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On convergences of measurable functions in Archimedean vector lattices

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Abstract For Archimedean vector lattices $\mathbf{X}, \mathbf{Y}$ and the positive cone $\mathbb{L}$ of all regular linear operators $L : \mathbf{X} \to \mathbf{Y}$, a theory of sequential convergences of functions connected with an $\mathbb{L}$-valued measure is introduced and investigated.

Keywords Archimedean vector lattices · Operator valued measure · (r)-Convergence · Convergence in measure · Almost everywhere convergence · Almost uniform convergence · Egoroff theorem

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1 Introduction

A vector lattice is a partially ordered (real) vector space which, as a partially ordered set, is a lattice. Many of the most important spaces of functions occurring in measure and integration theory, cf. [7], are vector lattices. Thus, it is not surprising that the

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abstract theory of vector lattices (also called the Riesz spaces) play a significant role in the study of such function spaces. It is also true that the vector lattice structure of function spaces has had a deep influence on the development of the abstract (measure and integration) theory, cf. [6]. Let us mention here the book [4], where the reader can find a generalization of the Riemann integral (called the Kurzweil-Henstock integral) in Riesz space context. Note that the effort to provide an integration technique in vector lattices (definition of the integral as the limit of a certain net) is already contained in paper [11] and others. Also, other types of integral, e.g. the so called Choquet integral with respect to Riesz space-valued (fuzzy) measures, cf. [5], a Bochner-type integral for Riesz space-valued functions, cf. [3], etc., have been introduced.

Usually four types of sequential convergences of functions are associated with a measure: convergence in measure, almost everywhere convergence, almost uniform convergence, and convergence in the mean, cf. [7]. The purpose of this article is to study the first three types of convergences in connection with an $\mathbb{L}$-valued charge (= a finitely additive measure), where $\mathbb{L}$ is the positive cone of the vector lattice of all regular linear operators $L : \mathbf{X} \to \mathbf{Y}$ and $\mathbf{X}, \mathbf{Y}$ are two Archimedean vector lattices. The convergence in mean is closely related to a kind of integral and, consequently, with convergence theorems for that integral. Note here that a Riemann-type integral for such $\mathbb{L}$-valued measures, when $\mathbf{X}$ is an Archimedean vector lattice and $\mathbf{Y}$ is a complete vector lattice, was constructed by the first author in [8]. The second author extended the Dobrakov submeasure to Banach lattices, cf. [9].

2 Preliminaries

Let $\mathbb{N}, \mathbb{R}$ denote the sets of all natural and real numbers, respectively. The notion of a vector lattice is applied here in the sense as in [2]. Alternatively, the term Riesz space is used in [10] and K-lineals in [12]. Let V be a vector lattice and let $S(V)$ be the system of all sequences with elements from V. These sequences will be denoted by $\langle v_i \rangle$, where $v_i \in V, i \in \mathbb{N}$.

Let $\mathbf{X}$ be an Archimedean vector lattice (A.V.L., for short). Denote by $\mathbf{X}^+$ the positive cone of $\mathbf{X}$, i.e. $\mathbf{X}^+ = \{\mathbf{x} \in \mathbf{X}, \mathbf{x} = \mathbf{x} \vee 0\}$. It is a distributive lattice with zero. A sequence $\langle \mathbf{x}_i \rangle \in S(\mathbf{X})$ is called to be (r)-*convergent* to $\mathbf{x} \in \mathbf{X}$ (we write $\mathbf{x} = (r)\text{-}\lim_{i\to\infty} \mathbf{x}_i$), if there exists $u \in \mathbf{X}^+$ (called *the regulator*), such that for each $\varepsilon > 0$ there exist $i_0 \in \mathbb{N}$, such that for all $i \geq i_0, i \in \mathbb{N}$ holds $|\mathbf{x} - \mathbf{x}_i| \leq \varepsilon u$.

Each linear solid subspace of a vector lattice is called *an ideal*. An ideal in an A.V.L. is an A.V.L. as well. If $u \in \mathbf{X}^+$, then the smallest ideal containing u is called *a* (u)*-ideal* in $\mathbf{X}$ (denoted by $\mathbf{X}(u)$). It is easy to see, that

$$\mathbf{X}(u) = \{\mathbf{x} \in \mathbf{X};\ \exists \lambda \in \mathbb{R}, 0 \leq \lambda < +\infty, |x| \leq \lambda u\},$$

and the Minkowski functional $\|\cdot\|_u$ of order interval $[-u, u]$, $u > 0$ is a monotone norm in $\mathbf{X}(u)$ (analogously for $w \in \mathbf{Y}^+$ we introduce a (w)-ideal $\mathbf{Y}(w)$ in $\mathbf{Y}$ and $\|\cdot\|_w$, respectively). It is well-known that in normed lattices the convergence with regulator implies the norm convergence, cf. [1].

Let $\mathbf{X}$, $\mathbf{Y}$ be two real A.V.L. Let $\mathcal{L}(\mathbf{X}, \mathbf{Y})$ be a space of all linear regular operators $L : \mathbf{X} \to \mathbf{Y}$ and $\mathbb{L}$ *the positive cone* of $\mathcal{L}(\mathbf{X}, \mathbf{Y})$, i.e. $L \in \mathbb{L}$ if and only if for every $\mathbf{x} \in \mathbf{X}^+$ there is $L\mathbf{x} \in \mathbf{Y}^+ = \{\mathbf{y} \in \mathbf{Y}; \mathbf{y} = \mathbf{y} \vee 0\}$. Every additive and positive homogeneous operator $L_0 : \mathbf{X}^+ \to \mathbf{Y}^+$ has a unique extension to a linear operator $L : \mathbf{X} \to \mathbf{Y}$, where the extension is defined by the formula $L\mathbf{x} = L_0\mathbf{x}^+ - L_0\mathbf{x}^-$, where $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-, \mathbf{x}^+, \mathbf{x}^- \in \mathbf{X}^+$.

For $(u, w) \in \mathbf{X}^+ \times \mathbf{Y}^+$ denote $\mathbf{L}(u, w) = \{L \in \mathcal{L}(\mathbf{X}, \mathbf{Y}); Lu \in \mathbf{Y}(w)\}$ and for $L \in \mathbf{L}(u, w)$ put $p_{u,w}(L) = \|Lu\|_w$. A sequence $(L_n)_{n\in\mathbb{N}} \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$ of operators *is said to be convergent* to $L \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$ in $\mathbf{L}(u, w)$ whenever $\lim_{n\to\infty} p_{u,w}(L_n - L) = 0$ in $\mathbf{Y}(w)$.

For $(u_1, w_1), (u_2, w_2) \in \mathbf{X}^+ \times \mathbf{Y}^+$ we write $(u_1, w_1) \ll (u_2, w_2)$ if and only if $u_1 \le u_2$ and $w_1 \ge w_2$. Let $\mathfrak{L} = \{\mathbf{L}(u, w);\ u \in \mathbf{X}^+, w \in \mathbf{Y}^+\}$ and on $\mathfrak{L}$ define the operations $\wedge$, $\vee$ and *an order* $\ll$ as follows:

$$\begin{aligned} \mathbf{L}(u_1, w_1) \vee \mathbf{L}(u_2, w_2) &\quad = \quad \mathbf{L}(u_1 \wedge u_2, w_1 \vee w_2), \\ \mathbf{L}(u_1, w_1) \wedge \mathbf{L}(u_2, w_2) &\quad = \quad \mathbf{L}(u_1 \vee u_2, w_1 \wedge w_2), \\ \mathbf{L}(u_2, w_2) \ll \mathbf{L}(u_1, w_1) &\quad \text{iff} \quad (u_1, w_1) \ll (u_2, w_2), \end{aligned}$$

where $(u_1, w_1), (u_2, w_2) \in \mathbf{X}^+ \times \mathbf{Y}^+$. It is easy to see that $\wedge$, $\vee$ are *lattice operations* in $\mathfrak{L}$.

Let T be a non-void set. Denote by 2^T the potential set of the set T and by Σ an algebra of subsets of T. We use χ_E to denote the characteristic function of the set E.

Let $\mathbf{m} : \Sigma \to \mathbb{L}$ be a charge (= a finitely additive measure). For $(u, w) \in \mathbf{X}^+ \times \mathbf{Y}^+$ define the set function $\mu_{u,w} : \Sigma \to [0, +\infty]$ as follows

$$\mu_{u,w}(E) = \|\mathbf{m}(E)u\|_w, \quad E \in \Sigma.$$

If $\mathbf{m}(E)u \notin \mathbf{Y}(w)$, we put $\mu_{u,w}(E) = +\infty$. The following assertion is obvious.

Lemma 2.1 *For every $(u, w) \in \mathbf{X}^+ \times \mathbf{Y}^+$ the set function $\mu_{u,w}$ is a monotone, subadditive, and $\mu_{u,w}(\emptyset) = 0$ (i.e., $\mu_{u,w}$ is a submeasure on Σ).*

Put $\mu = \{\mu_{u,w};\ u \in \mathbf{X}^+, w \in \mathbf{Y}^+\}$ and call it *the generalized submeasure*. A set $E \in \Sigma$ is said to be of *finite generalized submeasure* μ if there exist $(u, w) \in \mathbf{X}^+ \times \mathbf{Y}^+$, such that $\mu_{u,w}(E) < +\infty$, i.e., if $\mathbf{m}(E)u \in \mathbf{Y}(w)$.

For $(u, w) \in \mathbf{X}^+ \times \mathbf{Y}^+$ denote $\mathcal{N}_{u,w} = \{N \in \Sigma;\ \mu_{u,w}(N) = 0\}$. A set $N \in \Sigma$ is called *μ-null* if there exists a couple $(u, w) \in \mathbf{X}^+ \times \mathbf{Y}^+$, such that $N \in \mathcal{N}_{u,w}$. We say that an assertion holds *μ-almost everywhere*, shortly μ-a.e., if it holds everywhere except in a μ-null set.

3 A lattice structure of operator spaces

In this section we consider a lattice structure of the range space of the operator valued measure $\mathbf{m}$, the space $\mathcal{L}(\mathbf{X}, \mathbf{Y})$. The first assertion is trivial.

Lemma 3.1 *The space* $\mathbf{X}$ *is an inductive limit of A.V.L.* $\mathbf{X}(u)$*, where* $u \in \mathbf{X}^+$*, and we write*

$$\mathbf{X} = \operatorname*{injlim}_{u \in \mathbf{X}^+} \mathbf{X}(u).$$

Theorem 3.2 *The family* $\mathfrak{L}$ *of operator spaces is a distributive lattice.*

Proof For $(u_1, w_1), (u_2, w_2), (u_3, w_3) \in \mathbf{X}^+ \times \mathbf{Y}^+$, we have:

$$\begin{aligned}
&\mathbf{L}(u_1, w_1) \vee (\mathbf{L}(u_2, w_2) \wedge \mathbf{L}(u_3, w_3)) \\
&\quad = \mathbf{L}(u_1, w_1) \vee \mathbf{L}(u_2 \vee u_3, w_2 \wedge w_3) \\
&\quad = \mathbf{L}(u_1 \wedge (u_2 \vee u_3), w_1 \vee (w_2 \wedge w_3)) \\
&\quad = \mathbf{L}((u_1 \wedge u_2) \vee (u_1 \wedge u_3), (w_1 \vee w_2) \wedge (w_1 \vee w_3)) \\
&\quad = \mathbf{L}(u_1 \wedge u_2, w_1 \vee w_2) \wedge \mathbf{L}(u_1 \wedge u_3, w_1 \vee w_2) \\
&\quad = (\mathbf{L}(u_1, w_1) \vee \mathbf{L}(u_2, w_2)) \wedge (\mathbf{L}(u_1, w_1) \vee \mathbf{L}(u_3, w_2)).
\end{aligned}$$

By [2], Theorem 2.2, $\mathfrak{L}$ is a distributive lattice. □

The lattice $\mathfrak{L}$ introduces a topology of an inductive limit on $\mathcal{L}(\mathbf{X}, \mathbf{Y})$, i.e. there holds the following theorem.

Theorem 3.3 *Let* $\mathbf{X}$*,* $\mathbf{Y}$ *be two A.V.L. Then*

$$\mathcal{L}(\mathbf{X}, \mathbf{Y}) = \operatorname*{injlim}_{(u,w) \in \mathbf{X}^+ \times \mathbf{Y}^+} \mathbf{L}(u, w).$$

Proof For $(u, w) \in \mathbf{X}^+ \times \mathbf{Y}^+$ it is easy to verify that $\mathbf{L}(u, w)$ is a vector subspace of $\mathcal{L}(\mathbf{X}, \mathbf{Y})$ equipped with the topology given by the seminorm $p_{u,w}(L)$.

Show that

$$\bigcup_{(u,w) \in \mathbf{X}^+ \times \mathbf{Y}^+} \mathbf{L}(u, w) = \mathcal{L}(\mathbf{X}, \mathbf{Y}).$$

It is enough to prove the inclusion $\bigcup_{(u,w) \in \mathbf{X}^+ \times \mathbf{Y}^+} \mathbf{L}(u, w) \supset \mathcal{L}(\mathbf{X}, \mathbf{Y})$ (the reverse inclusion is trivial). Let $L \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$. So, to each $u \in \mathbf{X}^+$ there exists $w_{u,L} \in \mathbf{Y}^+$ such that $Lu \subset w_{u,L}$, i.e. $p_{u,w_{u,L}}(L) \leq 1 < +\infty$. Thus, $L \in \mathbf{L}(u, w_{u,L}) \subset \bigcup_{(u,w) \in \mathbf{X}^+ \times \mathbf{Y}^+} \mathbf{L}(u, w)$.

Let $(u_1, w_1), (u_2, w_2) \in \mathbf{X}^+ \times \mathbf{Y}^+$. Show now that if $\mathbf{L}(u_2, w_2) \ll \mathbf{L}(u_1, w_1)$, then $\mathbf{L}(u_2, w_2) \subset \mathbf{L}(u_1, w_1)$ and if a sequence $(L_n)_{n \in \mathbb{N}} \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$ of operators converges to $L \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$ in $\mathbf{L}(u_2, w_2)$, then the sequence $(L_n)_{n \in \mathbb{N}}$ converges to L in $\mathbf{L}(u_1, w_1)$ as well. Indeed, by definition $(u_1, w_1) \ll (u_2, w_2)$ iff $u_1 \leq u_2$ and $w_1 \geq w_2$. The relation $u_1 \leq u_2$ implies that $p_{u_1,w}(L) \leq p_{u_2,w}(L)$ for every $w \in \mathbf{Y}^+$. On the other hand the relation $w_2 \leq w_1$ implies that $\|L\mathbf{x}\|_{w_1} \leq \|L\mathbf{x}\|_{w_2}$ for every $\mathbf{x} \in \mathbf{X}$. From this we have $p_{u,w_1}(L) \leq p_{u,w_2}(L)$ for every $u \in \mathbf{X}^+$. Thus,

$$p_{u_1,w_1}(L) \leq p_{u_1,w_2}(L) \leq p_{u_2,w_2}(L).$$

So, if $(u_1, w_1) \ll (u_2, w_2)$ and $L \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$, then $p_{u_1,w_1}(L) \leq p_{u_2,w_2}(L)$. □

Theorem 3.4 *For every* $(u_1, w_1) \in \mathbf{X}^+ \times \mathbf{Y}^+$*, the set*

$$\mathfrak{I}_{u_1,w_1} = \{\mathbf{L}(u, w) \in \mathfrak{L};\ \mathbf{L}(u, w) \ll \mathbf{L}(u_1, w_1),\ u \in \mathbf{X}^+, w \in \mathbf{Y}^+\}$$

is an ideal in $\mathfrak{L}$*.*

Proof Let $p, u \in \mathbf{X}^+$ and $w, q \in \mathbf{Y}^+$, such that $(u_1, w_1) \ll (u, w)$ and $(u_1, w_1) \ll (p, q)$. Since $u \wedge p \geq u_1$ and $w \vee q \leq w_1$, then

$$\mathbf{L}(u, w) \vee \mathbf{L}(p, q) = \mathbf{L}(u \wedge p, w \vee q) \ll \mathbf{L}(u_1, w_1).$$

Also, if $p, u \in \mathbf{X}^+$ and $w, q \in \mathbf{Y}^+$, such that $(u_1, w_1) \ll (p, q)$, then

$$\mathbf{L}(u, w) \wedge \mathbf{L}(p, q) = \mathbf{L}(u \vee p, w \wedge q) \ll \mathbf{L}(u_1, w_1).$$

□

Dually to Theorem 3.4, we obtain the following direct consequence.

Corollary 3.5 *For every* $(u_2, w_2) \in \mathbf{X}^+ \times \mathbf{Y}^+$ *the set*

$$\mathfrak{F}_{u_2,w_2} = \{\mathbf{L}(u, w) \in \mathfrak{L};\ \mathbf{L}(u_2, w_2) \ll \mathbf{L}(u, w),\ u \in \mathbf{X}^+, w \in \mathbf{Y}^+\},$$

is a filter in $\mathfrak{L}$*.*

Theorem 3.6 *Let* $(u_1, w_1), (u_2, w_2) \in \mathbf{X}^+ \times \mathbf{Y}^+$*. If* $(u_1, w_1) \ll (u_2, w_2)$*, then the order interval*

$$[\mathbf{L}(u_2, w_2), \mathbf{L}(u_1, w_1)] = \mathfrak{I}_{u_1,w_1} \cap \mathfrak{F}_{u_2,w_2}$$

in $\mathfrak{L}$ *is a Boolean algebra with* $\mathbf{L}(u_2, w_2)$ *as null element and* $\mathbf{L}(u_1, w_1)$ *as unit element.*

Proof Let $u \in \mathbf{X}^+$, $w \in \mathbf{Y}^+$, such that $(u_1, w_1) \ll (u, w) \ll (u_2, w_2)$. Put

$$\mathbf{L}(u, w)^{\perp} = \mathbf{L}((u_2 \backslash u) \vee u_1, (w_1 \backslash w) \vee w_2) \in [\mathbf{L}(u_2, w_2), \mathbf{L}(u_1, w_1)]$$

and show that $\mathbf{L}(u, w)^{\perp}$ is a complement of $\mathbf{L}(u, w)$ in $[\mathbf{L}(u_2, w_2), \mathbf{L}(u_1, w_1)]$. We have

$$\begin{aligned}\mathbf{L}(u, w) \vee \mathbf{L}(u, w)^{\perp} &= \mathbf{L}(u, w) \vee \mathbf{L}((u_2 \backslash u) \vee u_1, (w_1 \backslash w) \vee w_2) \\ &= \mathbf{L}(u \wedge [(u_2 \backslash u) \vee u_1], w \vee [(w_1 \backslash w) \vee w_2]) \\ &= \mathbf{L}([u \wedge (u_2 \backslash u)] \vee [u \wedge u_1], w_1 \vee w_2) = \mathbf{L}(u_1, w_1).\end{aligned}$$

Analogously, $\mathbf{L}(u, w) \wedge \mathbf{L}^{\perp}(u, w) = \mathbf{L}(u_2, w_2)$. So, $\mathbf{L}(u_2, w_2)$ is the null element and $\mathbf{L}(u_1, w_1)$ is the unit element of the Boolean algebra $[\mathbf{L}(u_2, w_2), \mathbf{L}(u_1, w_1)]$. □

4 Convergences of measurable functions

Our basis space which we deal with is the space of all measurable functions in the following sense.

Definition 4.1 The largest vector space of functions $f : T \to \mathbf{X}$ with the property: there exists $r \in \mathbf{X}^+$, such that for every $u \geq r$, $u \in \mathbf{X}^+$, and $\delta > 0$ we have $\{t \in T;\ |f(t)| \geq \delta u\} \in \Sigma$, will be denoted by $\mathcal{M}$ and called *the space of all measurable functions*. We say that a function f is (r)-*measurable* if $f \in \mathcal{M}$.

A function $f : T \to \mathbf{X}$ is called Σ-*simple* if $f(T)$ is a finite set and $f^{-1}(\mathbf{x}) \in \Sigma$ for every $\mathbf{x} \in \mathbf{X} \backslash \{0\}$. The space of all Σ-simple functions is denoted by $\mathcal{S}$. The following result is obvious.

Lemma 4.2 *Let $\mathcal{F}$ be the set of functions $f : T \to \mathbf{X}$, such that there exists $r \in \mathbf{X}^+$ and $f_i \in \mathcal{S}$, $i \in \mathbb{N}$, such that for all $t \in T$ and for all $\varepsilon > 0$ there exists $i_0 \in \mathbb{N}$, such that for all $i \geq i_0$ holds $|f_i(t) - f(t)| < \varepsilon r$. Then $\mathcal{F} \subset \mathcal{M}$.*

The following three definitions introduce the analogies of the notions of convergences almost everywhere, almost uniformly, and in measure in the case of operator-valued charges in A.V.L.

Definition 4.3 Let $\mathbf{m} : \Sigma \to \mathbb{L}$ be a charge and $E \in \Sigma$.

(a) Let $r \in \mathbf{X}^+$ and $(u, w) \in \mathbf{X}^+ \times \mathbf{Y}^+$. We say that *a sequence $(f_i : T \to \mathbf{X})_{i \in \mathbb{N}}$ of functions (r, E)-converges $\mu_{u,w}$*-a.e. *to a function $f : T \to \mathbf{X}$* if for each $\varepsilon > 0$ there exists $i_0 \in \mathbb{N}$, such that for all $i \geq i_0$, $i \in \mathbb{N}$ and for all $t \in E \backslash N$ holds $|f_i(t) - f(t)| < \varepsilon r$, where $N \in \mathcal{N}_{u,w}$.

(b) We say that *a sequence $(f_i : T \to \mathbf{X})_{i \in \mathbb{N}}$ of functions $(r)_E$-converges μ*-a.e. *to a function $f : T \to \mathbf{X}$* if there exist $r \in \mathbf{X}^+$, $u \in \mathbf{X}^+$, $w \in \mathbf{Y}^+$, such that the sequence $(f_i)_{i \in \mathbb{N}}$ of functions (r, E)-converges $\mu_{u,w}$-a.e. to f.

Definition 4.4 Let $\mathbf{m} : \Sigma \to \mathbb{L}$ be a charge and $E \in \Sigma$.

(a) Let $r \in \mathbf{X}^+$ and $(u, w) \in \mathbf{X}^+ \times \mathbf{Y}^+$. We say that *a sequence $(f_i : T \to \mathbf{X})_{i \in \mathbb{N}}$ of functions (r, E)-converges uniformly to a function $f : T \to \mathbf{X}$* if for each $\varepsilon > 0$ and for each $t \in E$ there exists $i_0 \in \mathbb{N}$, such that for all $i \geq i_0$, $i \in \mathbb{N}$ holds $|f_i(t) - f(t)| < \varepsilon r$.

(b) Let $r \in \mathbf{X}^+$, $u \in \mathbf{X}^+$, $w \in \mathbf{Y}^+$. We say that *a sequence $(f_i : T \to \mathbf{X})_{i \in \mathbb{N}}$ of functions (r, E)-converges $\mu_{u,w}$-almost uniformly to a function $f : T \to \mathbf{X}$* if for every $\varepsilon > 0$ there exists a set $F \in \Sigma$, such that $\mu_{u,w}(F) < \varepsilon$ and the sequence $(f_i)_{i \in \mathbb{N}}$ of functions $(r, E \backslash F)$-converges uniformly to f.

(c) We say that *a sequence $(f_i : T \to \mathbf{X})_{i \in \mathbb{N}}$ of functions $(r)_E$-converges μ-almost uniformly to a function $f : T \to \mathbf{X}$* if there exist $r \in \mathbf{X}^+$, $u \in \mathbf{X}^+$, $w \in \mathbf{Y}^+$, such the sequence $(f_i)_{i \in \mathbb{N}}$ of functions (r, E)-converges $\mu_{u,w}$-almost uniformly to f.

Definition 4.5 Let $\mathbf{m} : \Sigma \to \mathbb{L}$ be a charge and $E \in \Sigma$.

(a) Let $r \in \mathbf{X}^+$ and $(u, w) \in \mathbf{X}^+ \times \mathbf{Y}^+$. We say that *a sequence* $(f_i : T \to \mathbf{X})_{i\in\mathbb{N}}$ *of functions* $\mu_{u,w}(r, E)$*-converges to a function* $f : T \to \mathbf{X}$ if for each $\varepsilon > 0$ and $\delta > 0$ there exists $i_{\varepsilon,\delta} \in \mathbb{N}$, such that for every $i \geq i_{\varepsilon,\delta}, i \in \mathbb{N}$ holds $\mu_{u,w}(\{t \in E;\ |f_i(t) - f(t)| \geq \delta r\}) < \varepsilon$.

(b) We say that *a sequence* $(f_i : T \to \mathbf{X})_{i\in\mathbb{N}}$ *of functions* $\mu(r)_E$*-converges to a function* $f : T \to \mathbf{X}$ if there exist $r \in \mathbf{X}^+$, $u \in \mathbf{X}^+$, $w \in \mathbf{Y}^+$, such the sequence $(f_i)_{i\in\mathbb{N}}$ of functions $\mu_{u,w}(r, E)$-converges to f.

The following lemma explains the nature of the sequential convergences given in above definitions.

Lemma 4.6 *Let* $E \in \Sigma$ *and* $r, r_1 \in \mathbf{X}^+$, $(u, w), (u_1, w_1) \in \mathbf{X}^+ \times \mathbf{Y}^+$, *such that* $r \leq r_1, u \geq u_1, w \leq w_1$. *If a sequence* $(f_i : T \to \mathbf{X})_{i\in\mathbb{N}}$ *of functions*

(a) (r, E)*-converges* $\mu_{u,w}$*-a.e.,*
(b) (r, E)*-converges* $\mu_{u,w}$*-almost uniformly,*
(c) $\mu_{u,w}(r, E)$*-converges*

to a function $f : T \to \mathbf{X}$*, then the sequence* $(f_i)_{i\in\mathbb{N}}$ *of functions*

(a) (r_1, E)*-converges* μ_{u_1,w_1}*-a.e.,*
(b) (r_1, E)*-converges* μ_{u_1,w_1}*-almost uniformly,*
(c) $\mu_{u_1,w_1}(r_1, E)$*-converges*

to f*, respectively.*

Proof By assumptions it is easy to see that $\mu_{u_1,w_1}(E) \leq \mu_{u,w}(E)$ for every $E \in \Sigma$, and

$$\sup_{|x|\leq\lambda r} \lambda \geq \sup_{|x|\leq\lambda r_1} \lambda$$

for every $\mathbf{x} \in \mathbf{X}$. The rest of the proof follows from Definition 4.3(a), 4.4(a), and 4.5(a), respectively. □

The remaining results of this section show that the introduced convergences of function are well-defined.

Lemma 4.7 *Let* $E \in \Sigma$. *If a sequence* $(f_i : T \to \mathbf{X})_{i\in\mathbb{N}}$ *of functions* $(r)_E$*-converges* μ*-a.e. to* $f : T \to \mathbf{X}$ *and* $g : T \to \mathbf{X}$*, then* $f = g$ μ*-a.e. on* E*.*

Proof Let $\varepsilon > 0$. By assumption there exist $i_1 \in \mathbb{N}, r_1 \in \mathbf{X}^+, N_1 \in \mathcal{N}_{u_1,w_1}, (u_1, w_1) \in \mathbf{X}^+ \times \mathbf{Y}^+$ and $i_2 \in \mathbb{N}$, $r_2 \in \mathbf{X}^+$, $N_2 \in \mathcal{N}_{u_2,w_2}$, $(u_2, w_2) \in \mathbf{X}^+ \times \mathbf{Y}^+$, such that for each $i \geq \max\{i_1, i_2\}$ holds $|f_i(t) - f(t)| < r_1\varepsilon$ for every $t \in E\backslash N_1$ and for each $i \geq \max\{i_1, i_2\}$ holds $|g_i(t) - g(t)| < r_2\varepsilon$ for every $t \in E\backslash N_2$.

The uniqueness of the (r)-limit in A.V.L. implies that $f(t) = g(t)$ for every $t \in E\backslash(N_1 \cup N_2)$. Further, $N_1 \cup N_2 = N \in \mathcal{N}_{u,w}$, where $u = u_1 \wedge u_2$, $w = w_1 \vee w_2$. □

Lemma 4.8 *Let* $E \in \Sigma$. *If a sequence* $(f_i)_{i\in\mathbb{N}} \in \mathcal{M}$ *of functions* $\mu(r)_E$*-converges to* $f \in \mathcal{M}$ *and* $g \in \mathcal{M}$*, then* $f = g$ μ*-a.e. on* $E \in \Sigma$*.*

Proof Let $\varepsilon > 0, \delta > 0$ be two arbitrary reals and let $r_1, r_2 \in \mathbf{X}^+$, $(u_1, w_1), (u_1, w_2) \in \mathbf{X}^+ \times \mathbf{Y}^+$ be as in Definition 4.5 for the functions f and g, respectively. Put $u = u_1 \wedge u_2$, $w = w_1 \vee w_2$, and $r = r_1 \vee r_2$. Then there exists $i_{\varepsilon,\delta} \in \mathbb{N}$, such that for every $i \geq i_{\varepsilon,\delta}$, $i \in \mathbb{N}$ we have

$$\begin{aligned}&\mu_{u,w}(\{t \in E;\ |f(t) - g(t)| \geq 2 \cdot \delta r\})\\&\quad \leq \mu_{u,w}(\{t \in E;\ |f(t) - f_i(t)| \geq \delta r\} \cup \{t \in E;\ |f_i(t) - g(t)| \geq \delta r\})\\&\quad \leq \mu_{u_1,w_1}(\{t \in E;\ |f(t) - g(t)| \geq \delta r_1\}) + \mu_{u_2,w_2}(\{t \in E;\ |f(t) - g(t)| \geq \delta r_2\})\\&\quad < 2 \cdot \varepsilon.\end{aligned}$$

Since $\delta > 0, \varepsilon > 0$ are arbitrary reals, there is

$$\mu_{u,w}(\{t \in E;\ |f(t) - g(t)| > r\}) = 0,$$

i.e. $f = g$ μ-a.e. on E. □

Remark 4.9 Note that Lemma 4.7 and Theorem 5.2 below imply that if $(f_i : T \to \mathbf{X})_{i\in\mathbb{N}}$ is a sequence of functions which $(r)_E$-converges μ-almost uniformly to $f : T \to \mathbf{X}$ and $g : T \to \mathbf{X}$, then $f = g$ μ-a.e. on $E \in \Sigma$.

In the sequel of this paper we suppose all functions to be (r)-measurable.

5 Relations between convergences of functions

In the previous section we introduced some convergences on the space $\mathcal{M}$ which are generalizations of the classical notions such as the almost uniform convergence, the convergence almost everywhere and the convergence in measure. The following theorems show how these relations are satisfied in the context of A.V.L.

Theorem 5.1 *Let* $\mathbf{m} : \Sigma \to \mathbb{L}$ *be a charge and* $E \in \Sigma$ *be a set of finite generalized submeasure* μ*. If a sequence* $(f_i)_{i\in\mathbb{N}} \in \mathcal{M}$ *of functions* $(r)_E$*-converges* μ*-a.e. to a function* $f \in \mathcal{M}$*, then the sequence* $(f_i)_{i\in\mathbb{N}}$ *of functions* $\mu(r)_E$*-converges to* f*.*

Proof By assumption there are $r \in \mathbf{X}^+$ and $(u_1, w_1) \in \mathbf{X}^+ \times \mathbf{Y}^+$, such that for every $\delta > 0$ there exists $i_\delta \in \mathbb{N}$, such that for every $i \geq i_\delta$, $i \in \mathbb{N}$, there holds

$$|f_i(t) - f(t)| < \delta r, \quad t \in E\backslash N,\ N \in \mathcal{N}_{u_1,w_1}.$$

Also, by assumption there exists a couple $(u_2, w_2) \in \mathbf{X}^+ \times \mathbf{Y}^+$, such that $\mu_{u_2,w_2}(E) < +\infty$. Denote by $u = u_1 \wedge u_2$, $w = w_1 \vee w_2$. Then for every $i \geq i_\delta$, $i \in \mathbb{N}$, we have

$$\begin{aligned}&\mu_{u,w}(\{t \in E;\ |f_i(t) - f(t)| \geq \delta r\})\\&\quad = \mu_{u,w}(\{t \in E\backslash N;\ |f_i(t) - f(t)| \geq \delta r\}) + \mu_{u,w}(N)\\&\quad \leq \mu_{u_2,w_2}(\{t \in E\backslash N;\ |f_i(t) - f(t)| \geq \delta r\}) + \mu_{u_1,w_1}(N)\\&\quad = \mu_{u_2,w_2}(\emptyset) = 0,\end{aligned}$$

hence the result. □

Theorem 5.2 *Let* $\mathbf{m} : \Sigma \to \mathbb{L}$ *be a charge and* $E \in \Sigma$. *If a sequence* $(f_i)_{i\in\mathbb{N}} \in \mathcal{M}$ *of functions* $(r)_E$*-converges* μ*-almost uniformly to a function* $f \in \mathcal{M}$*, then the sequence* $(f_i)_{i\in\mathbb{N}}$ *of functions* $(r)_E$*-converges* μ*-a.e. to* f.

Proof Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary. By Definition 4.4 there exist $r \in \mathbf{X}^+$, $u \in \mathbf{X}^+$, $w \in \mathbf{Y}^+$, and a set $E_\varepsilon \in \Sigma$, such that $\mu_{u,w}(E_\varepsilon) < \varepsilon$ and the sequence $(f_i)_{i\in\mathbb{N}}$ of functions $(r, E\backslash E_\varepsilon)$-converges uniformly to f, i.e., there exists $i_{\varepsilon,\delta} \in \mathbb{N}$, such that for every $i \geq i_{\varepsilon,\delta}$, $i \in \mathbb{N}$ and $t \in E\backslash E_\varepsilon$ holds $|f_i(t) - f(t)| < \delta r$.

Let $\varepsilon = \frac{1}{k}$, $k \in \mathbb{N}$ and put $F_k = E_\varepsilon$. Then $(F_k)_{k\in\mathbb{N}}$ is a sequence of sets from Σ, such that $\mu_{u,w}(F_k) < \frac{1}{k}$ and the sequence $(f_i)_{i\in\mathbb{N}}$ of functions $(r, E\backslash F_k)$-converges uniformly for every $k \in \mathbb{N}$. Put $\bigcap_{k=1}^{\infty} F_k = F$. Thus we have

$$0 \leq \mu_{u,w}(F) \leq \mu_{u,w}(F_k) < \frac{1}{k}. \tag{1}$$

Since $k \in \mathbb{N}$ is an arbitrary number, we conclude that $\mu_{u,w}(F) = 0$. It is easy to show that the sequence $(f_i)_{i\in\mathbb{N}}$ of functions $(r, E\backslash F)$-converges to f. Indeed, if $t \in E\backslash F$, then there exists $k = k(t)$, such that $t \in E\backslash F_k$. But $|f_i(t) - f(t)| < \delta r$ for every $t \in E\backslash F_k$ and $i \geq i_{\varepsilon,\delta}$, $i \in \mathbb{N}$. □

Theorem 5.3 *Let* $\mathbf{m} : \Sigma \to \mathbb{L}$ *be a charge and* $E \in \Sigma$. *If a sequence* $(f_i)_{i\in\mathbb{N}} \in \mathcal{M}$ *of functions* $(r)_E$*-converges* μ*-almost uniformly to a function* $f \in \mathcal{M}$*, then the sequence* $(f_i)_{i\in\mathbb{N}}$ *of functions* $\mu(r)_E$*-converges to* f.

Proof Let $\varepsilon > 0$ and $\delta > 0$ be two arbitrary numbers. By Definition 4.4 there exist $r \in \mathbf{X}^+$, $(u, w) \in \mathbf{X}^+ \times \mathbf{Y}^+$ and a set $F \in \Sigma$, such that $\mu_{u,w}(F) < \varepsilon$ and the sequence $(f_i)_{i\in\mathbb{N}}$ of functions $(r, E\backslash F)$-converges uniformly to f, i.e. there exists $i_0 = i_0(\varepsilon, \delta) \in \mathbb{N}$, such that $|f_i(t) - f(t)| < \delta r$ for every $t \in E\backslash F$ and $i \geq i_0$, $i \in \mathbb{N}$. So,

$$\{t \in E;\ |f_i(t) - f(t)| \geq \delta r\} \subset F$$

for every $i \geq i_0$ and

$$\mu_{u,w}(\{t \in E;\ |f_i(t) - f(t)| \geq \delta r\}) < \varepsilon,$$

which completes the proof. □

To prove the assertion of Theorem 5.5, we introduce the following notion of σ-subadditivity of the generalized submeasure μ.

Definition 5.4 Let $\mathbf{m} : \Sigma \to \mathbb{L}$ be a charge.

(a) For $(u, w) \in \mathbf{X}^+ \times \mathbf{Y}^+$ the set function $\mu_{u,w}$ is said to be σ-*subadditive* on Σ if for each sequence $(E_n)_{n\in\mathbb{N}} \in \Sigma$ holds

$$\mu_{u,w}\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu_{u,w}(E_n).$$

(b) A charge $\mathbf{m}$ is said to be of *σ-subadditive generalized submeasure* μ if for every $(u, w) \in \mathbf{X}^+ \times \mathbf{Y}^+$ the set function $\mu_{u,w}$ is σ-subadditive.

Theorem 5.5 *Let a charge $\mathbf{m} : \Sigma \to \mathbb{L}$ be of σ-subadditive generalized submeasure μ and $E \in \Sigma$. If a sequence $(f_i)_{i\in\mathbb{N}} \in \mathcal{M}$ of functions $\mu(r)_E$-converges to a function $f \in \mathcal{M}$, then there exists a subsequence $(f_{k_i})_{i\in\mathbb{N}}$ of $(f_i)_{i\in\mathbb{N}}$ which $(r)_E$-converges μ-almost uniformly to f.*

Proof Let $E \in \Sigma$. By Definition 4.5 there exist $r \in \mathbf{X}^+$ and $u \in \mathbf{X}^+$, $w \in \mathbf{Y}^+$, such that for every $\varepsilon > 0$ and $\delta > 0$ there exists $i_1 = i_1(\varepsilon, \delta) \in \mathbb{N}$, such that for every $i \geq i_1$, $i \in \mathbb{N}$ the following inequality is true

$$\mu_{u,w}\left(\left\{t \in E;\ |f_i(t) - f(t)| \geq \frac{\delta}{2} r\right\}\right) < \frac{\varepsilon}{2} r. \tag{2}$$

Since for every $i, j \in \mathbb{N}$ we have

$$\begin{aligned} &\{t \in E;\ |f_i(t) - f_j(t)| \geq \delta r\} \\ &\quad \subset \left\{t \in E;\ |f_i(t) - f(t)| \geq \frac{\delta}{2} r\right\} \cup \left\{t \in E;\ |f(t) - f_j(t)| \geq \frac{\delta}{2} r\right\}, \end{aligned} \tag{3}$$

the relations (2) and (3) imply

$$\mu_{u,w}(\{t \in E;\ |f_i(t) - f_j(t)| \geq \delta r\}) < \varepsilon \tag{4}$$

for every $i, j \geq i_1$, $i, j \in \mathbb{N}$.

For every $i \in \mathbb{N}$ there exists $m = m(i) \in \mathbb{N}$, such that if $k, l \geq m$, $k, l \in \mathbb{N}$, then

$$\mu_{u,w}\left(\left\{t \in E;\ |f_k(t) - f_l(t)| \geq \frac{1}{2^i} r\right\}\right) < \frac{1}{2^i}. \tag{5}$$

So, (5) implies that there exists a subsequence $(f_{k_i})_{i\in\mathbb{N}} \in \mathcal{M}$, such that

$$\mu_{u,w}\left(\left\{t \in E;\ |f_{k_{i+1}}(t) - f_{k_i}(t)| \geq \frac{1}{2^i} r\right\}\right) < \frac{1}{2^i}.$$

Without loss of generality suppose $f_{k_i} = f_i$, $i \in \mathbb{N}$.

Show that the sequence $(f_i)_{i\in\mathbb{N}}$ of functions $(r)_E$-converges μ-almost uniformly to f. Put

$$E_i = \left\{t \in E;\ |f_{i+1}(t) - f_i(t)| \geq \frac{1}{2^i} r\right\}.$$

Then there exists $i_2 = i_2(\varepsilon, \delta) \in \mathbb{N}$, such that $1\big/\left(2^{i_2-1}\right) < \varepsilon$. Put $F = \bigcup_{i=i_2}^{\infty} E_i$. By the σ-subadditivity of the generalized submeasure μ we obtain

$$\mu_{u,w}(F) \le \sum_{i=i_2}^{\infty} \mu_{u,w}(E_i) < \frac{1}{2^{i_2}-1} < \varepsilon.$$

We show that the sequence $(f_i)_{i\in\mathbb{N}}$ of functions $(r, E\backslash F)$-converges uniformly. Choose $i_3 = i_3(\varepsilon, \delta) \in \mathbb{N}$, such that $i_3 \ge i_2$ and $1\big/\left(2^{i_3-1}\right) < \delta$. Then

$$|f_i(t) - f_j(t)| \le \sum_{n=i}^{\infty} |f_{n+1}(t) - f_n(t)| \le \frac{1}{2^{i-1}} r \le \frac{1}{2^{i_3-1}} r < \delta r$$

for every $i, j \ge i_3$ and $t \in E\backslash F$. This completes the proof. □

6 Egoroff theorem

To prove the Egoroff theorem in our setting we suppose the charge $\mathbf{m} : \Sigma \to \mathbb{L}$ to be of σ-subadditive and continuous generalized submeasure μ, where the last expression is meant in the following sense.

Definition 6.1 Let $\mathbf{m} : \Sigma \to \mathbb{L}$ be a charge.

(a) For $(u, w) \in \mathbf{X}^+ \times \mathbf{Y}^+$ the set function $\mu_{u,w}$ is said to be *continuous on* Σ if for each sequence $(E_n)_{n\in\mathbb{N}} \in \Sigma$ such that $E_n \searrow \emptyset$ (i.e., $E_n \supset E_{n+1}$, $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} E_n = \emptyset$) with $\mu_{u,w}(E_1) < +\infty$ holds $\mu_{u,w}(E_n) \to 0$ as $n \to \infty$.
(b) We say that a charge $\mathbf{m}$ is of *continuous generalized submeasure* μ if for every couple $(u, w) \in \mathbf{X}^+ \times \mathbf{Y}^+$ the set function $\mu_{u,w}$ is continuous on Σ.

Theorem 6.2 (Egoroff) *Let a charge* $\mathbf{m} : \Sigma \to \mathbb{L}$ *be of* σ*-subadditive and continuous generalized submeasure* μ *and* $E \in \Sigma$ *be a set of finite generalized submeasure* μ. *If a sequence* $(f_i)_{i\in\mathbb{N}} \in \mathcal{M}$ *of functions* $(r)_E$*-converges* μ*-a.e. to a function* $f \in \mathcal{M}$, *then the sequence* $(f_i)_{i\in\mathbb{N}}$ *of functions* $(r)_E$*-converges* μ*-almost uniformly to* f.

Proof Let $\varepsilon > 0$ and $\delta > 0$ be given. By assumption there exists a couple $(u_1, w_1) \in \mathbf{X}^+ \times \mathbf{Y}^+$, such that $\mu_{u_1,w_1}(E) < +\infty$. Also, there exist $r \in \mathbf{X}^+$ and $(u_2, w_2) \in \mathbf{X}^+ \times \mathbf{Y}^+$, such that for given δ there exists $i_\delta \in \mathbb{N}$, such that for every $i \ge i_\delta$, $i \in \mathbb{N}$, there holds

$$|f_i(t) - f(t)| < \delta r \tag{6}$$

for every $t \in E\backslash N$, where $N \in \mathcal{N}_{u_2,w_2}$. Put $E_N = E\backslash N$.

Putting $u = u_1 \wedge u_2$ and $w = w_1 \vee w_2$ we have

$$\begin{aligned}\mu_{u,w}(E) &\le \mu_{u,w}(E_N) + \mu_{u,w}(N) \le \mu_{u_1,w_1}(E_N) + \mu_{u_2,w_2}(N) \le \mu_{u_1,w_1}(E_N) + 0 \\ &\le \mu_{u_1,w_1}(E) < +\infty.\end{aligned}$$

For every $j, l \in \mathbb{N}$ put

$$B_{l,j} = E_N \cap \left\{ t \in E;\ |f_i(t) - f(t)| < \frac{1}{l} r \text{ and } i \geq j \right\}$$
$$= E_N \cap \bigcap_{i=j}^{\infty} \left\{ t \in E;\ |f_i(t) - f(t)| < \frac{1}{l} r \right\},\ i \in \mathbb{N}. \tag{7}$$

Clearly, if $k < j$, then $B_{l,j} \subset B_{l,k}$ for every $j, k, l \in \mathbb{N}$. Put

$$E_l = \bigcup_{j=1}^{\infty} B_{l,j} \in \Sigma.$$

Clearly, the sequence $(E_l \backslash B_{l,j})_{j\in\mathbb{N}}$ tends to void set for every $l \in \mathbb{N}$. Since **m** is of continuous generalized submeasure μ, there exists an index $j_l = j_l(\varepsilon) \in \mathbb{N}$, such that for every $i \geq j_l$, $i \in \mathbb{N}$, there holds

$$\mu_{u,w}(E_l \backslash B_{l,i}) < \frac{\varepsilon}{2^l}.$$

Put

$$F = \bigcup_{l=1}^{\infty} (E_l \backslash B_{l,j_l}) \cup N. \tag{8}$$

By the σ-subadditivity of the generalized submeasure μ we get

$$\mu_{u,w}(F) = \mu_{u,w}\left(\bigcup_{l=1}^{\infty} (E_l \backslash B_{l,j_l}) \cup N \right)$$
$$\leq \sum_{l=1}^{\infty} \mu_{u,w}(E_l \backslash B_{l,j_l}) + \mu_{u,w}(N) \leq \sum_{l=1}^{\infty} \frac{\varepsilon}{2^l} + 0 = \varepsilon. \tag{9}$$

Let us show that the sequence $(f_i)_{i\in\mathbb{N}}$ of functions $(r)_{E\backslash F}$-converges uniformly to f. Without loss of generality suppose that $\delta \leq 1$. Then (6) and (7) imply

$$\bigcup_{n=1}^{\infty} E_n = E_N. \tag{10}$$

Choose $l_0 \in \mathbb{N}$, such that $\frac{1}{l_0} < \delta$. Since $B_{l,j_l} \subset E_l$, the inequalities (8) and (10) imply

$$E_N\backslash F = E_N\Big\backslash\bigcup_{l=1}^{\infty}(E_l\backslash B_{l,j_l}) = \bigcup_{n=1}^{\infty} E_n\Big\backslash\bigcup_{l=1}^{\infty}(E_l\backslash B_{l,j_l})$$
$$= \bigcap_{l=1}^{\infty}\left(\bigcup_{n=1}^{\infty} E_n\Big\backslash(E_l\backslash B_{l,j_l})\right) = \bigcap_{l=1}^{\infty} B_{l,j_l} \subset B_{l_0,j_{l_0}}. \quad (11)$$

By the definition of the set $B_{l_0,j_{l_0}}$, for every $t \in B_{l_0,j_{l_0}}$ and $i \geq j_{m_0}$ we have

$$|f_i(t) - f(t)| < \delta r. \quad (12)$$

So, (8), (9), (11), and (12) imply that for every $\varepsilon > 0$ and $\delta > 0$ there exists an index $j_{l_0} = j_{l_0}(\varepsilon, \delta) \in \mathbb{N}$, such that for every $i \geq j_{l_0}$, $i \in \mathbb{N}$, holds

$$|f_i(t) - f(t)| < \delta r,\ \ t \in E_N\backslash B_{l_0,i} \supset E\backslash F,\ \ \mu_{u,w}(F) < \varepsilon,$$

i.e., the sequence $(f_i)_{i\in\mathbb{N}}$ of functions $(r)_{E\backslash F}$-converges uniformly to f. □

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