Analysis 29, 17–27 (2009) / **DOI** 10.1524/anly.2009.1001

Approximate continuity and topological Boolean algebras

Ján Haluška, Ondrej Hutník

Received: December 11, 2007; Revised: April 29, 2008

Summary: In this paper we describe a net limit method for constructing topologies on given Boolean algebras. If functions in this limit procedure are monotone and approximately continuous in a generalized sense, then the obtained process is recursive.

1 Introduction

For monotone ring topologies on Boolean rings, see cf. [15]. As convenient, we identify a monotone ring topology on a Boolean ring Σ with the 0-neighborhood system $\mathbb{U}$ belonging to it. Then $N(\mathbb{U}) = \bigcap_{U \in \mathbb{U}} U$ is the closure of $\{0\}$ with respect to $\mathbb{U}$. In paper [15], H. Weber established an isomorphism between the lattice $\mathfrak{M}_s(\Sigma)$ of all s-bounded monotone ring topologies (or, FN-topologies) on a Boolean ring Σ and a suitable uniform completion of Σ. Recall that the monotone ring topology $\mathbb{U}$ is called s-bounded if every disjoint sequence in Σ converges to 0 with respect to $\mathbb{U}$. It follows that $\mathfrak{M}_s(\Sigma)$ itself is a complete Boolean algebra. Using these facts he studied s-bounded monotone ring topologies and topological Boolean rings.

In [7] and [8], the first author studied some special monotone ring topologies (given by systems of seminorms) continuous in a generalized sense, and gave a condition which is sufficient and necessary for that the pointwise (net) convergence of measurable function in locally convex spaces implies the convergence in semivariation.

The aim of this paper is to study net limit methods for constructing new topologies on given topological Boolean algebras. We show that if functions in this limit procedure are monotone and approximately continuous in a generalized sense, then we can obtain a recursive process.

Mathematics subject classification (2000): Primary: 28A10, 28A25, 28A60
Key words and phrases: Submeasure, approximately continuous functions, net convergence of functions, topological Boolean algebras

2 Elementary non-limit algorithms of creating monotone ring topologies

Non-additive set functions, as for example outer measures, semi-variations of vector measures, appeared naturally earlier in the classical measure theory concerning countable additive set functions or more general finitely additive set functions. Most of the naturally arising non-additive set functions satisfy some subadditivity conditions which fairly well recompense the lack of additivity. Much attention is paid there to develop a theory of submeasures.

If Σ is a (Boolean) ring on the real line $\mathbb{R}$, then by a *submeasure on* Σ we mean a set function $\nu : \Sigma \to [0, \infty)$ satisfying the following conditions:

(1) $\nu(\emptyset) = 0$;

(2) if $E, F \in \Sigma$ and $E \subset F$, then $\nu(E) \leq \nu(F)$;

(3) if $E, F \in \Sigma$, then $\nu(E \vee F) \leq \nu(E) + \nu(F)$.

The chief motivation for developing a theory of submeasures is that the submeasures are used as a convenient tool in investigating some properties of measures, especially those which can be expressed in terms of continuity with respect to the Fréchet–Nikodym topology (FN-topology, for short). The reader is referred to [4]–[6], and [9] for a more detailed discussion of FN-topologies, submeasures, etc. For miscelaneous reasons, some additional properties of continuity (or exhaustivity) are sometimes added to the property (1) when defining the notion of a submeasure (and/or other generalizations, e.g. a Dobrakov submeasure, cf. [2], and a semimeasure, cf. [3]). There are also many papers where authors consider various generalized settings (e.g. [7, 8, 11], and [15]).

In the following examples we describe some elementary (non-limit) algorithms of creating monotone ring topologies.

Example 2.1 Let $(\mathbb{R}, \Sigma, \lambda)$ be the Lebesgue measure space. If f is a λ-integrable function, then the set function ν_f given by

$$\nu_f(E) = \int_E |f|\, d\lambda, \quad E \in \Sigma,$$

is a measure, and hence a submeasure. Put $N(\lambda) = \{E \in \Sigma; \lambda(E) = 0\}$, and $N(\mathbb{U}_f) = \{E \in \Sigma; \nu_f(E) = 0\}$, where f is λ-integrable. Clearly, $N(\lambda) \subset N(\mathbb{U}_f)$ for every λ-integrable f.

Example 2.2 If $(\mathbb{R}, \Sigma)$ is a measurable space, f is a measurable function, then the set function

$$\nu_f(E) = \sup_{t \in E} |f(t)|, \quad E \in \Sigma,$$

is a submeasure on Σ.

Example 2.3 The proof of the following assertion is based on the triangle inequality in $\mathbb{R}^n$, $n \in \natural$ (the set of all natural numbers). If $\nu_1, \nu_2, \ldots, \nu_n$, are submeasures on Σ, then the set function

$$\nu(E) = \sqrt{\sum_{i=1}^{n} \nu_i^2(E)}, \quad E \in \Sigma,$$

is a submeasure, and $N(\nu) = \bigcap_{i=1}^{n} N(\nu_i)$.

Example 2.4 If $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function, δ a positive real number, and μ a submeasure on Σ, then the set function $\nu_{\delta, f}$ given by

$$\nu_{\delta, f}(E) = \mu(\{t \in E; |f(t)| \geq \delta\}), \quad E \in \Sigma,$$

is a submeasure.

Example 2.5 Let (X, Ξ) be a measurable space, $\mu : \Xi \to [0, \infty]$ be a submeasure, and f be a non-negative measurable function on (X, Ξ). The set function $\nu_f : \Xi \to X$ defined by

$$\nu_f(E) = \int_0^\infty \mu(E \cap F_x)\, dx, \quad E \in \Xi,$$

where $F_x = \{t; f(t) \geq x\}$ for any $x > 0$, is a submeasure on Ξ (as it follows from its structural properties, cf. [12]). The integral used in definition of ν_f is known as the Choquet integral of f on E with respect to μ, cf. [14].

Example 2.6 Let μ be a submeasure on Σ. Let $\mathcal{F}$ be a set of all non-decreasing Lebesgue measurable real functions f on $\mathbb{R}$, such that $f(0) = 0$ and $x \geq y \geq 0$ implies $f(x) - f(y) \leq f(x - y)$. Then the set function

$$\nu_f(E) = f(\mu(E)), \quad E \in \Sigma,$$

is a submeasure. Indeed, if $E, F \in \Sigma$, and $x = \mu(E) + \mu(F)$, $y = \mu(F)$, then

$$\begin{aligned} \mu_f(E \vee F) &= f(\mu(E \vee F)) \leq f(\mu(E) + \mu(F)) = f(x) \\ &\leq f(x - y) + f(y) = f(\mu(E)) + f(\mu(F)) \\ &= \nu_f(E) + \nu_f(F). \end{aligned}$$

If, moreover, f is differentiable, then $0 \leq f'(y) \leq f'(0)$ for every $y \in \mathbb{R}$. For instance, $f(\cdot) = \arctan(\cdot)$ is an example of such function. Indeed, for $x, y \in \mathbb{R}$ such that $x > y \geq 0$, it holds

$$\arctan(x) - \arctan(y) = \arctan\left(\frac{x - y}{1 + xy}\right) \leq \arctan(x - y).$$

The submeasure $\arctan(\mu(\cdot))$ in the last example gives the same ring topology on Σ as the submeasure μ, because the function arctan is continuous. A linear combination of submeasures (if it is a submeasure) yields a new ring topology on Σ if the components in it are linearly independent. To obtain new ring topologies on Σ, nonlinear operations, or noncontinuous functions, or a limit process can be used when creating new submeasures from given ones.

3 Submeasures with function parameters

By *a net* (with values in a set S) we mean a function from a directed partially ordered set Ω to S. A net $\{a_\omega\}_{\omega\in\Omega}$ is *eventually in* a set A iff there is an element $\omega_0 \in \Omega$ such that if $\omega \in \Omega$ and $\omega \geq \omega_0$, then $a_\omega \in A$. Also other terminology about nets (the notion of the subnet, etc.) is used in the standard sense, cf. [10].

Let $\mathcal{F}$ be an (additive) *l-group*, cf. [1], of Lebesgue measurable functions on $\mathbb{R}$ equipped with the following system of gauges

$$\|f\|_E = \sup_{t\in E} |f(t)|, \quad E \in \Sigma,\ f \in \mathcal{F},$$

such that for every $f, g \in \mathcal{F}$ and $E \in \Sigma$ it holds

$$|f| \leq |g| \Rightarrow \|f\|_E \leq \|g\|_E.$$

Shortly, we say that $\mathcal{F}$ is an $(l, \|\cdot\|)$-group.

Definition 3.1 *We say that a* class $\mathcal{S}_\mathcal{F} = \{\nu_f;\ f \in \mathcal{F}\}$ of submeasures on Σ is parametrized by an $(l, \|\cdot\|)$-group $\mathcal{F}$ *of real functions on* $\mathbb{R}$ *if it satisfies the following conditions:*

(a) $\nu_f \in \mathcal{S}_\mathcal{F}$ *implies* $\nu_{-f} \in \mathcal{S}_\mathcal{F}$ *and* $\nu_f(E) = \nu_{-f}(E)$*;*

(b) $\nu_f \in \mathcal{S}_\mathcal{F}$ *and* $\nu_g \in \mathcal{S}_\mathcal{F}$ *implies* $\nu_{f+g} \in \mathcal{S}_\mathcal{F}$ *and*

$$\nu_{f+g}(E) \leq \nu_f(E) + \nu_g(E)$$

for every $f, g \in \mathcal{F}$ *and* $E \in \Sigma$.

If, moreover, there exists a submeasure γ *on* Σ *such that*

(c) $\nu_f(E) \leq \gamma(E) \cdot \|f\|_E$ *for every open finite interval* $E \in \Sigma$*, then we say that* $\mathcal{S}_\mathcal{F}$ is γ-parametrized by $\mathcal{F}$ *on* Σ.

Remark 3.2 Note that if $\gamma(E) = 0$, then $\nu_f(E) = 0$ for every f, such that $\|f\|_E < \infty$. Thus, $N(\gamma) \subset N(\nu_f)$.

Example 3.3 Let $0 < \alpha$ be an ordinal number.

(i) Let $(\mathbb{R}, \Sigma, \lambda)$ be the Lebesgue measure space. If $\mathcal{F}$ is an $(l, \|\cdot\|)$-group of all Lebesgue integrable functions of the α-th Baire class, then the class $\mathcal{S}_\mathcal{F} = \{\nu_f;\ f \in \mathcal{F}\}$ of submeasures of the form

$$\nu_f(E) = \left|\int_E f\, d\lambda\right|, \quad E \in \Sigma,\ f \in \mathcal{F},$$

is λ-parametrized by $\mathcal{F}$.

(ii) If $\mathcal{F}$ is the space of all functions of the α-th Baire class and λ is the Borel measure, then the class $\mathcal{S}_\mathcal{F} = \{\nu_f;\ f \in \mathcal{F}\}$ of submeasures

$$\nu_f(E) = \mu(\{t \in E; |f(t)| \geq \delta\}), \quad E \in \Sigma,\ f \in \mathcal{F},$$

is μ-parametrized by $\mathcal{F}$, where $\mu(E) = \inf_{A\in\Sigma, E\subset A} \lambda(A)$

Denote by $\mathcal{S}_{\mathcal{F}}^{\gamma}$ the class of all γ-parametrized submeasures on Σ by an $(l, \|\cdot\|)$-group $\mathcal{F}$. Clearly, $\mathcal{S}_{\mathcal{F}}^{\gamma} \subset \mathcal{S}_{\mathcal{F}}$. Note to Definition 3.1 (c) that although both γ and $\|f\|_{\cdot}$ are submeasures, their product need not be a submeasure as the following easy example shows.

Example 3.4 Let $E = [1,3]$, $F = [2,4]$, $\gamma(E) = \gamma(F) = 2$, $\|f\|_E = \|f\|_F = 4$, $\gamma(E \vee F) = 3$, $\|f\|_{E\vee F} = 6$. Then $\gamma(E \vee F) \cdot \|f\|_{E\vee F} = 18$, but $\gamma(E) \cdot \|f\|_E + \gamma(F) \cdot \|f\|_F = 16$.

The following lemma shows a limit process of creating new submeasures. Its proof is easy and therefore omitted. The second statement follows immediately from the monotonicity of the considered set functions. However, we do not solve the question on existence of a limit on this place. A sufficient condition for the existence of a limit will be given in Theorem 3.9.

Lemma 3.5 *Let $\{\nu_{(\omega)}\}_{\omega\in\Omega}$ be a net of submeasures on Σ. If a limit*

$$\nu(E) = \lim_{\omega\in\Omega} \nu_{(\omega)}(E)$$

exists for every $E \in \Sigma$, then ν is a submeasure on Σ, and moreover, $\nu_{(\omega)}$, $\omega \in \Omega$, are uniformly continuous.

For a more sophisticated method of creating new submeasures (and new ring topologies on Σ) we need the following few notions.

Definition 3.6 *Let $\mathcal{F}_1, \mathcal{F}_2$ be two $(l, \|\cdot\|)$-groups of Lebesgue measurable functions on $\mathbb{R}$, and let β be a submeasure on Σ. A net of functions $\{f_\omega\}_{\omega\in\Omega} \in \mathcal{F}_1$* β-converges *to a function $f \in \mathcal{F}_2$ if for every $\delta > 0$,*

$$\lim_{\omega\in\Omega} \beta(\{t \in \mathbb{R}; |f_\omega(t) - f(t)| \geq \delta\}) = 0.$$

Definition 3.7 *Let γ be a submeasure on Σ. A net of submeasures $\{\nu_{(\omega)}\}_{\omega\in\Omega}$ is* γ-equicontinuous *if for every $\varepsilon > 0$ there exists an open finite interval $E \in \Sigma$ and $\kappa > 0$ such that $\alpha(E) < \kappa$ and the net $\{\nu_{(\omega)}(\mathbb{R} \setminus E)\}_{\omega\in\Omega}$ is eventually in the interval $[0; \varepsilon)$.*

Definition 3.8 *Let β be a submeasure on Σ.* A net of submeasures $\{\nu_{(\omega)}\}_{\omega\in\Omega}$ is uniformly absolutely β-continuous *if for every $\varepsilon > 0$ there exist $\eta > 0$, such that for every $A \in \Sigma$, with $\beta(A) < \eta$, the net $\{\nu_{(\omega)}(A)\}_{\omega\in\Omega}$ is eventually in the interval $[0; \varepsilon)$.*

Now we state the main theorem describing the limit procedure of creating new submeasures.

Theorem 3.9 *Let γ, β be submeasures on Σ. Let $\mathcal{F}_1, \mathcal{F}_2$, be two $(l, \|\cdot\|)$-groups of Lebesgue measurable functions on $\mathbb{R}$, and let a net of functions $\{f_\omega\}_{\omega\in\Omega} \in \mathcal{F}_1$ β-converge to a function $f \in \mathcal{F}_2$. If is a net of submeasures $\{\nu_{f_\omega}\}_{\omega\in\Omega} \in \mathcal{S}_{\mathcal{F}_1}^{\gamma}$ is*

(i) *uniformly absolutely β-continuous, and*

(ii) *γ-equicontinuous,*

then the limit

$$\overline{\nu}_f(F) = \lim_{\omega \in \Omega} \nu_{f_\omega}(F), \tag{3.1}$$

exists for every $F \in \Sigma$ and $\overline{\nu}_f(\cdot)$ is a submeasure on Σ.

Proof: Let $F \in \Sigma$. If the limit $\overline{\nu}_f(F)$ exists for every $F \in \Sigma$, then it is a submeasure on Σ by Lemma 3.5. Show that $\overline{\nu}_f(F)$ exists.

Since $\mathbb{R}$ is complete, it is enough to show that for every $\varepsilon > 0$ there exists $\omega_\varepsilon \in \Omega$, such that for every $\omega, \omega' \geq \omega_\varepsilon$, there is $|\nu_{f_\omega}(F) - \nu_{f_{\omega'}}(F)| < \varepsilon$.

By (ii) the net of submeasures $\{\nu_{f_\omega}\}_{\omega \in \Omega}$ is γ-equicontinuous. So, for a given $\varepsilon > 0$ there exist an open interval $E \in \Sigma$, $\kappa > 0$ and $\omega_2 \in \Omega$ such that $\gamma(E) < \kappa$ and for every $\omega \geq \omega_2$, $\omega \in \Omega$, there is

$$\nu_{f_\omega}(\mathbb{R} \setminus E) < \varepsilon. \tag{3.2}$$

By Definition 3.1(b) we have that

$$\nu_{f_\omega}(E \wedge F) \leq \nu_{f_\omega - f_{\omega'}}(E \wedge F) + \nu_{f_{\omega'}}(E \wedge F).$$

This implies

$$|\nu_{f_\omega}(E \wedge F) - \nu_{f_{\omega'}}(E \wedge F)| \leq \nu_{f_\omega - f_{\omega'}}(E \wedge F). \tag{3.3}$$

By (3.3), monotonicity, and subadditivity of ν_{f_ω} and $\nu_{f_{\omega'}}$ we get

$$\begin{aligned} &|\nu_{f_\omega}(F) - \nu_{f_{\omega'}}(F)| \\ &\quad \leq |\nu_{f_\omega}(F \wedge (\mathbb{R} \setminus E)) + \nu_{f_\omega}(F \wedge E) + \nu_{f_{\omega'}}(F \wedge (\mathbb{R} \setminus E)) - \nu_{f_{\omega'}}(F \wedge E)| \\ &\quad \leq |\nu_{f_\omega}(F \wedge (\mathbb{R} \setminus E))| + |\nu_{f_{\omega'}}(F \wedge (\mathbb{R} \setminus E))| + |\nu_{f_\omega - f_{\omega'}}(E \wedge F)|. \end{aligned}$$

Clearly, $F \wedge (\mathbb{R} \setminus E) \subset \mathbb{R} \setminus E$. By (3.2),

$$|\nu_{f_\omega}(F) - \nu_{f_{\omega'}}(F)| \leq 2\varepsilon + \nu_{f_\omega - f_{\omega'}}(E \wedge F)$$

for every $\omega, \omega' \geq \omega_2$. By Definition 3.1(c) we obtain

$$\nu_{f_\omega - f_{\omega'}}(E \wedge F) \leq \nu_{f_\omega - f_{\omega'}}(E) \leq \gamma(E) \cdot \| f_\omega - f_{\omega'} \|_E < \kappa \cdot \| f_\omega - f_{\omega'} \|_E.$$

Then for a given $\varepsilon > 0$ there exists $\delta = \varepsilon/\kappa > 0$, such that

$$\| f_\omega - f_{\omega'} \|_E < \delta \quad \text{implies} \quad \nu_{f_\omega - f_{\omega'}}(E \wedge F) < \varepsilon. \tag{3.4}$$

Put $G = \{t \in \mathbb{R}; |f_\omega(t) - f_{\omega'}(t)| < \delta\}$. From subadditivity of $\nu_{f_\omega - f_{\omega'}}$ we have

$$\nu_{f_\omega - f_{\omega'}}(F \wedge E) \leq \nu_{f_\omega - f_{\omega'}}(F \wedge E \wedge G) + \nu_{f_\omega - f_{\omega'}}((F \wedge E) \setminus G). \tag{3.5}$$

By (3.4) and (3.5) we get

$$|\nu_{f_\omega}(F) - \nu_{f_{\omega'}}(F)| \leq 3\varepsilon + \nu_{f_\omega - f_{\omega'}}((E \wedge F) \setminus G). \tag{3.6}$$

The net of functions $\{f_\omega\}_{\omega\in\Omega}$ β-converges to f. Denote by χ_A the characteristic function of the set $A \in \Sigma$. Since β is a monotone set function, the net of functions $\{f_\omega\chi_A\}_{\omega\in\Omega}$ β-converges to $f\chi_A$ as well, where $A \in \Sigma$. Therefore, for every $\eta > 0$ there exists $\omega_1 \in \Omega$ such that for every $\omega \geq \omega_1$, $\omega \in \Omega$,

$$\beta\{t \in A;\ |f_\omega(t) - f_{\omega'}(t)| \geq \delta\} < \eta. \tag{3.7}$$

From uniform absolute β-continuity of the net of submeasures $\{\nu_{f_\omega}\}_{\omega\in\Omega}$ we have that for every $\varepsilon > 0$ there exist $\eta > 0$ and $\omega_3 \in \Omega$ such that for every $\omega \geq \omega_3$, $\omega \in \Omega$,

$$A \in \Sigma, \quad \beta(A) < \eta \quad \text{implies} \quad \nu_{f_\omega}(A) < \varepsilon. \tag{3.8}$$

Further, if $\nu_{f_\omega}(A) < \varepsilon$, where $\omega \in \Omega$, $A \in \Sigma$, then

$$\nu_{f_\omega - f_{\omega'}}(A) \leq \nu_{f_\omega}(A) + \nu_{f_{\omega'}}(A) < 2\varepsilon \tag{3.9}$$

for every $\omega, \omega' \geq \omega_3$.

Put $A = (E \wedge F) \setminus G$ and take $\omega_\varepsilon \in \Omega$, such that $\omega_\varepsilon \geq \max\{\omega_1, \omega_2, \omega_3\}$. Then (3.6), (3.7), (3.8), and (3.9) imply that for every $F \in \Sigma$ and $\varepsilon > 0$ there exists $\omega_\varepsilon \in \Omega$ such that for every $\omega \geq \omega_\varepsilon$, $\omega \in \Omega$, there is

$$|\nu_{f_\omega}(F) - \nu_{f_{\omega'}}(F)| < 5\varepsilon.$$

Hence the existence of limit is proved. □

Remark 3.10 It is clear that the family $\{\overline{\nu}_f(\cdot)\} \vee \{\nu_{f_\omega}(\cdot);\ \omega \in \Omega\}$ is uniformly absolutely β-continuous and γ-equicontinuous. Also, it may be easily verified that for a fixed directed set Ω the limit (3.1) does not depend on the choice of net of functions $\{f_\omega\}_{\omega\in\Omega} \in \mathcal{F}_1$.

4 Approximate continuity and recursive process

For β a submeasure on Σ the following concept of β-approximate continuity is a generalization of the notion of approximate continuity, cf. [13], to submeasures.

Definition 4.1 *Let $\beta : \Sigma \to [0, \infty)$ be a submeasure. A β-*density *of a set $F \in \Sigma$ at $t \in \mathbb{R}$, written $\mathcal{D}_F^\beta(t)$, is* $\lim \beta(E \vee F)/\beta(E)$ *provided the limit exists. Here the limit is taken over intervals E, $t \in E$, and $\beta(E)$ approaching* 0. *A point t is a* point of β-density of F *if $\mathcal{D}_F^\beta(t) = 1$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be β-*approximately continuous at t *if t is a point of β-density of a set F and f is continuous at t with respect to F. A function f is* β-approximately continuous in (a, b), *where $a, b \in \mathbb{R}$, $a < b$, if f is β-approximately continuous at each $t \in (a, b)$.*

In the sequel of this paper we suppose that a submeasure β satisfies the following *Lebesgue density property* (LD-property, for short): β-almost every point of any measurable set $M \subset \mathbb{R}$ is a point of β-density of M (e.g. β is Lebesgue measure on $\mathbb{R}$, cf. [13, § 6.1, p. 150, Lebesgue density theorem]).

Theorem 4.2 *Let β be a submeasure on Σ satisfying LD-property. Let $\mathcal{F}$ be a space of all β-approximately continuous Lebesgue measurable functions on $\mathbb{R}$. If a net of monotone real functions $\{f_\omega\}_{\omega\in\Omega}$ β-converges to $f \in \mathcal{F}$ on a finite interval (a,b), $a < b$, then the net of functions $\{f_\omega\}_{\omega\in\Omega}$ β-a.e. converges to f in each point of the β-approximate continuity of f.*

Proof: Let $\{f_\omega\}_{\omega\in\Omega}$ be a net of nondecreasing functions, and $t_0 \in (a,b)$ be a point of β-approximate continuity of f.

Suppose the contrary, i.e. a net $\{f_\omega(t_0)\}_{\omega\in\Omega}$ does not β-converge to $f(t_0)$. Then there exists $\eta > 0$ such that

$$\limsup_{\omega':\omega\geq\omega'} |f_\omega(t_0) - f(t_0)| \geq \eta.$$

Define the following index sets:

$$\begin{aligned}
\Omega_\eta^+ &= \{\omega \in \Omega;\ f_\omega(t_0) \geq f(t_0) + \eta\},\\
\Omega_\eta^- &= \{\omega \in \Omega;\ f_\omega(t_0) \leq f(t_0) - \eta\},\\
\Omega_\eta &= \Omega_\eta^+ \cup \Omega_\eta^-
\end{aligned}$$

Observe that Ω and Ω_η are cofinal. Therefore either Ω and Ω_η^+, or Ω and Ω_η^- are cofinal. Suppose that Ω and Ω_η^+ are cofinal. Then the net $\{f_\omega\}_{\omega\in\Omega_\eta^+}$ is a subnet of the net $\{f_\omega\}_{\omega\in\Omega}$.

Since t_0 is a point of the β-approximate continuity of f, there exists a set $F \subset (a,b)$ such that t_0 is the point of its β-density and $f|_F$ is continuous at t_0, i.e. there exists $\delta > 0$ such that for all $t \in F$ we have

$$|f_\omega(t_0) - f(t_0)| < \frac{\eta}{2}$$

whenever $0 \leq t - t_0 < \delta$. By definition of Ω_η^+ we get

$$f_\omega(t) - f(t) \geq f_\omega(t_0) - f(t) \geq f_\omega(t_0) + \eta - f(t) \geq \frac{\eta}{2}$$

for every $t \in F$ and $\omega \in \Omega_\eta^+$. Clearly,

$$(t_0, t_0 + \delta) \wedge F \subset \bigcap_{\omega\in\Omega_\eta^+} \left\{t \in (a,b);\ |f_\omega(t) - f(t)| > \frac{\eta}{2}\right\}.$$

Since t_0 is the point of β-density of F, then $\beta((t_0, t_0+\delta) \wedge F) > 0$. By the monotonicity of β,

$$\inf_{\omega\in\Omega_\eta^+} \beta\left(\left\{t \in (a,b);\ |f_\omega(t) - f(t)| > \frac{\eta}{2}\right\}\right) \geq \beta((t_0, t_0+\delta) \wedge F) > 0,$$

but it denies the β-convergence of the net of functions $\{f_\omega\}_{\omega\in\Omega}$ to f.

Analogously we proceed in the case Ω and Ω_η^- cofinal sets. □

Since any measurable function is β-approximatively continuous, cf. [13], from Theorem 4.2 we have

Corollary 4.3 *Let β be a submeasure on Σ satisfying LD-property and let $\mathcal{F}$ be a space of all β-approximately continuous Lebesgue measurable functions on $\mathbb{R}$. If a net of monotone functions $\{f_\omega\}_{\omega\in\Omega}$ β-converges to $f \in \mathcal{F}$ on a finite interval (a, b), $a < b$, then the net of functions $\{f_\omega\}_{\omega\in\Omega}$ β-a.e. converges to f on (a, b).*

Theorem 4.4 *Let γ, β be submeasures on Σ satisfying LD-property. Let $\mathcal{F}_1$ be an $(l, \|\cdot\|)$-group of Lebesgue measurable functions, and $\mathcal{F}_2$ be an $(l, \|\cdot\|)$-group of Lebesgue measurable functions β-approximately continuous on each open finite interval, such that each $f \in \mathcal{F}_2$ is a β-limit of a net of monotone functions from $\mathcal{F}_1$. If the submeasure $\overline{\nu}_{\cdot}(\cdot)$ is defined as in Theorem 3.9, then the class $\mathcal{S}_{\mathcal{F}_2} = \{\overline{\nu}_f(\cdot);\ f \in \mathcal{F}_2\}$ of submeasures is γ-parametrized by $\mathcal{F}_2$ on Σ.*

Proof: Let $F \in \Sigma$ and let us verify conditions of Definition 3.1.

(a) The equality $\overline{\nu}_f(F) = \overline{\nu}_{-f}(F)$ is trivial.

(b) Let a net of functions $\{g_\omega\}_{\omega\in\Omega} \in \mathcal{F}_1$ β-converge to $g \in \mathcal{F}_2$, and let $\overline{\nu}_g(F) = \lim_{\omega\in\Omega} \nu_{g_\omega}(F)$ exist. Then $\overline{\nu}_{f+g}(F)$ exists, and from the equality

$$\overline{\nu}_{f+g}(F) = \lim_{\omega\in\Omega} \nu_{f_\omega+g_\omega}(F),$$

and the obvious inclusion

$$\left\{t \in F;\ \Big|[f_\omega(t) + g_\omega(t)] - [f(t) + g(t)]\Big| \geq \frac{\delta}{2}\right\}$$
$$\subset \{t \in F;\ |f_\omega(t) - f(t)| \geq \delta\} \vee \{t \in F;\ |g_\omega(t) - g(t)| \geq \delta\}, \quad \delta > 0,$$

we get $\overline{\nu}_{f+g}(F) = \overline{\nu}_f(F) + \overline{\nu}_g(F)$.

(c) Let a net of monotone functions $\{f_\omega\}_{\omega\in\Omega} \in \mathcal{F}_1$ β-converge to a function $f \in \mathcal{F}_2$. Let $\{\nu_{f_\omega}\}_{\omega\in\Omega}$ be a net of submeasures on Σ, such that it is uniformly absolutely β-continuous and γ-equicontinuous. We show that for $\overline{\nu}_f(F)$ given by (3.1) holds

$$\overline{\nu}_f(F) \leq \gamma(F)\|f\|_F,$$

where $F = (a, b)$ for $a, b \in \mathbb{R}$, $a < b$.

By Theorem 4.2, the net of functions $\{f_\omega\}_{\omega\in\Omega}$ β-a.e. converges to f on F. Hence, there exists $H \in \Sigma$, such that $\|f_\omega\|_{F\setminus H}$ converges to $\|f\|_{F\setminus H}$ and $\beta(H) = 0$. Then

$$\lim_{\omega\in\Omega} \nu_{f_\omega}(F \setminus H) \leq \gamma(F) \cdot \lim_{\omega\in\Omega} \|f_\omega\|_{F\setminus H},$$

i.e.

$$\overline{\nu}_{f_\omega}(F \setminus H) \leq \gamma(F) \cdot \|f\|_{F\setminus H}.$$

But $\overline{\nu}_f(H) = \lim_{\omega\in\Omega} \nu_{f_\omega}(H)$.

By uniform absolute β-continuity of a net of submeasures $\{\nu_{f_\omega}\}_{\omega\in\Omega}$ we have that $\beta(H) = 0$, and $\omega \in \Omega$ imply $\nu_{f_\omega}(H) = 0$. Thus,

$$\overline{\nu}_f(H) = \lim_{\omega\in\Omega} \nu_{f_\omega}(H) = 0.$$

So,

$$\begin{aligned}\overline{\nu}_f(F) &\le \overline{\nu}_f(F\setminus H)+\overline{\nu}_f(H)=\overline{\nu}_f(F\setminus H)\\ &\le \gamma(F)\cdot\|f\|_{F\setminus H}\le\gamma(F)\cdot\|f\|_F.\end{aligned}$$

This completes the proof. □

Corollary 4.5 *Combining Theorems 4.2 and 4.4, we see that we described a recursive procedure how to create new classes of submeasures from given ones.*

Example 4.6 Let $0<\alpha$ be an ordinal number. Let $\mathcal{F}_1,\mathcal{F}_2,\dots,\mathcal{F}_\alpha,\dots$ be the α-th Baire classes of real functions defined on $\mathbb{R}$. Let $\mathcal{M}$ be the class of all uniformly bounded, approximately continuous, monotone, nonnegative Lebesgue measurable functions on $(0,1)$. Denote by $\mathcal{G}_\alpha=\mathcal{F}_\alpha\wedge\mathcal{M}$.

Define

$$\gamma(E)=\begin{cases}1, & (0,1)\wedge E\neq\emptyset,\\ 0, & (0,1)\wedge E=\emptyset\end{cases}$$

for $E\in\Sigma$. It is easy to see that $\gamma:\Sigma\to\{0;1\}$ is a submeasure. Let $\beta=\lambda$ (the Lebesgue measure on $\mathbb{R}$). Clearly, β satisfies the LD-property.

Define a (sub)measure $\nu^\alpha(\cdot,\cdot)$ on Σ, parametrized by $\mathcal{G}_\alpha$, as follows

$$\nu^\alpha(E,f)=\int_{E\wedge[0,1]} f\,d\lambda\le\|f\|_E\cdot\lambda(E), \tag{4.1}$$

for $E\in\Sigma$, $f\in\mathcal{G}_\alpha$.

Since $\nu^\alpha(\mathbb{R}\setminus(0,1),\ f^{(\alpha)}_{\omega_\kappa,\omega_{\kappa+1},\dots})=0$, $f^{(\alpha)}_\omega\in\mathcal{G}_\alpha$ for every $\omega_\kappa\in\natural$, $0<\kappa$, the family $\nu^\alpha(\cdot,\ f^{(\alpha)}_{\omega_\kappa,\omega_{\kappa+1},\dots})$, $\omega_\kappa\in\natural$, is a γ-equicontinuous sequence of submeasures for every $0<\alpha$, and $0<\kappa$.

Consider the following Souslin tree of functions (i.e. an uncountable tree of countable height and countable width). The sequence $f^{(\alpha)}_{\omega_\kappa,\omega_{\kappa+1},\dots}\in\mathcal{G}_\alpha$, $0<\kappa$, $\omega_\kappa\in\natural$, of functions pointwise converges to the function $f^{(\alpha+1)}_{\omega_{\kappa+1},\omega_{\kappa+2},\dots}\in\mathcal{G}_\alpha$, $0<\alpha$. The direction in the net of functions, the described Souslin tree, is given by the pointwise convergence of functions.

From LD-property, the uniform boundedness of functions of $\mathcal{M}$ and (4.1), we obtain that $\nu^\alpha(\cdot,\cdot)$, $0<\alpha$, are uniformly absolutely λ-continuous.

So, describing a Souslin tree of functions we obtain the isomorphic Souslin tree $\nu^\alpha(\cdot,\ f^{(\alpha)}_{\omega_\kappa,\omega_{\kappa+1},\dots})$ of submeasures dominated by the Lebesgue measure on $\mathbb{R}$.

If we denote $\mathcal{N}_\alpha=\{\nu^\alpha(\cdot,\ f);\ f\in\mathcal{G}_\alpha\}$, then it is known that $\mathcal{N}_\alpha\subset\mathcal{N}_{\alpha+1}$, $\mathcal{N}_\alpha\neq\mathcal{N}_{\alpha+1}$.

Acknowledgements: This paper was supported by Grants VEGA 2/0097/08, APVT-51-006904 and CNR–SAS project 2007–2009 “Integration in abstract structures”.

References

[1] G. Birkhoff. *Lattice Theory*. Amer. Math. Soc. Colloq. Publ. Vol. XXV, AMS, Providence, Rhode Island, 1967.

[2] I. Dobrakov. On submeasures I. *Dissertationes Math.*, 112: 5–35, 1974.

[3] I. Dobrakov, and J. Farková. On submeasures II. *Math. Slovaca* 30: 65–81, 1980.

[4] L. Drewnowski. Topological rings of sets, continuous set functions, integration I. *Bull. Acad. Polon. Sci. Ser. Sci. Math., Astr. Phys.* 20: 269–276, 1972.

[5] L. Drewnowski. Topological rings of sets, continuous set functions, integration II. *Bull. Acad. Polon. Sci. Ser. Sci. Math., Astr. Phys.* 20: 277–286, 1972.

[6] L. Drewnowski. Topological rings of sets, continuous set functions, integration III. *Bull. Acad. Polon. Sci. Ser. Sci. Math., Astr. Phys.* 20: 439–445, 1972.

[7] J. Haluška. On the generalized continuity of the semivariation in locally convex spaces. *Acta Univer. Carolin. – Math. Phys.* 32: 23–28, 1991.

[8] J. Haluška. On the continuity of the semivariation in locally convex spaces. *Math. Slovaca* 43: 185–192, 1993.

[9] O. Hutník. On vector-valued Dobrakov submeasures (submitted).

[10] J. L. Kelley. *General Topology*. Van Nostrand, New York, 1955.

[11] V. M. Klimkin, and M. G. Svistula. Darboux property of a non-additive set functions. *Sb. Math.* 192: 969–978, 2001.

[12] G. J. Klir, Z. Wang, and W. Wang. Monotone set functions defined by Choquet integral. *Fuzzy Sets and Systems* 81: 241–250, 1996.

[13] J. Lukeš, J. Malý, and L. Zajíček. *Fine Topology Methods in Real Analysis and Potential Theory*. Springer-Verlag, Lectures Notes in Mathematics 1189, Berlin – Heidelberg – New York – London – Paris – Tokyo, 1986.

[14] R. Mesiar. Fuzzy measures and integrals. *Fuzzy Sets and Systems* 156: 365–370, 2005.

[15] H. Weber. Topological Boolean rings. Decomposition of finitely additive set functions. *Pacific J. Math.* 110: 471–495, 1984.

Ján Haluška
Mathematical Institute
Slovak Academy of Science
Grešákova 6
040 01 Košice
Slovakia
jhaluska@saske.sk

Ondrej Hutník
Institute of Mathematics
Faculty of Science
Pavol Jozef Šafárik University in Košice
Jesenná 5
041 54 Košice
Slovakia
ondrej.hutnik@upjs.sk