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AN EXAMPLE WHICH DISCERNS POROSITY AND SYMMETRIC POROSITY

Let A be a set of reals and let $I = (a, b)$ be an open interval. $\lambda(A, I)$ denotes the length of the largest open subinterval of the interval I which is disjoint with A. Similarly, $\lambda^*(A, I)$ is the largest real number δ such that $(a, a+\delta) \cup (b-\delta, b)$ is disjoint with A. The porosity and symmetric porosity of the set A at $c \in \mathbf{R}$ are defined by

$$p(A,c) = \limsup_{\varepsilon \to 0^+} \frac{\lambda(A,(c-\varepsilon, c+\varepsilon))}{\varepsilon}, \quad \text{and}$$
$$s(A,c) = \limsup_{\varepsilon \to 0^+} \frac{\lambda^*(A,(c-\varepsilon, c+\varepsilon))}{\varepsilon}$$

respectively. A is porous (resp. strongly porous) if $p(A, a) > 0$ (resp. $p(A, a) = 1$) for every $a \in A$; symmetrically porous (resp. strongly symmetrically porous) if $s(A, a) > 0$ (resp. $s(A, a) = 1$) for every $a \in A$. A is σ-porous if it is a countable union of porous sets. The notions of σ-strongly porous, σ-symmetrically porous, and σ-strongly symmetrically porous sets ([1]) are defined similarly.

Let $\mathbf{P}$, $\mathbf{P}^+$, $\mathbf{S}$, $\mathbf{S}^+$ denote the σ-ideals of σ-strongly porous sets, σ-porous sets, σ-strongly symmetrically porous sets and σ-symmetrically porous sets respectively. Evidently, $\mathbf{P} \subseteq \mathbf{P}^+$, $\mathbf{S} \subseteq \mathbf{S}^+$, $\mathbf{S} \subseteq \mathbf{P}$, $\mathbf{S}^+ \subseteq \mathbf{P}^+$. L. Zajíček [2] had constructed a perfect porous set which is not σ-strongly porous. Using this result together with the next theorem one can easily verify that the above four inclusion relationships are the only ones among these four σ-ideals.

The aim of this paper is to prove the following theorem:

Theorem 1. a) *There exists a G_δ strongly porous set which is not σ-symmetrically porous.*

b) *There exists a closed symmetrically porous set which is not σ-strongly symmetrically porous.*

There is one important difference between porosity and symmetric porosity: If $A \subseteq \mathbf{R}$ is a closed nowhere dense set then the set of reals at which A

is strongly porous is residual in A ([1, Proposition 2.7]). This is not valid for symmetric porosity since the following holds true.

Theorem 2. a) *There exists a closed nowhere dense set A which is not symmetrically porous at any point $a \in A$.*

b) *There exists a closed symmetrically porous set A which is not strongly symmetrically porous at any point $a \in A$.*

Using Theorem 2 we are able to prove Theorem 1.

Proof of Theorem 1. a) Let A be a closed nowhere dense set which is not symmetrically porous at any $a \in A$. There is a set $G \subseteq A$ which is a G_δ dense subset of A such that G is a subset of the set (residual in A) of reals at which A is strongly porous. The set G is strongly porous. Since G is dense in A, every symmetrically porous set $B \subseteq G$ is nowhere dense in G. The set G as a G_δ subset of a compact space is a Baire space (i.e. no open set in G is of first category in G). Therefore G is not σ-symmetrically porous.

b) Let A be a closed nowhere dense set which is not strongly symmetrically porous at any $a \in A$. The same ideas as in part a) of the proof show that A is not σ-strongly symmetrically porous. Another proof of this fact can be obtained by using the generalization of Foran's Lemma given in [1] (Lemma 4.3). □

In what follows we will see that symmetric perfect sets ([1]) are good enough for proving Theorem 2.

Let $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ be a sequence with $0 < \alpha_n < 1$. The symmetric perfect set $C(\alpha)$ is defined as the Cantor ternary set by deleting concentric open intervals of length $\alpha_n d_n$ from the 2^n remaining intervals of length d_n on the n-th step of the construction.

The following Lemma will show how the symmetric porosity of the set $C(\alpha)$ depends on the sequence α.

Lemma. *Let D, D' be arbitrary deleting open intervals from the construction of the symmetric perfect set $C(\alpha)$ such that $|D| \le |D'|$ and let both intervals D, D' be subintervals of the same remaining interval obtained on the n-th step of the construction of the set $C(\alpha)$. Also let d be the distance between the intervals D, D'. Then*

$$\frac{|D|}{\frac{d}{2} + |D|} \le \frac{4\alpha_m}{1 + 3\alpha_m}$$

for some $m \ge n$.

Proof of Theorem 2. The set $C(\alpha)$ is a perfect nowhere dense set.

a) Immediately from the Lemma we get that if $\lim_{n\to\infty} \alpha_n = 0$ then $C(\alpha)$ is nowhere symmetrically porous.

b) If $\alpha_n = \alpha_0 > \frac{1}{3}$ for every $n \in \omega$ then $C(\alpha)$ is symmetrically porous and by Lemma it is nowhere strongly symmetrically porous. □

For the proof of the Lemma we will need some analysis of the set $C(\alpha)$.

There is a natural homeomorphism φ from the Cantor space ${}^{\omega}2$ onto $C(\alpha)$ defined by

$$\varphi(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon(n) d_n (1+\alpha_n) = \sum_{n=0}^{\infty} \varepsilon(n) \frac{(1-\alpha_0)\dots(1-\alpha_{n-1})(1+\alpha_n)}{2^{n+1}}, \quad \varepsilon \in {}^{\omega}2.$$

Endpoints of the remaining and deleting intervals correspond with those sequences $\varepsilon \in {}^{\omega}2$ which are eventually constant (i.e. $\varepsilon(n) = \varepsilon(n+1)$ for almost every n). For example let us compute:

$$\begin{aligned}
\varphi(1,1,\dots) &= \sum_{n=0}^{\infty} \frac{(1-\alpha_0)\dots(1-\alpha_{n-1})(1+\alpha_n)}{2^{n+1}} \\
&= \sum_{n=0}^{\infty} \frac{(1-\alpha_0)\dots(1-\alpha_{n-1})}{2^{n+1}} + \sum_{n=0}^{\infty} \frac{(1-\alpha_0)\dots(1-\alpha_{n-1})}{2^{n+2}} \cdot 2\alpha_n \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(1-\alpha_0)\dots(1-\alpha_{n-1})}{2^{n+1}} + \sum_{n=0}^{\infty} \frac{(1-\alpha_0)\dots(1-\alpha_{n-1})}{2^{n+2}} \cdot 2\alpha_n \\
&= \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(1-\alpha_0)\dots(1-\alpha_n)}{2^{n+2}} + \sum_{n=0}^{\infty} \frac{(1-\alpha_0)\dots(1-\alpha_{n-1})}{2^{n+2}} \cdot 2\alpha_n \\
&= \frac{1}{2} + \frac{1}{2}\varphi(1,1,\dots).
\end{aligned}$$

From this equality we get $\varphi(1,1,\dots) = 1$. Using this trick again one can easily prove that for $\varepsilon \neq \varepsilon'$, $\varphi(\varepsilon) < \varphi(\varepsilon')$ iff $\varepsilon(i_0) = 0$ and $\varepsilon'(i_0) = 1$, where $i_0 = \min\{i : \varepsilon(i) \neq \varepsilon'(i)\}$. We will leave it to the reader to verify some simple facts.

On the n-th step of the construction there were deleted open intervals

$$D_s = (\varphi(s^-), \varphi(s^+)), \quad s \in {}^{n}2,$$

where

$$s^- \restriction n = s^+ \restriction n = s,$$
$$s^-(n) = 0,\ s^+(n) = 1,$$
$$s^-(i) = 1,\ s^+(i) = 0 \quad \text{for } i > n.$$

The length of the interval D_s is the number

$$\begin{aligned}\alpha_n d_n &= \varphi(s^+) - \varphi(s^-)\\ &= \frac{(1-\alpha_0)\dots(1-\alpha_{n-1})(1+\alpha_n)}{2^{n+1}} - \sum_{i=n+1}^{\infty} \frac{(1-\alpha_0)\dots(1-\alpha_{i-1})(1+\alpha_i)}{2^{i+1}}\\ &= \frac{(1-\alpha_0)\dots(1-\alpha_{n-1})(1+\alpha_n)}{2^{n+1}} - \frac{(1-\alpha_0)\dots(1-\alpha_n)}{2^{n+1}}\cdot 1\\ &= \frac{(1-\alpha_0)\dots(1-\alpha_{n-1})}{2^n}\cdot\alpha_n,\end{aligned}$$

where d_n is the length of the remaining closed intervals on the n-th step of the construction.

Now we will introduce several facts about the intervals D_s that we will use later:

(a) If $s \neq t$ then $D_s \cap D_t = \emptyset$.

(b) Let $s \subseteq t$, $s \neq t$. Then $t(|s|) = 1$ iff D_s is on the left of D_t. We write $D_s \prec D_t$ to denote this.

(c) For $s \neq t$ let us define $i_0 = \max\{i : s{\restriction}i = t{\restriction}i\}$. If $i_0 \in \operatorname{dom} s \cap \operatorname{dom} t$ then $D_s \prec D_t$ iff $s(i_0) = 0$ and $t(i_0) = 1$. This together with (b) gives: if $s \neq t$ then $D_s \prec D_t$ iff $s(i_0) = 0$ or $t(i_0) = 1$.

(d) Let $D_s \prec D_t$. Let us define

$$\begin{aligned}i_s &= \max\{i : s{\restriction}i = (s{\restriction}i_0)^-{\restriction}i\},\\ i_t &= \max\{i : t{\restriction}i = (t{\restriction}i_0)^+{\restriction}i\}.\end{aligned}$$

We already know that there are only these possibilities: $s(i_0) = 0$ or $t(i_0) = 1$. Moreover,

(d1) if $s(i_0) = 0$ then $i_0 < i_s$ and $D_s \preceq D_{s\restriction i_s} \prec D_{t\restriction i_0} \preceq D_t$;

(d2) if $t(i_0) = 1$ then $i_0 < i_t$ and $D_s \preceq D_{s\restriction i_0} \prec D_{t\restriction i_t} \preceq D_t$.

(e) Let $D_s \prec D_t$ and let D be the smaller of the two intervals D_s, D_t. Let d denote the distance between these two intervals and let $m = \max\{i_s, i_t\}$. Then

$$\begin{aligned}d = \varphi(t^-) - \varphi(s^+) = &\sum_{i=i_0}^{\infty} t^-(i)\frac{(1-\alpha_0)\dots(1-\alpha_{i-1})(1+\alpha_i)}{2^{i+1}}\\ &- \sum_{i=i_0}^{\infty} s^+(i)\frac{(1-\alpha_0)\dots(1-\alpha_{i-1})(1+\alpha_i)}{2^{i+1}}.\end{aligned}$$

In the next two particular cases we will compute the value of the expression

$$\frac{|D|}{\frac{d}{2}+|D|}.$$

(e1) If $t \subseteq s$ and $i_s = \operatorname{dom} s$ then $i_0 < i_s = m$, $D = D_s$ and

$$s^+(i_0) = 0, \quad s^+(i) = 1 \quad \text{for } i_0 < i \le m, \quad s^+(i) = 0 \quad \text{for } i > m;$$
$$t^-(i_0) = 0, \quad t^-(i) = 1 \quad \text{for } i > i_0.$$

Therefore

$$d = \sum_{i=m+1}^{\infty} \frac{(1-\alpha_0)\dots(1-\alpha_{i-1})(1+\alpha_i)}{2^{i+1}} = \frac{(1-\alpha_0)\dots(1-\alpha_m)}{2^{m+1}}$$
$$= |D| \cdot \frac{1-\alpha_m}{2\alpha_m} \qquad (*)$$

(e2) If $s \subseteq t$ and $i_t = \operatorname{dom} t$ then $i_0 < i_t = m$, $D = D_t$ and

$$s^+(i_0) = 1, \quad s^+(i) = 0 \quad \text{for } i > i_0;$$
$$t^-(i_0) = 1, \quad t^-(i) = 0 \quad \text{for } i_0 < i \le m, \quad t^-(i) = 1 \quad \text{for } i > m.$$

As in the case (e1), $(*)$ holds true again.

In both cases (e1), (e2) we easily obtain

$$\frac{|D|}{\frac{d}{2} + |D|} = \frac{4\alpha_m}{1+3\alpha_m}.$$

Now we are ready to prove the Lemma.

Proof of the Lemma. For two arbitrary different sequences s, t let

$$\sigma(s,t) = \frac{|D|}{\frac{d}{2} + |D|},$$

where d is the distance between the intervals D_s, D_t and D is the smaller interval from D_s, D_t.

Let R be arbitrary remaining interval obtained on the n-th step of the construction of the set $C(\alpha)$ such that both intervals D_s, D_t are subintervals of R. We use the notation introduced in paragraphs (a)–(e). Evidently $n \le i_0$. We may assume without lost of generality that $D_s \prec D_t$. By (c), $s(i_0) = 0$ or $t(i_0) = 1$.

First let us assume that $s(i_0) = 0$. Then according to (d1), $\sigma(s,t) \le \sigma(u,v)$, where $u = s \restriction i_s$, $v = t \restriction i_0$. It is evident that D_u, D_v are subintervals of R too. Therefore it is enough to prove the Lemma in the case when $s = u$, $t = v$, which is just the case (e1). But in this case we have

$$\sigma(u,v) = \frac{4\alpha_m}{1+3\alpha_m},$$

where $m = i_s > i_0 \geq n$.

The proof of the Lemma in the case $t(i_0) = 1$ is similar and involves (d2) and (e2). □

After writing the manuscript of this paper I have learnt from Prof. L. Zajíček about the paper of Michael J. Evans, *A symmetric porosity conjecture of Zajíček*, in which much easier proofs of the first parts of both our theorems were given.

References

[1] Zajíček L. *Porosity and σ-porosity*, Real Analysis Excange **13**, (1987–88), no. 2, 314–350.

[2] Zajíček L. *Sets of σ-porosity and sets of σ-porosity* (q), Časopis Pěst. Mat. **101** (1976), 350–359.