $\mathbb{R} \setminus Q \subseteq \mathrm{C}(f)$. For $x \in \mathbb{R} \setminus Q$ and $n \in \omega$ let $\delta > 0$ be the distance of x from $\bigcup_{m\leq n} \mathrm{cl}(R_m)$. Then $|f(y)| < 2^{-n}$ whenever $|x - y| < \delta$: If $y \notin \bigcup_{m\in\omega} R_m$, then $f(y) = 0$, and if $y \in R_m$ for some $m \in \omega$ and $|x - y| < \delta$, then $m > n$ and hence $f(y) = 2^{-m} < 2^{-n}$. □

It is a well-known fact that a set A in a Polish space is G_δ if and only if $\mathrm{C}(f) = A$ for some f. In Section 2 we prove several variants of this fact, and in particular we prove (Theorem 2.1) that G_δ sets in $\mathbb{R}$ are characterized by any of these properties: $(\exists f)\ \mathrm{C}^+(f) = A$, $(\exists f)\ \mathrm{C}(f) = \mathrm{C}^+(f) = A$, $(\exists f)\ \mathrm{C}^+(f) = \mathrm{SC}(f) = A$.

Let $\mathcal{SC}$ denote the family of all sets of the form $\mathrm{SC}(f)$ for a function $f : \mathbb{R} \to \mathbb{R}$. It is quite easy to see that subgroups of $(\mathbb{R}, +)$ and their translations belong to $\mathcal{SC}$ (see [14] or [12]). In Section 3 we show that every set of reals of cardinality $< \mathfrak{p}$ is in $\mathcal{SC}$ and if C is a weakly independent G_δ set, then $\mathcal{P}(C) \subseteq \mathcal{SC}$. The latter slightly improves the mentioned Darji's result. We also make a discussion about the sets of symmetric continuity for two-valued functions.

The first nontrivial result about the sets of symmetric continuity was the Fried's result [5] saying that $\mathrm{C}(f)$ is comeager whenever $\mathrm{SC}(f)$ is comeager. This means that $\mathrm{SC}(f) \setminus \mathrm{C}(f)$ does not contain a nonmeager Borel set. Belna [1] proved that the set $\mathrm{SC}(f) \setminus \mathrm{C}(f)$ has inner measure zero. One can easily observe that Belna's arguments work also in the case of category. In Section 4 we introduce a sequence of notions of smallness (stronger than 'meager' and 'measure zero') that depend on complexity of sets so that the slightly improved Belna's arguments ensure that subsets of $\mathrm{SC}(f) \setminus \mathrm{C}(f)$ of given complexity are small. As an application we prove that $\mathrm{SC}(f) \setminus \mathrm{C}(f)$ cannot contain the Cantor ternary set. This improves the result of Marcus which assumes $\mathrm{SC}(f) = \mathbb{R}$.

In Section 5 we look for conditions ensuring that a set of reals A has a perfect nowhere dense subset P such that $P - P$ contains an interval.

If $r \in \mathbb{R}$ and A and B are sets of reals then $A - B = \{a - b : a \in A \text{ and } b \in B\}$, $A + B = \{a + b : a \in A \text{ and } b \in B\}$, $rA = Ar = \{ra : a \in A\}$, and $A \setminus B$ is the set minus operation.

2. Continuity and symmetric continuity

We prove the following theorem.

Theorem 2.1. *Let $A \subseteq \mathbb{R}$. The following conditions are equivalent:*

(1) *A is a G_δ set.*
(2) *There is a Borel function $f : \mathbb{R} \to \mathbb{R}$ such that $\mathrm{C}(f) = \mathrm{C}^+(f) = A$.*
(3) *There is a function $f : \mathbb{R} \to \mathbb{R}$ such that $\mathrm{C}(f) = \mathrm{C}^+(f) = A$.*
(4) *There is a function $f : \mathbb{R} \to \mathbb{R}$ such that $\mathrm{C}(f) = A$.*
(5) *There is a function $f : \mathbb{R} \to \mathbb{R}$ such that $\mathrm{C}^+(f) = A$.*
(6) *There is $f : \mathbb{R} \to \mathbb{R}$ such that $\mathrm{C}^+(f) = \mathrm{SC}(f) = A$.*

Proof. The implications (1) → (6) and (1) → (2) hold by Lemma 2.2 and Theorem 2.6, respectively.

The implications (2) → (3) → (4), (3) → (5), and (6) → (5) are trivial.

The implications (4) → (1) and (5) → (1) hold by Lemma 2.5. □

We say that a set A in a space X is everywhere uncountable, if $U \cap A$ is uncountable whenever U is an open set such that $U \cap A \neq \emptyset$. For example, $\emptyset$ is everywhere uncountable. We say that a G_δ set $A \subseteq X$ is a strong G_δ set, if $X \setminus A$ is everywhere uncountable.

Lemma 2.2. *For any function $g : \mathbb{R} \to \mathbb{R}$ and any G_δ set $A \subseteq \mathbb{R}$ there is a function $f : \mathbb{R} \to \mathbb{R}$ such that $\mathrm{C}^+(f) = \mathrm{C}^+(g) \cap A$ and $\mathrm{SC}(f) = \mathrm{SC}(g) \cap A$.*

Proof. Assume that $A = \bigcap_{n\in\omega} U_n$ with $U_0 = \mathbb{R}$ and $U_{n+1} \subseteq U_n$ open. Let $V_n = \bigcup\{U : U \text{ is open and } |U \setminus U_n| \leq \omega\}$. The set $S = \bigcup_{n\in\omega}(V_n \setminus U_n)$ is a countable subset of $\mathbb{R} \setminus A$. Let $\{r_n : n < k\}$ be a one-to-one enumeration of S where $k = |S| \leq \omega$. By definition of the set S it follows that the sets $U_n \setminus (U_{n+1} \cup S)$ are everywhere uncountable and hence they are everywhere of cardinality $\mathfrak{c}$. Therefore by transfinite induction (through a well-ordering of $\mathbb{R}$ of type $\mathfrak{c}$) we can choose for each $x \in \mathbb{R}$ a sequence of reals $h_{x,m}$ for $m \in \omega$ such that

(i) $|h_{x,m}| \searrow 0$ for $m \in \omega$,
(ii) the reals $x \pm h_{x,m}$ for $x \in \mathbb{R}$ and $m \in \omega$ are all distinct, and
(iii) if $x \in U_n \setminus (U_{n+1} \cup S)$, then $x + h_{x,m} \in U_n \setminus (U_{n+1} \cup S)$ for all $m \in \omega$.

Let $\delta_{x,m} = 1$ if $g(x - h_{x,m}) \leq g(x + h_{x,m})$ and let $\delta_{x,m} = -1$ if $g(x - h_{x,m}) > g(x + h_{x,m})$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x + h_{x,m}) = g(x + h_{x,m}) + \delta_{x,m}2^{-n}, \text{ if } x \in U_n \setminus (U_{n+1} \cup S) \text{ and } m \in \omega,$$
$$f(r_n + h_{r_n,m}) = g(r_n + h_{r_n,m}) + \delta_{r_n,m}2^{-n}, \text{ if } n < k \text{ and } m \in \omega,$$
$$f(x) = g(x), \text{ otherwise.}$$

For each $x \in \mathbb{R} \setminus A$ there is $n \in \omega$ such that either $x \in U_n \setminus (U_{n+1} \setminus S)$ or $x = r_n$, and hence $|f(x + h_{x,m}) - f(x - h_{x,m})| = |g(x + h_{x,m}) - g(x - h_{x,m})| + 2^{-n} \geq 2^{-n}$ for all $m \in \omega$. Therefore $\mathrm{C}^+(f) \subseteq \mathrm{SC}(f) \subseteq A$.

Let $x \in A$ and $n \in \omega$ be arbitrary. Because $x \notin S$, there is $\delta > 0$ such that $B(x, \delta) \subseteq U_n$ and $B(x, \delta) \setminus \{x\}$ is disjoint from the closed set $\{r_i, r_i + h_{r_i,m} : m \in \omega$ and $i < \min\{n, k\}\}$. Let $y \in B(x, \delta) \setminus \{x\}$. If $y = z + h_{z,m}$ for some $z \in \mathbb{R} \setminus A$ and $m \in \omega$, then there is $i \geq n$ such that either $z \in U_i \setminus (U_{i+1} \cup S)$ or $z = r_i$ and hence $|f(y) - g(y)| = 2^{-i} \leq 2^{-n}$; otherwise $|f(y) - g(y)| = 0 \leq 2^{-n}$. This argument proves that $\mathrm{C}^+(f) \cap A = \mathrm{C}^+(g) \cap A$ and $\mathrm{SC}(f) \cap A = \mathrm{SC}(g) \cap A$. Consequently, $\mathrm{C}^+(f) = \mathrm{C}^+(g) \cap A$ and $\mathrm{SC}(f) = \mathrm{SC}(g) \cap A$. □

Lemma 2.3. *A set A in a Polish space X is a strong G_δ set if and only if there is a decreasing sequence of open sets U_n for $n \in \omega$ such that $A = \bigcap_{n\in\omega} U_n$, $U_0 = X$, and $U_n \setminus U_{n+1}$ is everywhere uncountable for all $n \in \omega$.*

Proof. We prove this claim: If A is a strong G_δ set in a Polish space X and F is a closed set disjoint from A, then there is a perfect set $H \supseteq F$ disjoint from A.

For every $x \in X \setminus A$ and $\varepsilon > 0$, there is a perfect set $P \subseteq X \setminus A$ of diameter $< \varepsilon$ such that $x \in P$. To see this, for every $n \in \omega$ choose a perfect set $P_n \subseteq B(x, 2^{-(n+1)}\varepsilon) \setminus A$ and take $P = \{x\} \cup \bigcup_{n\in\omega} P_n$. This is possible by applying the perfect set theorem because $X \setminus A$ is Borel and everywhere uncountable. Now let F be a closed set in X disjoint from A. The set $Q = \bigcup\{U \cap F : U \text{ is open and } |U \cap F| \leq \omega\}$ is countable because X has a countable base. Let $\{a_n : n \in \omega\} \subseteq F$ be a sequence containing all elements of Q. For every $n \in \omega$ choose a perfect set $H_n \subseteq X \setminus A$ of diameter $< 2^{-n}$ such that $x_n \in H_n$. The set $H = F \cup \bigcup_{n\in\omega} H_n$ is perfect, $F \subseteq H$, and $A \cap H = \emptyset$. The claim is proved.

Assume that A is a strong G_δ set and $A = \bigcap_{n\in\omega} V_n$ with $V_0 = X$ and V_n open. By induction we construct a decreasing sequence of open sets U_n such that $A \subseteq U_n \subseteq V_n$ and $U_n \setminus U_{n+1}$ is everywhere uncountable. Assume that U_n has been constructed. Let $F = U_n \setminus V_{n+1}$. If $F = \emptyset$, set $U_{n+1} = U_n$. Otherwise, U_n is a Polish space in which A is a strong G_δ set and F is a closed set disjoint from A. By the above claim there is a perfect set $H \supseteq F$ (relatively in U_n) disjoint from A. Set $U_{n+1} = U_n \setminus H$.

Conversely, assume that $A = \bigcap_{n\in\omega} U_n$ satisfies conditions of the lemma and let V be any open set such that $V \cap (X \setminus A) \neq \emptyset$. Then $V \cap (X \setminus A)$ is uncountable because $V \cap (X \setminus A) = V \cap \bigcup_{n\in\omega}(U_n \setminus U_{n+1})$. □

Lemma 2.4. *Let X be a Polish space. For every G_δ set $A \subseteq X$ such that the subspace $X \setminus A$ has no isolated points there is a Borel function $f : X \to \mathbb{R}$ such that $\mathrm{C}(f) = \mathrm{C}^+(f) = A$.*

Proof. The case $A = X$ is trivial. Therefore we assume that $A \neq X$. Let $V = \bigcup\{U : U \text{ is open and } |U \setminus A| \leq \omega\}$. Then V is open, $|V \setminus A| \leq \omega$, $V \subseteq \mathrm{cl}(A)$, and $A \cup V$ is a strong G_δ set. Therefore $A \cup V = \bigcap_{n\in\omega} U_n$ where $U_0 = X$ and $U_{n+1} \subseteq U_n$ are open such that $U_n \setminus U_{n+1}$ are everywhere uncountable for all $n \in \omega$. Let $\mathcal{B}$ be the countable system of all nonempty sets of the form $I \cap (U_n \setminus U_{n+1})$ where I is a basic open set and $n \in \omega$. Then $|\mathcal{B}| = \omega$ whenever $\mathcal{B} \neq \emptyset$. Choose $Q, S \subseteq X \setminus A$ satisfying the following conditions:

(i) Q is an infinite countable set dense in itself containing $V \setminus A$.
(ii) S is a countable set disjoint from Q such that $B \cap S \neq \emptyset$ for all $B \in \mathcal{B}$.

Then $S = \emptyset$, whenever $\mathcal{B} = \emptyset$ and $B \setminus (Q \cup S) \neq \emptyset$ for all $B \in \mathcal{B}$ because the sets in $\mathcal{B}$ are uncountable. Fix a one-to-one enumeration $\{r_n : n \in \omega\}$ of Q and choose a disjoint system of discrete sets $\{R_n : n \in \omega\} \subseteq [Q]^\omega$ such that r_n is the unique limit point of R_n for all $n \in \omega$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 2^{-n}, & \text{if } x \in R_n \text{ and } n \in \omega, \\ 2^{-n}, & \text{if } x \in (U_n \setminus U_{n+1}) \cap S, \\ 2^{-n-1}, & \text{if } x \in (U_n \setminus U_{n+1}) \setminus (Q \cup S), \\ 0, & \text{otherwise.} \end{cases}$$

If $x \in V \setminus A$, then $x = r_n$ for some $n \in \omega$ and $x \notin \mathrm{C}^+(f)$ because $f{\restriction}R_n$ has the constant value 2^{-n} while all neighbourhoods of x contain zero-points of f since $x \in \mathrm{cl}(A)$. If $x \in U_n \setminus U_{n+1}$ for some $n \in \omega$, then $x \notin \mathrm{C}^+(f)$ because by the choice of S it follows that in every neighbourhood of x the function f has both values 2^{-n} and 2^{-n-1}. Therefore $\mathrm{C}(f) \subseteq \mathrm{C}^+(f) \subseteq A$. Let $x \in A$ and $n \in \omega$. Let U be an open neighbourhood of x such that $U \subseteq U_n$ and U is disjoint from the closed set $\bigcup_{m<n} R_m \cup \{r_m\}$. Then $|f(y)| \leq 2^{-n}$ for all $y \in U$ by definition of f. It follows that $A \subseteq \mathrm{C}(f)$ and hence $\mathrm{C}(f) = \mathrm{C}^+(f) = A$. □

Lemma 2.5. *Let X be a separable metric space and $f : X \to \mathbb{R}$. Then $\mathrm{C}(f) \subseteq \mathrm{C}^+(f)$ are G_δ sets with the countable difference $\mathrm{C}^+(f) \setminus \mathrm{C}(f)$.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be arbitrary. Like in [10], for $n \in \omega$ let

$$U_n = \bigcup\{V : V \text{ is open and } \mathrm{diam}(f(V)) \leq 2^{-n}\},$$
$$U_n^+ = \bigcup\{V : V \text{ is open and } (\exists x \in X)\ \mathrm{diam}(f(V \setminus \{x\})) \leq 2^{-n}\}.$$

Then $\mathrm{C}(f) = \bigcap_{n\in\omega} U_n$ and we prove that $\mathrm{C}^+(f) = \bigcap_{n\in\omega} U_n^+$.

If $x \in \mathrm{C}^+(f)$, then for every $n \in \omega$ there exists $\delta > 0$ such that $B(x,\delta) \subseteq U_n^+$ and hence $x \in \bigcap_{n\in\omega} U_n^+$. Assume that $x \in \bigcap_{n\in\omega} U_n^+$. Then for each $n \in \omega$ there are an open neighbourhood V_n of x and $x_n \in X$ such that $\operatorname{diam}(f(V_n \setminus \{x_n\})) \leq 2^{-n}$. We can choose V_n so that $x_n \notin V_n$ whenever $x_n \neq x$ and hence $\operatorname{diam}(f(V_n \setminus \{x\})) \leq 2^{-n}$ for all $n \in \omega$ and hence $x \in \mathrm{C}^+(f)$.

Obviously, $U_n \subseteq U_n^+$. If $x \in U_n^+ \setminus U_n$, then there is a basic open neighbourhood V of x such that $V \setminus \{x\} \subseteq U_n$. It follows that $U_n^+ \setminus U_n$ is discrete and hence countable for all $n \in \omega$ because X has a countable base. Consequently $\mathrm{C}^+(f) \setminus \mathrm{C}(f)$ is countable because $(\bigcap_{n\in\omega} U_n^+) \setminus (\bigcap_{m\in\omega} U_m) = \bigcup_{m\in\omega}(\bigcap_{n\in\omega} U_n^+) \setminus U_m$. □

Theorem 2.6. *Let A be a subset of a Polish space X. The following conditions are equivalent:*

1. *A is a G_δ set containing all isolated points of X.*
2. *There is a Borel function $f : X \to \mathbb{R}$ such that $\mathrm{C}(f) = \mathrm{C}^+(f) = A$.*

Proof. The implication (2) → (1) holds by Lemma 2.5.

(1) → (2). For a non-isolated point $r \in X$ and $\varepsilon > 0$ let $g_{r,\varepsilon} : X \setminus \{r\} \to [-1,1]$ be a continuous function such that $g_{r,\varepsilon}(x) = 0$ for $x \in X \setminus B(r,\varepsilon)$ and for some sequence $\{x_n\}_{n\in\omega}$ of elements of $B(r,\varepsilon)$ converging to r, $g_{r,\varepsilon}(x_n) = (-1)^n$ for all $n \in \omega$ (apply the Tietze extension theorem, see [10]).

Let $A \subseteq X$ be a G_δ set containing all isloated points of X. By transfinite induction we define

(i) $A_\alpha = A \cup \bigcup_{\beta<\alpha} E_\beta$,
(ii) $E_\alpha = \{x \in X : x$ is an isolated point of $X \setminus A_\alpha\}$, and
(iii) positive reals ε_x for $x \in E_\alpha$ such that $B(x,\varepsilon_x) \setminus A_\alpha = \{x\}$ for $x \in E_\alpha$ and the open balls $B(x, 2\varepsilon_x)$ for $x \in E_\alpha$ are pairwise disjoint.

This is possible because X has a countable base of open sets and by the same reason there is a countable ordinal α_0 such that $A_{\alpha_0+1} = A_{\alpha_0}$, i.e., $X \setminus A_{\alpha_0}$ has no isolated points. Every A_α is a G_δ set because $A_\alpha = A \cup \bigcup_{\beta<\alpha}\bigcup_{x\in E_\beta} B(x,\varepsilon_x)$. In particular, A_{α_0} is a G_δ set such that $X \setminus A_{\alpha_0}$ has no isolated points. By Lemma 2.4 there is $f_0 : X \to \mathbb{R}$ such that $\mathrm{C}(f_0) = \mathrm{C}^+(f_0) = A_{\alpha_0}$.

We define $f : X \to \mathbb{R}$ such that $\mathrm{C}(f) = \mathrm{C}^+(f) = A$. If $\alpha_0 = 0$, then $A = A_{\alpha_0}$ and $f = f_0$ works. Let $\alpha_0 > 0$. Denote $E = \bigcup_{\alpha<\alpha_0} E_\alpha$. For each $x \in E_\alpha$ for $0 < \alpha < \alpha_0$ there is a sequence of points in $\bigcup_{\beta<\alpha} E_\beta$ converging to x. It follows that for every $x \in E \setminus E_0$ there exists a convergent sequence of points in E_0 with limit x. Since E is countable it is possible to find a system of pairwise disjoint sets $R_x \in [E_0]^\omega$ for $x \in E \setminus E_0$ such that $\operatorname{cl}(R_x) = R_x \cup \{x\}$. By removing infinitely members from each R_x if necessary we can assume that the set $E_0' = \bigcup\{R_x : x \in E \setminus E_0\}$ is infinite whenever E_0 is infinite (this helps to avoid the consideration of various subcases concerned with the cardinalities of the sets $E \setminus E_0$ and E_0; notice that $E \setminus E_0 = \emptyset$, if E_0 is finite). For every $x \in E_0'$ let $R_x = \{x\}$. Let $\{R_n : n < |E_0|\}$ be a one-to-one enumeration of the partition $\{R_x : x \in (E \setminus E_0) \cup E_0'\}$ of the set E_0. Define

$$f(x) = \begin{cases} f_0(x) + 2^{-n} g_{r,\varepsilon_r}(x), & \text{if } r \in R_n \text{ and } x \in B(r, 2\varepsilon_r) \setminus \{r\}, \\ f_0(x), & \text{otherwise.} \end{cases}$$

We show that $\mathrm{C}(f) = \mathrm{C}^+(f) = A$.

Let $x \in E$. There is $n < |E_0|$ such that either $x \in R_n$ or $x \in \mathrm{cl}(R_n) \setminus R_n$. f_0 is continuous on $B(x, \varepsilon_x)$ because $B(x, \varepsilon_x) \subseteq A_{\alpha_0}$. If $x \in R_n$, then (by definition of f and g_{x,ε_x}) each neighbourhood of x contains some y and z such that $f(y) - f_0(y) = -2^{-n}$ and $f(z) - f_0(z) = 2^{-n}$. If $x \in \mathrm{cl}(R_n) \setminus R_n$, then each neighbourhood of x contains an element of R_n together with its y's and z's as in the previous case. Therefore $x \notin C^+(f)$.

Let $x \in X \setminus E$. Either there are $n < |E_0|$ and $r \in R_n$ such that $x \in B(r, 2\varepsilon_r) \setminus \{r\}$, or $x \in X \setminus \bigcup_{n<|E_0|} \bigcup_{r \in R_n} B(r, 2\varepsilon_r)$. In the former case, $x \in \mathrm{C}(f) \leftrightarrow x \in \mathrm{C}(f_0)$ and $x \in \mathrm{C}^+(f) \leftrightarrow x \in \mathrm{C}^+(f_0)$ because the difference $f - f_0$, is continuous on $B(r, 2\varepsilon_r) \setminus \{r\}$. In the latter case for every $n < |E_0|$, $x \notin \bigcup_{r \in R_n} \mathrm{cl}(B(r, \varepsilon_r))$ and then $x \notin \mathrm{cl}(\bigcup_{r \in R_n} B(r, \varepsilon_r))$ because $x \notin \mathrm{cl}(E_n) \setminus E_n$. Then for every $m \in \omega$ there is $\varepsilon > 0$ such that $B(x, \varepsilon)$ is disjoint from the closed set $\bigcup_{n<m} \mathrm{cl}(\bigcup_{r \in R_n} B(r, \varepsilon_r))$, and hence $|f - f_0| \leq 2^{-m}$ on $B(x, \varepsilon)$ (if $|E_0|$ is finite and $m > |E_0|$, then $f = f_0$ on $B(x, \varepsilon)$). In both cases, $x \in \mathrm{C}(f) \leftrightarrow x \in \mathrm{C}(f_0)$ and $x \in \mathrm{C}^+(f) \leftrightarrow x \in \mathrm{C}^+(f_0)$. ☐

Lemma 2.7. *Let $f : X \to \mathbb{R}$ be any function and let $f^* : X \to \mathbb{R}$ be defined by*

$$f^*(x) = \begin{cases} \lim_{y \to x} f(y), & \text{if } x \in \mathrm{C}^+(f), \\ f(x), & \text{otherwise.} \end{cases}$$

Then $\mathrm{C}^+(f) \subseteq \mathrm{C}(f^*)$.

Proof. Let $x \in \mathrm{C}^+(f)$. For any neighbourhood U of $f^*(x)$ there is an open neighbourhood V of x such that $f(y) \in U$ for $y \in V \setminus \{x\}$. Then $f^*(y) \in \mathrm{cl}(U)$ for $y \in V \setminus \{x\}$ and by the choice of U also for $y = x$. It follows that $x \in \mathrm{C}(f^*)$. ☐

Given function $f : X \to \mathbb{R}$ define by induction on ordinals α the functions $(f)_\alpha = f_\alpha : X \to \mathbb{R}$ as follows:

$$f_0 = f, \quad f_{\alpha+1} = (f_\alpha)^*, \quad \text{and}$$

$$f_\alpha(x) = \begin{cases} f_\beta(x), & \text{if } \beta < \alpha \text{ is first such that } x \in \mathrm{C}(f_\beta), \\ f(x), & \text{otherwise,} \end{cases} \qquad \text{for } \alpha \text{ limit.}$$

Lemma 2.8. *For any ordinal γ the following conditions hold:*

(i)$_\gamma$ $(\forall\beta)$ $f_{\beta+\gamma} = (f_\beta)_\gamma$.
(ii)$_\gamma$ $(\forall\beta)$ $f_{\beta+\gamma}(U) \subseteq \mathrm{cl}(f_\beta(U))$.
(iii)$_\gamma$ $(\forall\beta)$ $\mathrm{C}(f_\beta) \subseteq \mathrm{C}(f_{\beta+\gamma})$.
(iv)$_\gamma$ $(\forall\beta)$ $f_{\beta+\gamma}(x) = f_\beta(x)$ *for* $x \in \mathrm{C}(f_\beta) \cup (X \setminus \mathrm{C}(f_{\beta+\gamma}))$.

Proof. One can easily verify that for every ordinal γ, $f_\gamma(U) \subseteq \mathrm{cl}(f(U))$ for any set $U \subseteq X$, $f_\gamma(x) = f(x)$ for $x \in \mathrm{C}(f)$, and $f_\gamma(x) = f(x)$ for $x \in X \setminus \bigcup_{\xi \leq \gamma} \mathrm{C}(f_\xi)$. By inserting f_β for f in these formulas it is easy to see that (i)$_\gamma$ $\to$ (ii)$_\gamma$ and (i)$_\gamma \wedge$ (iii)$_\gamma \to$ (iv)$_\gamma$.

Assume that the assertions (i)$_\gamma$ and (iii)$_\gamma$ hold for all $\gamma < \alpha$ and we prove them for $\gamma = \alpha$.

If $\alpha = \gamma + 1$, then by the induction hypothesis and the previous lemma, $f_{\beta+\gamma+1} = (f_{\beta+\gamma})^* = ((f_\beta)_\gamma)^* = (f_\beta)_{\gamma+1}$ and $\mathrm{C}(f_\beta) \subseteq \mathrm{C}(f_{\beta+\gamma}) \subseteq \mathrm{C}^+(f_{\beta+\gamma}) \subseteq \mathrm{C}(f_{\beta+\gamma+1})$.

Let α be a limit ordinal. We use the induction hypothesis. For every $x \in X$, either there is $\gamma < \alpha$ such that $x \in \mathrm{C}((f_\beta)_\gamma) = \mathrm{C}(f_{\beta+\gamma})$, and then $(f_\beta)_\alpha(x) = (f_\beta)_\gamma(x) = f_{\beta+\gamma}(x) = f_{\beta+\alpha}(x)$, or $x \notin \bigcup_{\gamma<\alpha} \mathrm{C}((f_\beta)_\gamma) = \bigcup_{\gamma<\alpha} \mathrm{C}((f_{\beta+\gamma})$, and then by definition in the limit case, $(f_\beta)_\alpha(x) = f_\beta(x)$, $f_{\beta+\alpha}(x) = f(x)$, and by the note at the beginning of the proof, $f_\beta(x) = f(x)$. Therefore $(\mathrm{i})_\alpha$ holds.

We prove $(\mathrm{iii})_\alpha$. Let $x \in \mathrm{C}(f_\beta)$ and let U be any open neighbourhood of $f_{\beta+\alpha}(x)$. Using $(\mathrm{i})_\alpha$ and definition of $(f_\beta)_\alpha$, $f_{\beta+\alpha}(x) = (f_\beta)_\alpha(x) = f_\beta(x)$. Therefore $f_\beta(x) \in U$, and then there is an open neighbourhood V of x such that $f_\beta(V) \subseteq U$. Then $f_{\beta+\alpha}(V) \subseteq \mathrm{cl}(U)$ by $(\mathrm{ii})_\alpha$. It follows that $x \in \mathrm{C}(f_{\beta+\alpha})$. □

Question 2.9. Let $\alpha(f) = \min\{\alpha : f_{\alpha+1} = f_\alpha\}$ for $f : X \to \mathbb{R}$. If X is an uncountable Polish space, then for every $\alpha < \omega_1$ there is a set $A \subseteq X$ such that $\alpha(\chi_A) = \alpha$ (the construction is by induction). Is there $f : X \to \mathbb{R}$ such that $\alpha(f) \geq \omega_1$?

3. Symmetric sets, independent sets, and symmetric continuity

Let $\mathcal{SC}$ denote the family of all sets of the form $\mathrm{SC}(f)$ for a function $f : \mathbb{R} \to \mathbb{R}$. It is a still open question of S. Marcus (see [15, p. 422]) to characterize the sets that belong to $\mathcal{SC}$.

For $a \in \mathbb{R}$ the symmetry about a is the function $\mathrm{sym}_a : \mathbb{R} \to \mathbb{R}$ defined by $\mathrm{sym}_a(x) = 2a - x$, for $x \in \mathbb{R}$. For $a \in \mathbb{R}$ and $\varepsilon > 0$ let ε-symmetry about a be the function $\mathrm{sym}_{a,\varepsilon} : \mathbb{R} \to \mathbb{R}$ defined by $\mathrm{sym}_{a,\varepsilon}(x) = \mathrm{sym}_a(x)$, if $x \in B(a, \varepsilon)$, and $\mathrm{sym}_{a,\varepsilon}(x) = x$, if $x \notin B(a, \varepsilon)$.

A set $X \subseteq \mathbb{R}$ is said to be symmetric about $A \subseteq \mathbb{R}$, if

$$(\forall a \in A)(\forall x \in X)\ \mathrm{sym}_a(x) \in X.$$

Let $\mathrm{sym}_A(X)$ be the closure of the set X under the functions sym_a for $a \in A$. Clearly, $\mathrm{sym}_A(X) = \bigcup_{x \in X} \mathrm{sym}_A(\{x\})$, $y \in \mathrm{sym}_A(\{x\}) \leftrightarrow x \in \mathrm{sym}_A(\{y\})$, and $Y \cap \mathrm{sym}_A(X) \neq \emptyset \leftrightarrow \mathrm{sym}_A(Y) \cap X \neq \emptyset \leftrightarrow \mathrm{sym}_A(Y) \cap \mathrm{sym}_A(X) \neq \emptyset$.

A set $X \subseteq \mathbb{R}$ is said to be locally symmetric about $A \subseteq \mathbb{R}$, if there is a function $\delta : A \to (0, 1)$ such that,

$$(\forall a \in A)(\forall x \in A)\ \mathrm{sym}_{a,\delta(a)}(x) \in A.$$

A set $X \subseteq \mathbb{R}$ is said to be symmetric (locally symmetric), if X is symmetric (locally symmetric) about X. Then X is symmetric if and only if $X = \mathrm{sym}_X(X)$. For example, $\emptyset$, $\mathbb{R}$, and every subgroup of $(\mathbb{R}, +)$ are symmetric sets, open sets are locally symmetric, symmetric sets are locally symmetric, translations of symmetric sets are symmetric, and translations of locally symmetric sets are locally symmetric. The sets X and $\mathbb{R} \setminus X$ are locally symmetric about $\mathrm{SC}(\chi_X)$.

For a set $A \subseteq \mathbb{R}$ denote

$$\langle A \rangle = \left\{\textstyle\sum_{i<n} u_i x_i : n \in \omega \text{ and } u_i \in \mathbb{Z} \text{ and } x_i \in A\right\},$$
$$\langle A \rangle_0 = \{x_0 - x_1 + \cdots + x_{2n} - x_{2n+1} : n \in \omega \text{ and } x_i \in A\},$$
$$\langle A \rangle_1 = \{x_0 - x_1 + \cdots - x_{2n-1} + x_{2n} : n \in \omega \text{ and } x_i \in A\},$$
$$\langle A \rangle_s = \bigcap\{X : X \text{ is symmetric and } A \subseteq X\}.$$

Hence $\langle A \rangle$ is the subgroup of $(\mathbb{R}, +)$ generated by A and $\langle A \rangle_s$ is the smallest symmetric set containing A.

Lemma 3.1. *Let $A, X \subseteq \mathbb{R}$.*

1. $A \subseteq \langle A \rangle_1 = \langle A \rangle_0 + a$ *for every* $a \in A$.
2. $\mathrm{int}(\langle A \rangle_0) \neq \emptyset \leftrightarrow \mathrm{int}(\langle A \rangle_1) \neq \emptyset \leftrightarrow \langle A \rangle_0 = \langle A \rangle_1 = \mathbb{R}$.
3. $\langle A + x \rangle_0 = \langle A \rangle_0$, $\langle A + x \rangle_1 = \langle A \rangle_1 + x$, $\langle A + x \rangle_s = \langle A \rangle_s + x$ *for* $x \in \mathbb{R}$.

(4) $\langle A\rangle_0$ *is a subgroup of* $\langle A\rangle$ *and* $\langle A\rangle = \langle A\rangle_0 + \mathbb{Z}a$ *for every* $a \in A$.
(5) $\langle A\rangle_0$ *and* $\langle A\rangle_1$ *are symmetric and* $\langle A\rangle_s \subseteq \langle A\rangle_1$.
(6) $\operatorname{sym}_A(X) = \operatorname{sym}_{\langle A\rangle_1}(X) = (2\langle A\rangle_0 + X) \cup (2\langle A\rangle_1 - X)$.
(7) $\operatorname{sym}_X(X) = 2\langle X\rangle_1 - X = 2\langle X\rangle_0 + X$.

Proof. (1)–(2) are obvious. For the last identity in (3) the equality $\operatorname{sym}_{a+x}(b+x) = \operatorname{sym}_a(b) + x$ can be used.

(4) $\langle A\rangle_0$ and $\langle A\rangle_0 + \mathbb{Z}a$ are closed under $-$ and the latter set includes A because $x = (x-a)+a$ for $x \in A$.

(5) The sets $\langle A\rangle_0$ and $\langle A\rangle_1$ are symmetric because $\langle A\rangle_0$ is a subgroup of $(\mathbb{R}, +)$ and $\langle A\rangle_1$ is a translation of $\langle A\rangle_0$. Then $\langle A\rangle_s \subseteq \langle A\rangle_1$ because $A \subseteq \langle A\rangle_1$.

(6) One can verify that the odd compositions of symmetries about members of A are symmetries about members of $\langle A\rangle_1$ and the even compositions of symmetries about members of A are translations by $2a$ for $a \in \langle A\rangle_0$. Therefore $\operatorname{sym}_A(X) = (2\langle A\rangle_1 - X) \cup (2\langle A\rangle_0 + X)$. Then also $\operatorname{sym}_{\langle A\rangle_1}(X) = (2\langle A\rangle_0 + X) \cup (2\langle A\rangle_1 - X)$ because $\langle\langle A\rangle_1\rangle_0 = \langle A\rangle_0$ and $\langle\langle A\rangle_1\rangle_1 = \langle A\rangle_1$.

(7) This follows by (6) because $2\langle X\rangle_1 - X = 2(\langle X\rangle_1 - X) + X = 2\langle X\rangle_0 + X$. □

Hence subgroups of $(\mathbb{R}, +)$ and their translations are exactly sets of the form $\langle A\rangle_0$ or $\langle A\rangle_1$ for an $A \subseteq \mathbb{R}$, respectively.

Every infinite set $A \subseteq \mathbb{R}$ is a subset of a symmetric set of the same cardinality.

The following lemma is easy.

Lemma 3.2. *Let* $A \subseteq \mathbb{R}$

(1) $\mathrm{SC}(\chi_A)$ *is locally symmetric and* $\mathrm{SC}(\chi_A) = \mathrm{SC}(\chi_{\mathbb{R}\setminus A})$.
(2) *A is locally symmetric about B if and only if* $B \subseteq \mathrm{SC}(\chi_A)$.
(3) $\mathrm{SC}(\chi_A) = \mathbb{R}$ *if and only if A and $\mathbb{R} \setminus A$ are locally symmetric.*
(4) *If A is discrete, then* $\mathrm{SC}(\chi_A) = \mathbb{R}$*; if A is symmetric, then* $\langle A\rangle_1 \subseteq \mathrm{SC}(\chi_A)$*; if A is dense, then* $\mathrm{SC}(\chi_A) \subseteq \left\{\frac{x+y}{2} : x, y \in A\right\}$.
(5) *If A is a dense subgroup of $(\mathbb{R}, +)$ or a translation of a dense subgroup, then* $\mathrm{SC}(\chi_A) = \left\{\frac{x+y}{2} : x, y \in A\right\}$.
(6) *If A is a dense subgroup of $(\mathbb{R}, +)$ divisible by 2, then* $\mathrm{SC}(\chi_A) = A$. □

At the presence we are not able to decide the question whether $\mathcal{SC}$ contains all locally symmetric sets, and neither whether $\mathcal{SC}$ contains all symmetric sets. On the other hand we have no example of a set of reals that is not in $\mathcal{SC}$. The referee kindly informed us about several other papers on symmetric continuity and two of them [7] and [8] have connections to these questions.

In Theorem 3.8 (4) below we reprove and a bit strengthen the result of U. B. Darji [2] who for a given perfect linearly independent set $C \subseteq \mathbb{R}$ over $\mathbb{Q}$ proved that for every set $A \subseteq C$ there is $f : \mathbb{R} \to \mathbb{R}$ such that $\mathrm{SC}(f) = A$. For a simple construction of a perfect linearly independent set see [9] and the same proof gives the following:

Lemma 3.3. *Every perfect set of reals has a perfect linearly independent subset.* □

Let A and B be sets of reals. We say that B is weakly independent from A, if $(\langle A\rangle_0 \cup \langle A\rangle_1) \cap B = \emptyset$. A set $C \subseteq \mathbb{R}$ is said to be weakly independent, if every subset of C is weakly independent from its complement. Obviously, a linearly independent set of reals over $\mathbb{Q}$ is weakly independent.

Lemma 3.4. *Let A and C be sets of reals. Then $C \setminus \langle A\rangle$ is weakly independent from $C \cap \langle A\rangle$. If $\langle A\rangle_0 \cap C = \emptyset$, then $C \setminus \langle A\rangle_1$ is weakly independent from $C \cap \langle A\rangle_1$.*

Proof. $\langle C\cap\langle A\rangle\rangle_0 \cup \langle C\cap\langle A\rangle\rangle_1 \subseteq \langle\langle A\rangle\rangle_0 \cup \langle\langle A\rangle\rangle_1 = \langle A\rangle$ and $\langle C\cap\langle A\rangle_1\rangle_0 \cup \langle C\cap\langle A\rangle_1\rangle_1 \subseteq \langle\langle A\rangle_1\rangle_0 \cup \langle\langle A\rangle_1\rangle_1 = \langle A\rangle_0 \cup \langle A\rangle_1$. □

Lemma 3.5. *Let $C \subseteq \mathbb{R}$ and let $G = \mathbb{Q}\cdot\langle C\rangle$. Then G is the vector space over $\mathbb{Q}$ generated by C and the following holds.*

(1) *G is meager (has measure zero) if and only if $\langle C\rangle$ is meager (has measure zero) if and only if $\langle C\rangle_0$ is meager (has measure zero).*
(2) *If C is analytic weakly independent, then G is meager and has measure zero.*
(3) *If C is analytic, then G is meager and has measure zero if and only if C does not contain a Hamel base.*

Proof. (1) By Lemma 3.1, $\langle C\rangle = \bigcup_{n\in\mathbb{Z}}(\langle C\rangle_0 + na)$ for any $a \in C$.

(2) Choose distinct reals $x_n \in C$ for $n \in \omega$ and let $A_n = C \setminus \{x_n\}$. If C is weakly independent, then $\langle A_n\rangle_0$ is a proper analytic group because $x_n \notin \langle A_n\rangle_0$ and therefore $\langle A_n\rangle_0$ is meager and has measure zero. Then by (1), G is meager nad has measure zero because $\langle C\rangle_0 = \bigcup_{n\in\omega}\langle A_n\rangle_0$.

(3) If C does not contain a Hamel base, then G is a proper analytic group and therefore is meager and has measure zero. □

Lemma 3.7 below is a simplified version of a result of Darji (on the function h in [2]). Differently from Darji's result, this simplified version gives no useful information for the study of sets of points where a given function is smooth, symmetrically differentiable, or symmetric. We state the lemma for weakly independent sets and not only for linearly independent sets.

Lemma 3.6. *Let $A \subseteq C \subseteq \mathbb{R}$ be such that $C \setminus A$ is weakly independent from A and $\langle C\rangle_0$ is meager or has measure zero. Then there exists a set $L \subseteq \mathbb{R}$ such that $\mathrm{SC}(\chi_L) \cap C = A$ and $\mathrm{C}^+(\chi_L) = \emptyset$.*

Proof. Choose any $e \in \mathbb{R} \setminus (\mathbb{Q}\cdot\langle C\rangle)$. By Lemma 3.5 it exists and the vector space $G = \mathbb{Q}\cdot\langle C\rangle + \mathbb{Q}\cdot e$ is meager or has measure zero. First assume that $A \neq C$. Denote $B = C \setminus A$ and let $L = \mathrm{sym}_A(2B - (\mathbb{Q}\setminus\{0\})e)$. Then

$$\begin{aligned} L &= (2\langle A\rangle_0 + 2B - (\mathbb{Q}\setminus\{0\})e) \cup (2\langle A\rangle_1 - 2B + (\mathbb{Q}\setminus\{0\})e) \\ &= 2[(-\langle A\rangle_0 + B) \cup (\langle A\rangle_1 - B)] + (\mathbb{Q}\setminus\{0\})e \end{aligned}$$

because $\langle A\rangle_0 = -\langle A\rangle_0$ and $\mathbb{Q} = -\mathbb{Q}$; and $0 \notin (-\langle A\rangle_0 + B) \cup (\langle A\rangle_1 - B)]$ because B is weakly independent from A. It follows that $re \notin L$ for all $r \in \mathbb{Q}$.

The sets L and $\mathbb{R}\setminus L$ are dense subsets of $\mathbb{R}$ because $L \subseteq G$ and L contains the dense set $2B - (\mathbb{Q}\setminus\{0\})e$. Therefore $\mathrm{C}^+(\chi_L) = \emptyset$.

Certainly $A \subseteq \mathrm{SC}(\chi_L)$ because $L = \mathrm{sym}_A(L)$. We show that $\mathrm{SC}(\chi_L) \cap B = \emptyset$. Let $x \in B$ be arbitrary. Choose any sequence $\{r_k\}_{k\in\omega}$ in $\mathbb{Q}\setminus\{0\}$ such that the sequence $\{r_k e\}_{k\in\omega}$ converges to x. Then $x \notin \mathrm{SC}(\chi_L)$ because $2x - r_k e \in L$ and $r_k e \notin L$ for all $k \in \omega$.

In case $A = C$ the conclusion of the lemma holds with $L = \mathrm{sym}_A(A)$. □

Lemma 3.7. *Let $C \subseteq \mathbb{R}$ be a weakly independent set such that $\langle C\rangle_0$ is meager or has measure zero. Then for every set $A \subseteq C$ there exists a set $L \subseteq \mathbb{R}$ such that $\mathrm{SC}(\chi_L) \cap C = A$ and $\mathrm{C}^+(\chi_L) = \emptyset$.*

Proof. By Lemma 3.5 and Lemma 3.6. □

Theorem 3.8 below summarizes partial results about $\mathcal{SC}$.

Recall that $\mathfrak{p}$ is the least cardinality of a family $A \subseteq [\omega]^\omega$ without an infinite pseudo-intersection such that each finite subfamily of A has an infinite pseudo-intersection. Equivalently, $\mathfrak{p} \geq \omega_1$ is the least cardinal number for which Martin's axiom for σ-centered partial orders does not hold, see [3] and [4]. If X is a second-countable T_1-space of cardinality $< \mathfrak{p}$, then X is a Q-space, i.e., every subset of X is F_σ and G_δ (see [4, 23B]).

Theorem 3.8.

(1) *$\mathcal{SC}$ contains all subgroups of $(\mathbb{R}, +)$ and their translations.*
(2) $[\mathbb{R}]^{<\mathfrak{p}} \subseteq \mathcal{SC}$.
(3) $\{A \subseteq C : C \setminus A$ *is weakly independent from* $A\} \subseteq \mathcal{SC}$ *whenever* $C \subseteq \mathbb{R}$ *is a* G_δ *set not containing a Hamel base.*
(4) $\mathcal{P}(C) \subseteq \mathcal{SC}$ *whenever* $C \subseteq \mathbb{R}$ *is a weakly independent* G_δ *set.*
(5) *Every perfect set in* $\mathbb{R}$ *has a perfect subset* C *such that* $\mathcal{P}(C) \subseteq \mathcal{SC}$.
(6) *$\mathcal{SC}$ is closed under the following operations:*
   (a) *If* $A \in \mathcal{SC}$ *and* $B \in G_\delta$, *then* $A \cap B \in \mathcal{SC}$.
   (b) *If* $A \in \mathcal{SC}$ *and* $\varphi : \mathbb{R} \to \mathbb{R}$ *is linear, then* $\varphi(A) \in \mathcal{SC}$.
   (c) *If* $A_n \in \mathcal{SC}$ *are separated by pairwise disjoint open sets* U_n *for* $n \in \omega$, *then* $\bigcup_{n\in\omega} A_n \in \mathcal{SC}$.

Proof. (1) If A is a discrete subgroup, then A is a closed subset of $\mathbb{R}$ and since $\mathbb{R} = \mathrm{SC}(\chi_A) \in \mathcal{SC}$, by Theorem 2.1 there is f such that $\mathrm{SC}(f) = A$. If A is a dense subgroup, then $B = \{2x : x \in A\}$ is a dense subgroup of $(\mathbb{R}, +)$ and $\mathrm{SC}(\chi_B) = A$ by Lemma 3.2 (5). The translations are shown in (6b).

(2) Let $A \in [\mathbb{R}]^{<\mathfrak{p}}$ be arbitrary. Let B be a subgroup of $(\mathbb{R}, +)$ divisible by 2 such that $A \subseteq B$ and $|B| < \mathfrak{p}$. Then $B = \mathrm{SC}(\chi_B)$. The set B is a Q-space because $|B| < \mathfrak{p}$ Therefore such that $A = B \cap G$ for some G_δ set G. By Lemma 2.2 there is a function $f : \mathbb{R} \to \mathbb{R}$ such that $\mathrm{SC}(f) = B \cap G = A$.

(3)–(4) If C is a G_δ set not containing a Hamel base and $A \subseteq C$ is such that $C \setminus A$ is weakly independent from A, then by Lemma 3.6 there is a set $L \subseteq \mathbb{R}$ such that $\mathrm{SC}(\chi_L) \cap C = A$. By Lemma 2.2 there is a function f such that $\mathrm{SC}(f) = A$.

(5) This is a consequence of (4) and Lemma 3.3.

(6a) holds by Lemma 2.2.

(6b) If $\varphi : \mathbb{R} \to \mathbb{R}$ is a linear transformation and $A = \mathrm{SC}(f)$ for some $f : \mathbb{R} \to \mathbb{R}$, then $\varphi(A) = \mathrm{SC}(g)$ where $g(x) = f(\varphi^{-1}(x))$ for $x \in \mathbb{R}$. Therefore $\varphi(A) \in \mathcal{SC}$.

(6c) Denote $A = \bigcup_{n\in\omega} A_n$ and let $f_n : \mathbb{R} \to \mathbb{R}$ be such that $A_n = \mathrm{SC}(f_n)$ for $n \in \omega$. Define $g : \mathbb{R} \to \mathbb{R}$ by $f(x) = f_n(x)$, if $x \in U_n$, and $g(x) = 0$, if $x \in \mathbb{R} \setminus \bigcup_{n\in\omega} U_n$. Then $A \subseteq \mathrm{SC}(g)$ and $A = \mathrm{SC}(g) \cap \bigcup \mathcal{U}$. Therefore by Lemma 2.2 there is a function $f : \mathbb{R} \to \mathbb{R}$ such that $A = \mathrm{SC}(f)$. □

Theorem 3.9. *Let* $A \subseteq \mathbb{R}$ *and either* $|A| < \mathfrak{p}$, *or* A *is a subset of a weakly independent* G_δ *set, or* A *is a proper subgroup of* $(\mathbb{R}, +)$, *or a translation of such sets. Then there is* $f : \mathbb{R} \to \mathbb{R}$ *such that* $A = \mathrm{SC}(f) \setminus \mathrm{C}(f)$.

Proof. If $|A| < \mathfrak{p}$, then continue with the proof of Theorem 3.8 (2): Since the group G is divisible by 2 it is a dense subset of $\mathbb{R}$ and hence $\mathrm{C}^+(\chi_B) = \emptyset$. Therefore for the function f in the proof we have $\mathrm{C}(f) \subseteq \mathrm{C}^+(f) = \emptyset$ and $\mathrm{SC}(f) = A$.

The case of weakly independent sets follows by Lemma 3.6 and Lemma 2.2.

The case when A is a proper subgroup of $\mathbb{R}$ is a result of K. Muthuvel [12] and it is obvious by the above arguments. $\square$

It is not clear whether the class $\mathcal{SC}$ is closed under intersection of two sets. We can see this only for some special sets.

Lemma 3.10. *The class* $\mathcal{SC}_0 = \{\mathrm{SC}(f) : \mathrm{rng}(f)$ *is bounded nowhere dense*$\}$ *is closed under countable intersections.*

Proof. Observe first that $\mathcal{SC}_0 = \{\mathrm{SC}(f) : \mathrm{cl}(\mathrm{rng}(f)) \subseteq [0,1] \setminus \mathbb{Q}\}$. To see this let $f : \mathbb{R} \to \mathbb{R}$ be any function such that $\mathrm{cl}(\mathrm{rng}(f))$ is a subset of a bounded open interval (a,b) and choose a countable dense subset $S \subseteq (a,b) \setminus \mathrm{cl}(\mathrm{rng}(f))$. Let $\varphi : [a,b] \to [0,1]$ be any increasing homeomorphism that maps S onto $(0,1) \cap \mathbb{Q}$ and define $g : \mathbb{R} \to (0,1)$ by $g(x) = \varphi(f(x))$. Then $\mathrm{cl}(\mathrm{rng}(g)) \subseteq (0,1) \setminus \mathbb{Q}$ and $\mathrm{SC}(f) = \mathrm{SC}(g)$.

Now, fix any homeomorphism $F = \langle F_n : n \in \omega \rangle : {}^{\omega}([0,1] \setminus \mathbb{Q}) \to [0,1] \setminus \mathbb{Q}$ and let $f_n : \mathbb{R} \to (0,1) \setminus \mathbb{Q}$ for $n \in \omega$ be given functions such that $\mathrm{cl}(\mathrm{rng}(f_n))$ are compact subsets of $(0,1) \setminus \mathbb{Q}$. Define $f : \mathbb{R} \to [0,1] \setminus \mathbb{Q}$ by $f(x) = F(\langle f_n(x) : n \in \omega \rangle)$. The set $C = \prod_{n \in \omega} \mathrm{cl}(\mathrm{rng}(f_n))$ is a compact subset of ${}^{\omega}([0,1] \setminus \mathbb{Q})$ and $\mathrm{rng}(f) \subseteq F(C)$. Therefore $\mathrm{cl}(\mathrm{rng}(f)) \subseteq [0,1] \setminus \mathbb{Q}$ is compact. We show that $\mathrm{SC}(f) = \bigcap_{n \in \omega} \mathrm{SC}(f_n)$. Assume that there is $x \in \bigcap_{n \in \omega} \mathrm{SC}(f_n) \setminus \mathrm{SC}(f)$. Then, using the compactness assumption, there are a sequence of reals $\{h_k\}_{k \in \omega}$ converging to 0 and $r = F(\langle r_n : n \in \omega \rangle)$ in $[0,1] \setminus \mathbb{Q}$ such that $\lim_{k \in \omega} f(x + h_k) = r$ and $\lim_{k \in \omega} f(x - h_k) \neq r$. Then for some $n \in \omega$, $\lim_{k \in \omega} f_n(x + h_k) = r_n$ and $\lim_{k \in \omega} f_n(x - h_k) \neq r_n$. This contradiction proves the inclusion $\bigcap_{n \in \omega} \mathrm{SC}(f_n) \subseteq \mathrm{SC}(f)$. Similar arguments work for the inverse inclusion. $\square$

By Theorem 3.8 for every dense subgroup $A \subseteq \mathbb{R}$ there is $B \subseteq \mathbb{R}$ such that $A = \mathrm{SC}(\chi_B)$. We show that if A is a translation of a subgroup of small cardinality, then there is a Bernstein set $B \subseteq \mathbb{R}$ such that $A = \mathrm{SC}(\chi_B)$.

Lemma 3.11. *Assume that* $Y, Z, A \in [\mathbb{R}]^{<\mathfrak{c}}$ *and* $Y \cap \mathrm{sym}_A(Z) = \emptyset$. *Then*

$$(\forall z \in \mathbb{R} \setminus \langle A \rangle_1)\ |\{h \in \mathbb{R} : (Y \cup \{z - h\}) \cap \mathrm{sym}_A(Z \cup \{z + h\}) \neq \emptyset\}| < \mathfrak{c}.$$

Proof. Let $z \in \mathbb{R} \setminus \langle A \rangle_1$. The set $C = (\mathrm{sym}_A(Y) - \{z\}) \cup (\{z\} - \mathrm{sym}_A(Z))$ has cardinality $< \mathfrak{c}$ and $(Y \cup \{z - h\}) \cap \mathrm{sym}_A(Z \cup \{z + h\}) \neq \emptyset$ if and only if $z - h \in \mathrm{sym}_A(Z)$ or $z - h \in \mathrm{sym}_A(\{z + h\})$ or $z + h \in \mathrm{sym}_A(Y)$.

If $z - h \in \mathrm{sym}_A(Z)$ or $z + h \in \mathrm{sym}_A(Y)$, then $h \in C$. If $z - h \in \mathrm{sym}_A(\{z + h\})$, then either (i) there is $x \in \langle A \rangle_1$ such that $z - h = 2x - (z + h)$, or else (ii) there is $x \in \langle A \rangle_0$ such that $z - h = 2x + (z + h)$. Case (i) is not possible because it implies $z \in \langle A \rangle_1$, and in case (ii), $h \in \langle A \rangle_0$ where $|\langle A \rangle_0| < \mathfrak{c}$. $\square$

Theorem 3.12. *For every set* $A \in [\mathbb{R}]^{<\mathfrak{c}}$ *there is a Bernstein set* $B \subseteq \mathbb{R}$ *such that* $\mathrm{SC}(\chi_B) = \langle A \rangle_1$. *In particular, if* $A = \emptyset$, *then* $\mathrm{SC}(\chi_B) = \emptyset$.

Proof. Let $\{x_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of $\mathbb{R} \setminus \langle A \rangle_1$ and let $\{B_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of uncountable Borel sets in $\mathbb{R}$. By induction on $\alpha < \mathfrak{c}$ we choose $y_\alpha, z_\alpha \in B_\alpha$ and a decreasing sequence of positive reals $h_{\alpha,n} \searrow 0$ for $n \in \omega$ and define

$$
\begin{aligned}
&X^-_{\alpha,n} = \{x_\beta - h_{\beta,k} : \beta < \alpha \text{ and } k \in \omega\} \cup \{x_\alpha - h_{\alpha,k}, k : k < n\},\\
&X^+_{\alpha,n} = \{x_\beta + h_{\beta,k} : \beta < \alpha \text{ and } k \in \omega\} \cup \{x_\alpha + h_{\alpha,k}, k : k < n\},\\
&X^- = \textstyle\bigcup_{\alpha<\mathfrak{c}} X^-_{\alpha,0}, \quad X^+ = \textstyle\bigcup_{\alpha<\mathfrak{c}} X^+_{\alpha,0},\\
&Y_\alpha = \{y_\beta : \beta < \alpha\}, \quad Y = \{y_\alpha : \alpha < \mathfrak{c}\},\\
&Z_\alpha = \{z_\beta : \beta < \alpha\}, \quad Z = \{z_\alpha : \alpha < \mathfrak{c}\}
\end{aligned}
$$

so that $y_\beta, z_\beta \in B_\beta$ for $\beta < \alpha$ and $(Y_\alpha \cup X^-_{\alpha,0}) \cap \operatorname{sym}_A(Z_\alpha \cup X^+_{\alpha,0}) = \emptyset$.

Assume that y_β, z_β, $h_{\beta,n}$ for $\beta < \alpha$ and $n \in \omega$ have been defined and have the required properties. Choose any $y_\alpha \in B_\alpha \setminus \operatorname{sym}_A(Z_\alpha \cup X^+_{\alpha,0})$ and by induction on $n \in \omega$ using Lemma 3.11 because $x_\alpha \notin \langle A\rangle_1$, choose $h_{\alpha,n} \searrow 0$ so that

$$(Y_{\alpha+1} \cup X^-_{\alpha,n+1}) \cap \operatorname{sym}_A(Z_\alpha \cup X^+_{\alpha,n+1}) = \emptyset$$

for all $n \in \omega$. This defines $X^-_{\alpha+1,0}$ and $X^+_{\alpha+1,0}$ so that

$$(Y_{\alpha+1} \cup X^-_{\alpha+1,0}) \cap \operatorname{sym}_A(Z_\alpha \cup X^+_{\alpha+1,0}) = \emptyset.$$

Now choose any $z_\alpha \in B_\alpha \setminus \operatorname{sym}_A(Y_{\alpha+1} \cup X^-_{\alpha+1,0})$. Then

$$(Y_{\alpha+1} \cup X^-_{\alpha+1,0}) \cap \operatorname{sym}_A(Z_{\alpha+1} \cup X^+_{\alpha+1,0}) = \emptyset.$$

Let $B = \operatorname{sym}_A(Z \cup X^+) = \operatorname{sym}_{\langle A\rangle_1}(Z \cup X^+)$. Then B is a Bernstein set because $Z \subseteq B$ and $Y \cap B = \emptyset$. The inclusion $\langle A\rangle_1 \subseteq \operatorname{SC}(\chi_B)$ holds because B is symmetric about $\langle A\rangle_1$ and the inclusion $\operatorname{SC}(\chi_B) \subseteq \langle A\rangle_1$ holds because $X^+ \subseteq B$ and $X^- \cap B = \emptyset$. Therefore $\operatorname{SC}(\chi_B) = \langle A\rangle_1$. □

Theorem 3.13.

(1) *There is a Bernstein set* $B \subseteq \mathbb{R}$ *such that* $\operatorname{SC}(\chi_B) = B$.
(2) *There is a Bernstein set* $B \subseteq \mathbb{R}$ *such that* $\operatorname{SC}(\chi_B) = \mathbb{R} \setminus B$.

Proof. (1) We prove that there is a Bernstein set B which is an additive subgroup of $(\mathbb{R}, +)$ divisible by 2. For $A \subseteq \mathbb{R}$ let $\langle A\rangle_2$ denote the smallest subgroup of $(\mathbb{R}, +)$ containing A and divisible by 2. Let $\{B_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all uncountable Borel sets. By induction on $\alpha < \mathfrak{c}$ choose $y_\alpha, z_\alpha \in B_\alpha$ so that for $Y_\alpha = \{y_\beta : \beta < \alpha\}$ and $Z_\alpha = \{z_\beta : \beta < \alpha\}$ we have $Z_\alpha \cap \langle Y_\alpha\rangle_2 = \emptyset$. Then the subgroup $B = \langle\{y_\alpha : \alpha < \mathfrak{c}\}\rangle_2$ has the required properties.

The αth step of the induction: Assume that Y_α and Z_α have been constructed. Choose any $z_\alpha \in B_\alpha \setminus \langle Y_\alpha\rangle_2$; hence $Z_{\alpha+1} \cap \langle Y_\alpha\rangle_2 = \emptyset$. The set

$$
\begin{aligned}
C &= \{y \in \mathbb{R} : Z_{\alpha+1} \cap \langle Y_\alpha \cup \{y\}\rangle_2 \neq \emptyset\}\\
&= \{(2^m z - a)/k : m \in \omega \text{ and } k \in \mathbb{Z} \text{ and } a \in \langle Y_\alpha\rangle \text{ and } z \in Z_{\alpha+1}\}
\end{aligned}
$$

has cardinality $< \mathfrak{c}$ and hence we can choose $y_\alpha \in B_\alpha \setminus C$.

(2) Let $A \subseteq \mathbb{R}$ be a Bernstein set such that $\operatorname{SC}(\chi_A) = A$. Then for $B = \mathbb{R} \setminus A$ we have $\operatorname{SC}(\chi_B) = A = \mathbb{R} \setminus B$. □

4. A generalization of Belna's and Fried's theorems

By H. Fried [5] and C. L. Belna [1] it is known that Borel subsets of $\mathrm{SC}(f) \setminus \mathrm{C}(f)$ are meager and have Lebesgue measure zero. We prove a straightforward generalization of these results (Theorem 4.2).

For $A \subseteq \mathbb{R}$ define by induction:

$$\sigma_0(A) = \{0\} \quad \text{and} \quad \sigma_{k+1}(A) = \sigma_k(A) + (A - A).$$

By the family of sets that are essentially in a $\mathcal{J} \subseteq \mathcal{P}(\mathbb{R})$ we mean

$$\mathrm{ess}(\mathcal{J}) = \{B \subseteq \mathbb{R} : (\exists \delta > 0)\ B \cap I \in \mathcal{J} \text{ for every interval } I \text{ of length} < \delta\}.$$

Let Γ denote any pointclass containing closed sets that is closed under intersections with closed sets (e.g., $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^0_{\alpha+1}$, $\mathbf{\Delta}^0_{\alpha+1}$ for $\alpha \geq 1$, $\mathbf{\Delta}^1_1$, $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$, etc.). For $k \geq 1$ let $\mathcal{M}_k(\Gamma)$ be the intersection of all systems $\mathcal{J} \subseteq \mathcal{P}(\mathbb{R})$ satisfying the following conditions:

- $(0)_k$ If $A \subseteq \mathbb{R}$ and $\mathrm{int}(\mathrm{cl}(\sigma_k(A))) = \emptyset$, then $A \in \mathcal{J}$.
- $(1)_k$ If $A \subseteq \mathbb{R}$ is in Γ and $\mathrm{int}(\sigma_k(A)) = \emptyset$, then $A \in \mathcal{J}$.
- $(2)_k$ If $A \subseteq B$ and $B \in \mathcal{J}$, then $A \subseteq \mathcal{J}$.
- $(3)_k$ If $A \subseteq \mathbb{R}$ is in Γ and A is the union of a countable increasing chain of sets from $\mathrm{ess}(\mathcal{J})$, then $A \in \mathcal{J}$.

Notice that $(0)_k$ is a consequence of $(1)_k$ and $(2)_k$ because $\sigma_k(\mathrm{cl}(A)) = \mathrm{cl}(\sigma_k(A))$ and $\mathrm{cl}(A)$ is in Γ. It is easy to see that $\mathcal{M}_k(\Gamma)$ satisfies $(0)_k$–$(3)_k$ and hence $\mathcal{M}_k(\Gamma)$ is the least family satisfying $(0)_k$–$(3)_k$. Observe also that $\mathcal{M}_{k+1}(\Gamma) \subseteq \mathcal{M}_k(\Gamma)$ because $\sigma_k(A) \subseteq \sigma_{k+1}(A)$ for all sets $A \subseteq \mathbb{R}$. Let $\mathcal{M}_\infty(\Gamma) = \bigcap_{k\geq 1} \mathcal{M}_k(\Gamma)$. If $\Gamma \subseteq \Gamma'$, then $\mathcal{M}_k(\Gamma) \subseteq \mathcal{M}_k(\Gamma')$ and $\mathcal{M}_\infty(\Gamma) \subseteq \mathcal{M}_\infty(\Gamma')$.

The class $\mathcal{M}_k(\Gamma)$ can be constructed by transfinite induction as follows:

$$\mathcal{M}_{k,0}(\Gamma) = \{A \subseteq \mathbb{R} : (\exists B \in \Gamma)\ A \subseteq B \text{ and } \mathrm{int}(\sigma_k(B)) = \emptyset\},$$

$$\mathcal{M}_{k,\alpha+1}(\Gamma) = \{A \subseteq \mathbb{R} : (\exists B \in \Gamma)\ A \subseteq B \text{ and } B \text{ is an increasing union of sets from } \mathrm{ess}(\mathcal{M}_{k,\alpha}(\gamma))\},$$

$$\mathcal{M}_{k,\alpha}(\Gamma) = \{A \subseteq \mathbb{R} : (\exists \beta < \alpha)\ A \in \mathcal{M}_{k,\beta}(\Gamma)\}, \quad \alpha \text{ limit}.$$

Then $\mathcal{M}_k(\Gamma) = \mathcal{M}_{k,\omega_1}(\Gamma)$. By transfinite induction using this scheme we obtain the following lemma:

Lemma 4.1. *$\mathcal{M}_k(\Gamma)$ has a cofinal subfamily consisting of sets from Γ. Consequently, if Γ is closed under countable intersections, then $\mathcal{M}_\infty(\Gamma)$ has a cofinal subfamily consisting of sets from Γ.* □

We say that a set $A \subseteq \mathbb{R}$ is $(\Gamma, \mathcal{J})$-thin, if $\mathcal{P}(A) \cap \Gamma \subseteq \mathcal{J}$.

Theorem 4.2. *Let $f : \mathbb{R} \to \mathbb{R}$. Then $\mathrm{SC}(f) \setminus \mathrm{C}(f)$ is $(\Gamma, \mathcal{M}_\infty(\Gamma))$-thin for every Γ containing closed sets that is closed under intersections with closed sets.*

The proof of this theorem relies on the following two lemmas that slightly modify and generalize Lemma 1 and Lemma 2 of [1]. In fact the proofs from [1] work in case $k = 1$.

Let $f : \mathbb{R} \to \mathbb{R}$ be any given function. For $x \in \mathbb{R}$ and $\varepsilon > 0$ define

$$\delta(x, \varepsilon) = \sup\{\delta \geq 0 : (\forall h,\ |h| \leq \delta)\ |f(x+h) - f(x-h)| < \varepsilon\}.$$

Lemma 4.3 ([1, Lemma 1])**.** *Let ε and δ be positive numbers, $k \geq 1$, I be an interval such that $0 < |I| < \delta/8$, and $E = \{x \in I : \delta(x, \varepsilon) \geq \delta\}$. If $0 \in \operatorname{int}(\operatorname{cl}(\sigma_k(E)))$, then there exists $\tau > 0$ such that $\delta(x, (4k+1)\varepsilon) \geq \tau$ for each $x \in \mathrm{SC}(f) \cap I$.*

Proof. Let $0 < \tau \leq \delta/2$ be such that $\sigma_k(E)$ is dense in the interval $(-\tau, \tau)$. Choose any point $x \in \mathrm{SC}(f) \cap I$. Then $\delta(x, \varepsilon) > 0$. We shall show that for every $h \in (0, \tau)$,

$$|f(x+h) - f(x-h)| < (4k+1)\varepsilon. \tag{4.1}$$

This is certainly satisfied if $h < \delta(x, \varepsilon)$. Suppose $\delta(x, \varepsilon) \leq h$. Because $\sigma_k(E)$ is dense in $(0, \tau)$, there exists a sequence $\langle (z_j, y_j) : j < k \rangle$ in $E \times E$ such that

$$0 < h - 2\textstyle\sum_{j<k}(z_j - y_j) < \delta(x, \varepsilon). \tag{4.2}$$

Since $\delta(x, \varepsilon) \leq h < \delta/2$ and $2|z - y| < \delta/4$ for all $z, y \in E$, we can reorder this sequence if necessary so that

$$(\forall i \leq k)\ 0 < h - 2\textstyle\sum_{j<i}(z_j - y_j) < \delta/2. \tag{4.3}$$

Let $h_i = h - 2\sum_{j<i}(z_j - y_j) = h + 2\sum_{j<i}(y_j - z_j)$ for $i \leq k$ and let

$$x^+_{2i} = x + h_i, \quad x^-_{2i} = x - h_i, \quad x^+_{2i+1} = 2z_i - x^+_{2i}, \quad x^-_{2i+1} = 2y_i - x^-_{2i}.$$

Then x^+_{2i+1} is the reflection of x^+_{2i} in z_i and x^+_{2i+2} is the reflection of x^+_{2i+1} in y_i because $2y_i - x^+_{2i+1} = x^+_{2i+2}$. Similarly, x^-_{2i+1} is the reflection of x^-_{2i} in y_i and x^-_{2i+2} is the reflection of x^-_{2i+1} in z_i because $2z_i - x^-_{2i+1} = x^-_{2i+2}$. By (4.3),

$$\begin{aligned}
|x^+_{2i+1} - z_i| &= |x^+_{2i} - z_i| = |x - z_i + h_i| \leq |x - z_i| + h_i < \delta, \\
|x^+_{2i+1} - y_i| &= |x^+_{2i+2} - y_i| = |x - y_i + h_{i+1}| \leq |x - y_i| + h_{i+1} < \delta, \\
|x^-_{2i+1} - y_i| &= |x^-_{2i} - y_i| = |x - y_i - h_i| \leq |x - y_i| + h_i < \delta, \\
|x^-_{2i+1} - z_i| &= |x^-_{2i+2} - z_i| = |x - z_i - h_{i+1}| \leq |x - z_i| + h_{i+1} < \delta.
\end{aligned}$$

Then, since $x + h = x^+_0$, $x - h = x^-_0$, and $y_i, z_i \in E$ for all $i < k$, we get

$$\begin{aligned}
|f(x+h) - f(x^+_{2k})| &\leq \textstyle\sum_{i<2k} |f(x^+_i) - f(x^+_{i+1})| < 2k\varepsilon, \\
|f(x-h) - f(x^-_{2k})| &\leq \textstyle\sum_{i<2k} |f(x^-_i) - f(x^-_{i+1})| < 2k\varepsilon.
\end{aligned}$$

Also $|f(x^+_{2k}) - f(x^-_{2k})| < \varepsilon$ because by (4.2), $0 < h_k < \delta(x, \varepsilon)$. Now, (4.1) follows by these inequalities which finishes the proof. ☐

Lemma 4.4 ([1, Lemma 2])**.** *Let ε, δ, d be positive numbers, $k \geq 1$, I be an interval such that $0 < |I| < \delta/8$, and $E = \{x \in I : \delta(x, \varepsilon) \geq \delta\}$. If $(-d, d) \subseteq \sigma_k(E)$, then $|f(a) - f(b)| < 2k\varepsilon$ for any two points $a, b \in I$ with $|a - b| < 2d$.*

Proof. Let $a, b \in I$ be such that $0 < b - a < 2d$. Then there exists a sequence $\langle (y_i, x_i) : i < k \rangle$ in $E \times E$ such that $b - a = 2\sum_{j<k}(y_j - x_j)$. Since $b - a < \delta/4$ and $2|y - x| < \delta/4$ for all $y, x \in E$ we can reorder this sequence if necessary so that

$$(\forall i \leq k)\ 0 \leq 2\textstyle\sum_{j<i}(y_j - x_j) < \delta/2. \tag{4.4}$$

Define $a_{2i} = a + 2\sum_{j<i}(y_j - x_j)$ and $a_{2i+1} = 2x_i - a_{2i}$ for $i \leq k$, i.e., $a_0 = a$, $a_{2k} = b$, a_{2i+1} is the reflection of a_{2i} in x_i, a_{2i+2} is the reflection of a_{2i+1} in y_i, and by (4.4), $|a_{2i} - a| < \delta/2$. Therefore $|a_{2i+1} - x_i| = |a_{2i} - x_i| \leq |a_{2i} - a| + |a - x_i| < \delta$ and $|a_{2i-1} - y_{i-1}| = |a_{2i} - y_{i-1}| \leq |a_{2i} - a| + |a - y_{i-1}| < \delta$ because $a, x_i, y_{i-1} \in I$. Then $|f(a_i) - f(a_{i+1})| < \varepsilon$ for all $i < 2k$ because all x_j, y_j are in E. It follows that $|f(a) - f(b)| < 2k\varepsilon$. ☐

Proof of Theorem 4.2. To obtain a contradiction assume that $\mathrm{SC}(f) \setminus \mathrm{C}(f)$ is not $\mathcal{M}_\infty(\Gamma)$-thin. Then there are a set $B \subseteq \mathrm{SC}(f) \setminus \mathrm{C}(f)$ in Γ and $k \geq 1$ such that $B \notin \mathcal{M}_k(\Gamma)$. Denote $D_t(f) = \{x \in \mathbb{R} : (\forall d > 0)\ \mathrm{diam}(f[(x-d, x+d)]) \geq t\}$. Then $B \subseteq \bigcup_{t>0} D_t(f)$ and by $(3)_k$ there is $t > 0$ such that $M = B \cap D_t(f) \notin \mathcal{M}_k(\Gamma)$. $M \in \Gamma$ because $D_t(f)$ is closed. Let $p > 4k(8k+1)$ and let $\varepsilon = t/p$. Denote $M_\delta = \{x \in M : \delta(x, \varepsilon) \geq \delta\}$. Then $M = \bigcup_{\delta>0} M_\delta$ because $M \subseteq \mathrm{SC}(f)$ and by $(3)_k$ there are a $\delta > 0$ and a closed interval I_0 of length $< \delta/8$ such that $M_\delta \cap I_0 \notin \mathcal{M}_k(\Gamma)$. Let $I \supseteq I_0$ be an open interval of length $< \delta/8$ and let $E_0 = \{x \in I : \delta(x, \varepsilon) \geq \delta\}$. Then, $E_0 \notin \mathcal{M}_k(\Gamma)$ because $M_\delta \cap I_0 \subseteq E_0$, and by $(0)_k$, $\sigma_k(E_0)$ is somewhere dense, and hence $\sigma_{2k}(E_0)$ is dense in a neighbourhood of 0. By Lemma 4.3 there exists $\tau > 0$ such that $\delta(x, (8k+1)\varepsilon) \geq \tau$ for each $x \in \mathrm{SC}(f) \cap I$. Now, $M \cap I_0 \notin \mathcal{M}_k(\Gamma)$ and $M \cap I_0 \in \Gamma$ because $M_\delta \cap I_0 \subseteq M \cap I_0$ and I_0 is closed. By $(3)_k$ there is a closed interval $J_0 \subseteq I_0$ of length $< \tau/8$ such that $M \cap J_0 \notin \mathcal{M}_k(\Gamma)$ and by $(1)_k$, $\mathrm{int}(\sigma_k(M \cap J_0)) \neq \emptyset$ and hence $0 \in \mathrm{int}(\sigma_{2k}(M \cap J_0))$. Let J be an open interval of length $< \tau/8$ such that $J_0 \subseteq J \subseteq I$ and let $E_1 = \{x \in J : \delta(x, (8k+1)\varepsilon) \geq \tau\}$. Then $0 \in \mathrm{int}(\sigma_{2k}(E_1))$ because $M \cap J_0 \subseteq \mathrm{SC}(f) \cap J$ and $\mathrm{SC}(f) \cap J \subseteq E_1$ by the choice of τ. By Lemma 4.4 there is $d > 0$ such that $|f(a) - f(b)| < 4k(8k+1)\varepsilon < t$ for all $a, b \in J$ with $|a - b| < 2d$. This contradicts the fact that $\emptyset \neq M \cap J \subseteq D_t(f)$. □

The following is a strengthening of a result of S. Marcus (Theorem 2.39 in [15]) which we prove without the assumption that a function is symmetrically continuous.

Corollary 4.5. *For any function $f : \mathbb{R} \to \mathbb{R}$ the set $\mathrm{SC}(f) \setminus \mathrm{C}(f)$ does not contain a linear image of the Cantor ternary set.*

Proof. Let $C \subseteq [0,1]$ be the Cantor ternary set and let C_s for $s \in {}^{<\omega}2$ denote be the standard clopen basic subsets of C. Then $\mathrm{diam}(C_s) = 3^{-n}$ and $C_s - C_s = [-3^{-n}, 3^{-n}]$ for $s \in {}^n2$. Let $\mathcal{J} = \{A \in \mathcal{M}_1(\mathbf{\Pi}^0_1) : A \cap C$ is nowhere dense in $C\}$. We show that $\mathcal{J}$ satisfies $(1)_1$–$(3)_1$ for $\Gamma = \mathbf{\Pi}^0_1$. If A is a closed set such that $\mathrm{int}(\sigma_1(A)) = \emptyset$, then $A \in \mathcal{M}_1(\mathbf{\Pi}^0_1)$ and $A \in \mathcal{J}$, i.e., $A \cap C$ is nowhere dense in C, because $0 \in \mathrm{int}(C_s - C_s)$ for every $s \in {}^{<\omega}2$. Therefore $\mathcal{J}$ satisfies $(1)_1$, and obviously, $\mathcal{J}$ satisfies also $(2)_1$. To verify that $\mathcal{J}$ satisfies $(3)_1$ use the Baire category theorem and the fact that $\mathrm{ess}(\mathcal{J}) = \mathcal{J}$ and $\mathcal{J}$ has a cofinal subfamily consisting of closed sets. It follows that $\mathcal{J} = \mathcal{M}_1(\mathbf{\Pi}^0_1)$ and therefore $C \notin \mathcal{M}_1(\mathbf{\Pi}^0_1)$. The same is true for any linear image of C. Hence, Theorem 4.2 can be applied. □

Let $\mathcal{M}$ and $\mathcal{N}$ denote the σ-ideal of meager subsets of $\mathbb{R}$ and the σ-ideal of subsets of $\mathbb{R}$ of Lebesgue measure zero, respectively. Let $\mathcal{N}(\Gamma)$ be the family of subsets of sets of reals from $\Gamma \cap \mathcal{M} \cap \mathcal{N}$. In particular, let $\mathcal{N}_0 = \mathcal{N}(\mathbf{\Sigma}^0_2)$ and $\mathcal{N}_1 = \mathcal{N}(\mathbf{\Pi}^0_2)$, i.e., $\mathcal{N}_0$ is the σ-ideal generated by closed sets of measure zero and $\mathcal{N}_1$ is the family of subsets of nowhere dense G_δ sets of measure zero.

Lemma 4.6.

(1) $\mathcal{M}_1(\mathbf{\Sigma}^1_1) \subseteq \mathcal{M} \cap \mathcal{N}$, $\mathcal{M}_1(\mathbf{\Sigma}^0_2) \subseteq \mathcal{N}_0$, *and* $\mathcal{M}_1(\mathbf{\Pi}^0_2) \subseteq \mathcal{N}_1$.

(2) $\mathcal{M}_\infty(\mathbf{\Sigma}^0_2) \nsubseteq \mathcal{N}_1$ *and* $\mathcal{M}_1(\mathbf{\Pi}^0_2) \nsubseteq \mathcal{N}_0$.

Proof. (1) The families $\mathcal{J} = \mathcal{M}, \mathcal{N}$ are σ-ideals such that $\mathcal{J} = \mathrm{ess}(\mathcal{J})$ and therefore they satisfy $(2)_1$ and $(3)_1$ for $\Gamma = \mathbf{\Sigma}^1_1$. To prove $\mathcal{M}_1(\mathbf{\Sigma}^1_1) \subseteq \mathcal{M} \cap \mathcal{N}$ it remains to verify $(1)_1$; the other two inclusions of the form $\mathcal{M}_1(\Gamma) \subseteq \mathcal{N}(\Gamma)$ follow because $\mathcal{M}_1(\Gamma) \subseteq \mathcal{M}_1(\mathbf{\Sigma}^1_1)$ whenever $\Gamma \subseteq \mathbf{\Sigma}^1_1$ and $\Gamma \cap \mathcal{M}_1(\Gamma)$ is cofinal in $\mathcal{M}_1(\Gamma)$.

Let $A \subseteq \mathbb{R}$ be $\mathbf{\Sigma}^1_1$; it is measurable and has the Baire property. If $\mu(A) > 0$, then there is an arbitrary small interval I such that $\mu(A \cap I) > 3|I|/4$, and then $(-|I|/2, |I|/2) \subseteq (A \cap I) - (A \cap I)$. If A is not meager, then there is an arbitrary small open interval I such that $A \cap I$ is comeager in I, and then $I - I = (A \cap I) - (A \cap I)$.

(2) Certainly, $\mathbb{Q} \in \mathcal{M}_1(\mathbf{\Pi}^0_2) \setminus \mathcal{N}_1$. We find a G_δ set $G \in \mathcal{M}_1(\mathbf{\Pi}^0_2) \setminus \mathcal{N}_0$. Fix a decreasing sequence of reals $0 < \delta_n < 1$ such that $\prod_{n\in\omega}(1 - \delta_n) > 0$ and let $M \subseteq [0, 1]$ be the symmetric perfect set of positive measure defined by

$$M = \bigcap_{n\in\omega} \bigcup \mathcal{F}_n,$$

where $\mathcal{F}_n$ are finite systems of pairwise disjoint closed intervals inductively defined for $n \in \omega$ so that $\mathcal{F}_0 = \{[0, 1]\}$ and for each $I \in \mathcal{F}_n$ the system $\mathcal{F}_{n+1}$ contains exactly two subintervals of I of length $\frac{1}{2}(1 - \delta_n)|I|$ which are obtained by removing the middle open subinterval of I of length $\delta_n|I|$. Let $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$.

The set M is a closed nowhere dense set and $\mu(M \cap U) > 0$ for every open set U such that $M \cap U \neq \emptyset$. The family $\{I \cap M : I \in \mathcal{F}\}$ is a base of the relativized topology on M. Let L denote the set of left endpoints of all intervals in $\mathcal{F}$. Then $M \cap (M + r)$ is nowhere dense in M for every $r \in (-1, 1) \setminus (L - L)$. To see this, assume that $I \cap M \subseteq M + r$ for some $I \in \mathcal{F}_n$. Then the interval $J = I - r$ belongs to $\mathcal{F}_n$ because $|J| = |I|$, the endpoints of J are in M, no hole in $M \cap J$ has length bigger than $\delta_n|I|$, and the holes separating neighbouring intervals from $\mathcal{F}_n$ have lengths bigger than $\delta_n|I|$. It follows that $r \in L - L$. Let $Q \subseteq (-1, 1) \setminus (L - L)$ be a countable dense subset of $(-1, 1)$ and let $G = M \setminus \bigcup_{r\in Q}(M + r)$. The set G is a G_δ dense subset of M and $G - G$ does not contain an interval because $Q \cap (G - G) = \emptyset$. The set G cannot be covered by a sequence of closed sets of measure zero since by the Baire category theorem some of these closed sets should have nonempty interior relatively in M, and hence should be of positive measure. □

Question 4.7. Is $\mathcal{M}_\infty(\text{Borel}) \neq \mathcal{M} \cap \mathcal{N}$ and is $\mathcal{M}_\infty(\mathbf{\Pi}^0_2) \subseteq \mathcal{N}_0$?

Lemma 4.8. *The following assertions are equivalent:* (1) $\mathcal{M}_\infty(\mathbf{\Pi}^0_2) \subseteq \mathcal{N}_0$. (2) *For every G_δ set $A \in \mathcal{M}_\infty(\mathbf{\Pi}^0_2)$ there is an open set $U \subseteq \mathbb{R}$ such that $A \subseteq U$ and $U \cap \mathrm{cl}(A)$ has measure zero.* (3) *For every nonempty G_δ set $A \in \mathcal{M}_\infty(\mathbf{\Pi}^0_2)$ there is an open set $U \subseteq \mathbb{R}$ such that $A \cap U \neq \emptyset$ and $U \cap \mathrm{cl}(A)$ has measure zero.*

Proof. Assume that $\mathcal{M}_\infty(\mathbf{\Pi}^0_2) \subseteq \mathcal{N}_0$ and let $A \in \mathcal{M}_\infty(\mathbf{\Pi}^0_2)$ be G_δ. Then $A \subseteq \bigcup_{n\in\omega} F_n$ where F_n's are closed sets of measure zero. Let $U \subseteq \mathbb{R}$ be the largest open set such that $U \cap \mathrm{cl}(A)$ has measure zero. Then $A \subseteq U$ since otherwise by the Baire category theorem there is $n \in \omega$ such that $F_n \setminus U$ contains a relatively open subset of $\mathrm{cl}(A) \setminus U$ and hence F_n has positive measure. This is a contradiction and hence (1) → (2) holds. The implication (2) → (3) is trivial. We prove (3) → (1).

If there is a G_δ set $A \in \mathcal{M}_\infty(\mathbf{\Pi}^0_2) \setminus \mathcal{N}_0$, then $\mathrm{cl}(A)$ has positive measure. Let U be the largest open set of reals such that $U \cap A \in \mathcal{N}_0$ and let $A_0 = A \setminus U$. Then $\mathrm{cl}(A_0) = \mathrm{cl}(A) \setminus U$ and $\mathrm{cl}(A_0) \cap V$ has positive measure whenever V is open and $V \cap A_0 \neq \emptyset$. Therefore (3) fails for the G_δ set $A_0 \in \mathcal{M}_\infty(\mathbf{\Pi}^0_2)$. □

Remark 4.9. The definition of $\mathcal{M}_\infty(\Gamma)$ can be refined by replacing "$\mathrm{int}(\sigma_k(A)) = \emptyset$" in $(1)_k$ with "$\mathrm{int}(p(A)) = \emptyset$" for a polynomial $p = p(x_1, \ldots, x_m) = \sum_{i=1}^m u_i x_i$ with integer coefficients u_i, where $p(A) = \{p(a_1, \ldots, a_m) : \text{all } a_i \in A\}$. Denote

$|p| = \sum_{i=1}^{m} |u_i|$. Inductively define

$$\mathcal{M}^*_{p,0}(\Gamma) = \{A \subseteq \mathbb{R} : (\exists B \in \Gamma)\ A \subseteq B \text{ and } \operatorname{int}(p(A)) = \emptyset\},$$
$$\mathcal{M}^*_{p,\alpha+1}(\Gamma) = \{A \subseteq \mathbb{R} : (\exists B \in \Gamma)\ A \subseteq B \text{ and } B \text{ is an increasing union of sets from } \operatorname{ess}(\mathcal{M}^*_{p,\alpha}(\gamma))\},$$
$$\mathcal{M}^*_{p,\alpha}(\Gamma) = \{A \subseteq \mathbb{R} : (\exists \beta < \alpha)\ A \in \mathcal{M}^*_{p,\beta}(\Gamma)\}, \quad \alpha \text{ limit.}$$

Let $\mathcal{M}^*_p(\Gamma) = \mathcal{M}^*_{p,\omega_1}(\Gamma)$ and let $\mathcal{M}^*_\infty(\Gamma) = \bigcap\{\mathcal{M}^*_p(\Gamma) : |p| > 1\}$. Then $\mathcal{M}_{|p|}(\Gamma) \subseteq \mathcal{M}^*_p(\Gamma)$ because $p(A) - p(A) \subseteq \sigma_{|p|}(A)$. On the other hand for each $k \geq 1$ there is p with $|p| = 2k$ such that $(p(A) = \sigma_k(A)$ and hence$)$ $\mathcal{M}^*_p(\Gamma) = \mathcal{M}_k(\Gamma)$. Therefore $\mathcal{M}^*_\infty(\Gamma) = \mathcal{M}_\infty(\Gamma)$.

5. Perfect subsets with large distance sets

In this section we are interested in the following property of a set $A \subseteq \mathbb{R}$: There exists a perfect set $P \subseteq A$ such that $\operatorname{int}(P - P) \neq \emptyset$. Let us start with the following observation:

Lemma 5.1.

(1) *There are perfect sets $P, Q \subseteq [0,1]$ such that $(P - P) \cup (Q - Q)$ is nowhere dense, $P - Q = [-1, 0]$, $Q - P = [0, 1]$, and hence $(P \cup Q) - (P \cup Q) = [-1, 1]$.*

(2) *There are perfect sets $P, Q \subseteq [0,1]$ such that $P - P = Q - Q = [-1, 1]$ and $(P - Q) \cup (Q - P)$ is nowhere dense.*

(3) *There is a perfect set $A \subseteq [0,1]$ such that $0 \notin \operatorname{int}(A - A) \neq \emptyset$.*

Proof. For a set of integers A and a natural number $p \geq 2$ let A_p denote the set of reals $\{\sum_{i\in\omega} x_i p^{-(i+1)} : x_i \in A\}$. For example, $\{0, 1, \ldots, p-1\}_p = [0, 1]$, $\{0\}_p = \{0\}$, and $\{p - 1\}_p = \{1\}$.

(1) Let $P = \{0, 1\}_4$ and $Q = \{1, 3\}_4$. Then $P - P$ and $Q - Q$ are nowhere dense because $P - P = \{-1, 0, 1\}_4 = \{0, 1, 2\}_4 - \{1\}_4$ and $Q - Q = \{-2, 0, 2\}_4 = 2 \cdot (P - P)$. $Q - P = \{0, 1, 2, 3\}_4 = [0, 1]$ and $P - Q = -(Q - P) = [-1, 0]$.

(2) Let $P = \{0, 2, 6\}_7$ and $Q = \{0, 4, 6\}_7$. Then $P - P = Q - Q$ because $P = 1 - Q$ and $P - P = [-1, 1]$ because $[-1, 1] \supseteq P - P = (P + Q) - 1 \supseteq 2 \cdot \{0, 1, 2, 3, 4, 5, 6\}_7 - 1 = [-1, 1]$. The set $(P - Q) \cup (Q - P)$ is nowhere dense because $P + P = 2 \cdot \{0, 1, 2, 3, 4, 6\}_7$ is nowhere dense and $P - Q = (P + P) - 1$.

(3) Let $P, Q \subseteq [0, 1]$ be perfect nowhere dense sets such that $(P - P) \cup (Q - Q)$ is nowhere dense and $\operatorname{int}(P - Q) \neq \emptyset$. Denote $P' = \frac{1}{3}P$ and $Q' = \frac{1}{3}Q + \frac{2}{3}$ and let $A = P' \cup Q'$. Then $A - A = (P' - P') \cup (Q' - Q') \cup (P' - Q') \cup (Q' - P')$ where $(P' - P') \cup (Q' - Q')$ is nowhere dense and $(P' - Q') \cup (Q' - P') \subseteq [-\frac{3}{3}, -\frac{1}{3}] \cup [\frac{1}{3}, \frac{3}{3}]$. □

Lemma 5.2. *If $(U \setminus A) \cup (V \setminus B)$ is meager for some open sets U and V, then $U - V \subseteq A - B$.*

Proof. If there is $x \in (U - V) \setminus (A - B)$, then the open set $W = U \cap (V + x)$ is nonempty, $A \cap (B + x) = \emptyset$, and $(W \setminus A) \cup (W \setminus (B + x))$ is meager, and then $A \cap (B + x) \neq \emptyset$. This is impossible and hence $U - V \subseteq A - B$. □

For sets $A, I, J \subseteq \mathbb{R}$ denote $A_{I,J} = (A \cap I) - (A \cap J)$. We consider the following properties of a nonempty set $A \subseteq \mathbb{R}$:

(s_1) $A_{I,J}$ is open for any open intervals I and J.

(s_2) $\operatorname{int}(A_{I,J}) = \bigcup\{\operatorname{int}(A_{K,L}) : K \subseteq I \cap U$ and $L \subseteq J \cap V$ are compact intervals$\}$ and $\operatorname{int}(A_{I,J}) \neq \emptyset$ whenever I and J are open intervals such that $A_{I,J} \neq \emptyset$ and $U \supseteq A$ is open.

(s_3) $\operatorname{int}(A_{I,J}) \neq \emptyset$ whenever I and J are open intervals such that $A_{I,J} \neq \emptyset$.

(s_4) $\operatorname{int}(A - A) \neq \emptyset$.

Obviously, (s_1) $\Rightarrow$ (s_2) $\Rightarrow$ (s_3) $\Rightarrow$ (s_4) and (s_3) is equivalent to (s'_3):

(s'_3) $A_{I,J} \subseteq \operatorname{cl}(\operatorname{int}(A_{I,J}))$ for any open intervals I and J.

We say that the set $A \subseteq \mathbb{R}$ is s_i-spread if A satisfies condition (s_i).

For example, the Cantor ternary set $C \subseteq [0, 1]$ is s_2-spread. If A is s_1-spread, then $0 \in \operatorname{int}(A - A)$ because $A - A$ is open.

Lemma 5.3. (1) *Every nonmeager subset of $\mathbb{R}$ with the Baire property has a G_δ s_1-spread subset and* (2) *every measurable subset of $\mathbb{R}$ of positive measure has an F_σ s_1-spread subset and a closed nowhere dense s_3-spread subset.*

Proof. (1) Let $A \subseteq \mathbb{R}$ be a nonmeager set with the Baire property. Let $U \neq \emptyset$ be an open interval such that $U \setminus A$ is meager and let $G \subseteq U \cap A$ be a G_δ dense subset of U. Then G is α_1-spread because for any open intervals I and J the set $U_{I,J}$ is an open interval (if $U_{I,J} \neq \emptyset$) and $G_{I,J} \neq \emptyset \leftrightarrow U_{I,J} \neq \emptyset$, and by Lemma 5.2, $G_{I,J} = U_{I,J}$.

(2) Let $A \subseteq \mathbb{R}$ be a closed set of positive measure. The set

$$B = \{x \in A : (\exists \delta > 0)(\forall h < \delta)\ \mu(A \cap (x - h, x + h)) \geq 3h/2\}.$$

is F_σ and $\mu(B) = \mu(A) > 0$ by Lebesgue density theoreom (see [13]). We prove that B is s_1-spread. Let I and J be open intervals and let $z \in B_{I,J}$. Then $z = x - y$ for some $x \in B \cap I$ and $y \in B \cap J$. Let $h > 0$ be such that $(x - h, x + h) \subseteq I$, $(y - h, y + h) \subseteq J$, $\mu(B \cap (x - h, x + h)) \geq 3h/2$, $\mu(B \cap (y - h, y + h)) \geq 3h/2$. Then $(z - h, z + h) \subseteq B_{I,J}$. Therefore $B_{I,J}$ is open.

Every closed set $A \subseteq \mathbb{R}$ such that $\mu(A \cap I) > 0$ whenever $A \cap I \neq \emptyset$ with I open is an s_3-spread set. □

Theorem 5.4. *Every F_σ s_4-spread set of reals has a perfect compact nowhere dense s_4-spread subset.*

Proof. F_σ sets in $\mathbb{R}$ are σ-compact and therefore $A = \bigcup_{n\in\omega} A_n$ for some compact sets A_n such that $A_n \subseteq A_{n+1}$ for all $n \in \omega$. Then $A - A = \bigcup_{n\in\omega}(A_n - A_n)$ and $A_n - A_n$ is compact for all $n \in \omega$ since $A_n - A_n$ is a continuous image of $A_n \times A_n$. Hence, there is $n \in \omega$ such that $\operatorname{int}(A_n - A_n) \neq \emptyset$ and it is enough to prove the assertion for A compact.

Let $A \subseteq \mathbb{R}$ be compact such that $\operatorname{int}(A - A) \neq \emptyset$. A closed interval I contains a perfect nowhere dense set P such that $P - P = I - I$ (take P isomorphic to the perfect set in Lemma 5.1 (3)). Therefore assume that A is nowhere dense. Let $U = \bigcup\{I : I$ is open and $(A \cap I) - A$ is meager$\}$. Then $P = A \setminus U$ is perfect compact nowhere dense, the sets $(A \cap U) - A$ and $A - (A \cap U)$ are meager, and $A - A = (P - P) \cup ((A \cap U) - A) \cup (A - (A \cap U))$. It follows that $P - P$ is nonmeager compact and hence $\operatorname{int}(P - P) \neq \emptyset$. □

Theorem 5.5. *Every G_δ s_2-spread set of reals has a perfect compact nowhere dense s_3-spread subset.*

Proof. If $\mathrm{int}(A) \neq \emptyset$, then the theorem is certainly true (with the Cantor ternary subset). Without loss of generality we can assume that A is a G_δ s_2-spread subset of an open interval $I_0 = (a, b)$ of length ≤ 1 and $\mathrm{int}(A) = \emptyset$. Let $A = \bigcap_{n\in\omega} U_n$ where $\mathrm{cl}(I_0) \subseteq U_0$ and $U_{n+1} \subseteq U_n$ are open sets. By induction on $k \in \omega$ we define an increasing sequence of natural numbers $\{n_k\}_{k\in\omega}$ and a sequence $\{\mathcal{V}_k\}_{k\in\omega}$ of finite collections of open subintervals of I_0 and we define $V_k = \bigcup \mathcal{V}_k$ and for every $K, L \in \mathcal{V}_k$ a nontrivial closed interval $J^{K,L}$ so that for all $k \in \omega$,

(1) $\mathrm{cl}(V_k) \subseteq U_{n_k}$ and $(\forall i < k)\ n_k > n_i$,
(2) $(\forall K \in \mathcal{V}_k)\ |K| \leq 2^{-k}$ and $A \cap K \neq \emptyset$,
(3) if $K, L \in \mathcal{V}_k$ and $K \neq L$, then $\mathrm{cl}(K) \cap \mathrm{cl}(L) = \emptyset$,
(4) $(\forall i \leq k)(\forall K, L \in \mathcal{V}_i)\ J^{K,L} \subseteq \bigcup\{\mathrm{int}(A_{I,J}) : I, J \in \mathcal{V}_k$ and $I \subseteq K$ and $J \subseteq L\}$,
(5) $(\forall K \in \mathcal{V}_{k+1})(\exists L \in \mathcal{V}_k)\ \mathrm{cl}(K) \subseteq L$, and hence, $\mathrm{cl}(V_{k+1}) \subseteq V_k$,
(6) each interval from $\mathcal{V}_k$ has at least two subintervals in $\mathcal{V}_{k+1}$.

Let $n_0 = 0$, $\mathcal{V}_0 = \{I_0\}$, $V_0 = I_0$, and let $J^{I_0,I_0} \subseteq \mathrm{int}(A_{I_0,I_0})$ be a closed interval which exists by the assumption. Then conditions (1)–(4) are satisfied for $k = 0$. Let $k \in \omega$ be arbitrary and assume that we have defined n_k, $\mathcal{V}_k$, and $J^{K,L}$ for $K, L \in \mathcal{V}_i$ and $i \leq k$ so that (1)–(4) hold. We shall define n_{k+1}, $\mathcal{V}_{k+1}$, and $J^{K,L}$ for $K, L \in \mathcal{V}_{k+1}$.

For every $K \in \mathcal{V}_k$, $A \cap K$ is infinite because $\mathrm{int}(A_{K,K}) \neq \emptyset$. Since A does not contain an interval, there is a finite sequence $a = a_0 < a_1 < \cdots < a_m = b$ of reals from $[a, b] \setminus A$ such that $|a_{i+1} - a_i| \leq 2^{-(k+1)}$ for all $i < m$ and

(7) $(\forall K \in \mathcal{V}_k)(\exists i < m)\ A \cap K \cap (a, a_i) \neq \emptyset$ and $A \cap K \cap (a_i, b) \neq \emptyset$.

Let $n_{k+1} > n_k$ be such that $U_{n_{k+1}} \cap \{a_i : i < m\} = \emptyset$ and let

$$\mathcal{V}^*_{k+1} = \{I : I \text{ is an open interval, } A \cap I \neq \emptyset, \text{ and } (\exists K \in \mathcal{V}_k)\ \mathrm{cl}(I) \subseteq K \cap U_{n_{k+1}}\}.$$

Whenever I is an open interval for which there exists a finite subfamily $\mathcal{V} \subseteq \mathcal{V}^*_{k+1}$ such that $I \cap \bigcup \mathcal{V}$ is a dense subset of I, then $I \in \mathcal{V}^*_{k+1}$. By (4) and by ($s_2$), because $J^{K,L}$ are compact intervals for all $K, L \in \mathcal{V}_i$ and $i \leq k$, there is a finite subfamily $\mathcal{V} \subseteq \mathcal{V}^*_{k+1}$ such that

(8) $(\forall i \leq k)(\forall K, L \in \mathcal{V}_i)\ J^{K,L} \subseteq \bigcup\{\mathrm{int}(A_{I,J}) : I, J \in \mathcal{V}$ and $I \subseteq K$ and $J \subseteq L\}$,
(9) whenever $K \in \mathcal{V}_k$ and $i < m$ are such that $A \cap K \cap (a_i, a_{i+1}) \neq \emptyset$, then $\mathcal{V}$ contains a subinterval of $K \cap (a_i, a_{i+1})$.

Now let $\mathcal{V}_{k+1}$ be the family obtained from $\mathcal{V}$ by gluing together the intervals whose closures have nonempty intersection (then $\mathcal{V}_{k+1} \subseteq \mathcal{V}^*_{k+1}$). Then conditions (1)–(3) and (5) are satisfied by the definition of $\mathcal{V}_{k+1}$, condition (6) is satisfied due to (7) and (9). At last for every $K, L \in \mathcal{V}_{k+1}$ choose any closed interval $J^{K,L} \subseteq \mathrm{int}(A_{K,L})$. This choice together with (8) ensures that condition (4) is satisfied for $k + 1$.

The set $P = \bigcap_{n\in\omega} \mathrm{cl}(V_n)$ is a compact perfect nowhere dense subset of A. Let $n \in \omega$ and $K, L \in \mathcal{V}_n$. We prove that $J^{K,L} \subseteq P_{K,L}$. Let $z \in J^{K,L}$. By (4), for each $k \geq n$ there are $x_k \in K$ and $y_k \in L$ such that $z = x_k - y_k$. By compactness of K and L there is an increasing sequence $\{k_i\}_{i\in\omega}$ of natural numbers such that $\{x_{k_i}\}_{n\in\omega}$ converges. Let $x = \lim_{i\in\omega} x_{k_i}$ and $y = \lim_{i\in\omega} y_{k_i} = x - z$. Let $k \geq n$ be arbitrary. For all $i > k$, $x_{k_i} \in K \subseteq V_k$, and hence $x \in \mathrm{cl}(V_k)$. It follows that $x \in P \cap K$. The same argument proves that $y \in P \cap L$ and hence $z \in P_{K,L}$. □

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