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ON ALGEBRAS OF SYMMETRICAL ASSOCIATIVE AGGREGATION OPERATORS RELATED TO MEANS

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ABSTRACT. The aim of this paper is to introduce and study finite dimensional algebras of symmetrical associative aggregation operators related to means.

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1. n-DIMENSIONAL CASE

Definition 1. Let $\mathbb{T}$ be a subset of the set $\mathcal{T} \neq \emptyset$. Let $\varphi : \mathbb{T} \to \Lambda$ be a function where Λ is a commutative l-semigroup. The set

$$\mathcal{R}^{\mathbb{T}} = \{\varphi(\tau); \quad \tau \in \mathbb{T}\} \tag{1}$$

is said to be a *tone system (in a narrower sense)* (cf. [3]).

The tone system in a narrower sense is an image of a partially defined function with values in an l-semigroup (a non-empty ordered set with an associative binary operation).

Tone systems were first introduced in School of Pythagoras and later used in research of partial differential equations and Fourier analysis. Studying the relevant literature we identify, that the reference denotes a unified theory defining the tone system in functional terms based on the principles and forms of uncertainty theory, [3]. We found out that the mathematical theory of tone systems is a baseline for every type of uncertainty objects (for a unified uncertainty theory, cf. [6]). Psycho-acoustical properties of tone systems provide a good motivation to study algebras of symmetric associative aggregation operators related to means and operations over them. Let us explain the musical motivation.

A set $\mathcal{T}$ is a universe of all objects called tones. Since a tone is a fundamental notion, from the mathematical point of view, there is no need to determine the precise character of tones, however, there are both psycho-acoustical and natural scientific imaginations. A subset $\mathbb{T} \subset \mathcal{T}$ is chosen by a musician from the set $\mathcal{T}$ to manifest that the act of sampling is immanent to art (in general). However, the aim of this paper is not to describe the so-called tone systems *in a broader sense* which notion includes the following concepts for a given tone system: the sampling algorithm, basic equation/relation, notion of symmetry, and fuzzy measure (cf. [3, 5]). Thus, due to some outside reasons, let us only emphasize that we cannot skip $\mathbb{T}$ and only consider the set $\mathcal{T}$ directly. In (1) the function $\varphi : \mathbb{T} \to \Lambda$ is called a *pitch function*. Hence $\varphi(\tau) \in \Lambda$ is a pitch of the tone $\tau \in \mathbb{T}$. We often have:

$\Lambda = \mathbb{R}$ (the set of all real numbers) or

$\mathbb{Z}$ (the set of all integer numbers) or

$\mathbb{N}$ (the set of all natural numbers) or

$\mathbb{C}$ (the set of all complex numbers) or

$\mathbb{Q}$ (the set of all rational numbers) or

or $\mathbb{Q}_{p,q}$ ("spirals of the fifths", the set of all numbers of type $p^\alpha q^\beta$, where α, β are rational numbers and p, q are two algebraic numbers) or

$\mathbb{F}_\triangle$ (the set of all triangular fuzzy numbers).

All the sets have their usual orders.

Example 1. One of the most known and simplest tone systems is *12-tone Equal Temperament* $E_{12} = \{(\sqrt[12]{2})^\tau; \quad \tau \in \mathbb{T} = \mathbb{Z}\}$. Here, $\Lambda = E_{12}$ is equipped with the operation of multiplication and the order induced by $\mathbb{R}$. This tone system reflects rather well *the opinion that the tone is of a spiritual nature* (all numbers only, the octaves are irrational).

Example 2. Perhaps the tone systems called *Pure Tunings* are the most typical representatives of *an opinion that hearing is primary and spirit secondary* as they are constructed with respect to the existence of a higher harmonics of the vibrating string. Since the presence of harmonics is objective, it does not depend on our psyché. An l-semigroup Λ of the simplest tone system of this kind is generated by the set of generators $\langle 1, 2, 3\rangle$ and the group operation of multiplication having the lexicographical ordering. For instance, $1/2 = 2^{-1} \leq 8 = 2^3$ and $9 = 3^2 \leq 1/2 = 2^{-1}$.

Example 3. V. Liern [7] developed a pitch of tones as fuzzy *triangular* numbers (cf. Figure 1). Here, the l-semigroup $\Lambda \subset \mathbb{F}_\triangle$ is not an l-group but an l-grupoid (cf. [2]) with respect to the operation

$$(a \wedge b)(t) = \frac{\min(a(t), b(t))}{\max_{t\in\mathbb{R}} \min(a(t), b(t))}, a \in \mathbb{F}_\triangle, b \in \mathbb{F}_\triangle, t \in \mathbb{R}.$$

In [3], Definition 1, a pitch function is introduced in general. Without considering the loudness, duration, and the other tone attributes incorporated into a tone system notion, the pitch is a one-dimensional object. In the following definition we specify the pitch function φ and l-semigroup Λ as n-dimensional objects from Definition 1.

Definition 2. Let $\mathbb{T} \subset \mathcal{T}$ be a set. Let $n \in \mathbb{N}$. Let $\varphi : \mathbb{T} \to \Lambda$ be a vector function of the vector argument, where $\varphi(\tau) = (\varphi_1, \ldots, \varphi_n)(\tau) = (\varphi_1(\tau), \ldots, \varphi_n(\tau))$, $\tau \in \mathbb{T}$, and $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$ where Λ_i, $i = 1, \ldots, n$, are commutative l-semigroups and the l-semigroup Λ is equipped with a lexicographical order. Then the set

$$\mathcal{R}^{\mathbb{T},n} = \{\varphi(\tau); \quad \tau \in \mathbb{T}\} \tag{2}$$

is said to be a *complex tone system (in a narrower sense)*.

In particular, we put $\mathcal{R}^{\mathbb{T}} = \mathcal{R}^{\mathbb{T},1}$, $\mathcal{C}^{\mathbb{T}} = \mathcal{R}^{\mathbb{T},2}$.

In Sections 4 and 5 we will construct the operations over the set of all tone systems introduced in Definition 2 and we will demonstrate the need for this definition.

2. Composite vector pitch functions

The individual coordinates $\varphi_1, \varphi_2, \ldots, \varphi_n$ (cf. Definition 2) of the pitch function $\varphi : \mathbb{T} \to \Lambda$ represent the tone attributes. They can be of psychological or physical nature. With each *psychological tone attribute* corresponds its *physical counterpart*, and vice–versa. As a result we get couples: *pitch* $\leftrightarrow$ frequency, *loudness* $\leftrightarrow$ amplitude, *duration* $\leftrightarrow$ time interval, etc. Correspondences between psychological and physical attributes within these pairs are not precious. Thus we set questions about the nature of l-semigroups Λ_i, $i = 1, \ldots, n$ (semigroup operations, orders, additional structures) from Definition 2 which reflect both the psychological and/or physical sides of tones. Concerning the couple *timbre* $\leftrightarrow$ Fourier (or wavelet) series, W. Sethares [9] constructs the tone systems optimal in some sense for a given timbre, and vice–versa.

An idea how to define operations over tone systems is a result of the observation that every attempt to define such operations based exclusively on psychological or exclusively on physical attributes has failed. But how to integrate the two seemingly mutually fighting parts – qualitative and quantitative tone attributes – into a whole? The solution: to construct the pitch functions as composite vector functions. Mathematically speaking, the principal qualitative "jump" will occur when expanding one dimension into two dimensions, from $\mathcal{R}^{\mathbb{T},1}$ to $\mathcal{R}^{\mathbb{T},2}$ (cf. Section 4). The generalization to $n > 2$ dimensions is easy then.

For instance, let us deal with two psychological tone attributes, the pitch $\varphi(\tau)$ and the loudness $\psi(\tau)$, where $\tau \in \mathbb{T}$. Now, besides psychological attributes, let us mention two physical attributes of this tone. In general, they need not be the counterparts to the considered psychological tone attributes, φ and ψ in our case. For instance, we may consider $(\varphi(\tau), \psi(\tau)) \in \mathcal{C}^{\mathbb{T}}$, where $\varphi(\tau) = [\Phi \circ (\xi, t)](\tau) = \Phi(\xi(\tau), t(\tau))$, $\psi(\tau) = [\Psi \circ (\xi, t)](\tau) = \Psi(\xi(\tau), t(\tau))$. Here, the *time* $t \in \mathbb{R}$ is a physical duration of the tone (in seconds), and the tone *frequency* $\xi \in \mathbb{R}$ (in Hz) and are supposed to be equal to the time duration and the frequency of the first harmonic in Fourier series representation of the tone $\tau \in \mathbb{T}$.

In this way, we can (or according to [5] we have to) introduce a fuzzy set structure as a refining of the mathematical structure of a tone system from Definition 2. Therefore it is legitimized/reasonable to call such parameterized tone systems the *fuzzy tone systems*. Of course, we can also consider the frequency and time to be functions of a pitch and a loudness, i.e., to make the superposition in a reverse order, starting from physical attributes and fuzzifying them by psychological tone attributes.

There exist some physical and biological limits of human hearing (maximal heard frequency and threshold of pain). However, they are individual and depend on age, mood, situation, etc. Therefore, despite the principled possibility, we do not suppose the normalization of fuzzy sets.

3. Aggregation operator algebras

Let us recall some facts about aggregation operators which are the main tools when defining operations over the set of all tone systems.

Let $\langle \Lambda, \otimes \leq, 0\rangle$ be a commutative l-semigroup *with the minimal element* 0. Under an *aggregation operator* we will understand the mapping

$$A : \bigcup_{m\in\mathbb{N}} \Lambda^m \to \Lambda,$$

satisfying the following two axioms. For every $m \in \mathbb{N}$,

$$A_{(m)}(0, \dots, 0) = 0 \tag{3}$$

and

$$\begin{gathered} \varphi'_{(1)} \leq \varphi''_{(1)}, \dots, \varphi'_{(m)} \leq \varphi''_{(m)} \\ \Rightarrow A_{(m)}(\varphi'_{(1)}, \dots, \varphi''_{(m)}) \leq A_{(m)}(\varphi''_{(1)}, \dots, \varphi''_{(m)}), \end{gathered} \tag{4}$$

where $A_{(m)}$ is a restriction $A_{(m)} = A|_{\Lambda^m}$ and $\varphi'_{(n)} \in \Lambda_n, \varphi''_{(n)} \in \Lambda_n$ (the first m coordinates), where Λ_n is the n-th copy of Λ in the Cartesian product Λ^m, $1 \leq n \leq m$.

For further reading about aggregation operators cf. [1]. It is known that the assertions of binary associative commutative (= symmetrical) operators (operations) can be easily extended to a general case of the associative symmetrical aggregation operators. That is why in this paper, a binary operator A will be denoted by $\otimes_A$ as a binary operation i.e., $A(\varphi', \varphi'') = \varphi' \otimes_A \varphi'' = \varphi'''$. Hence, in the case of associative and symmetrical operators we write:

$$(\varphi' \otimes_A \varphi'') \otimes_A \varphi''' = \varphi' \otimes_A (\varphi'' \otimes_A \varphi''')$$

and

$$\varphi' \otimes_A \varphi'' = \varphi'' \otimes_A \varphi',$$

respectively, for every $\varphi', \varphi'', \varphi''' \in \Lambda$.

Example 4. The operations of minimum, maximum, (truncated) addition, and multiplication of fuzzy sets define the associative and symmetrical aggregation operators.

Example 5. Let the binary operation $\otimes : \Lambda \times \Lambda \to \Lambda$ generate a symmetrical associative aggregation operator. Let $\beta : \Lambda \to \Lambda$ be a monotone bijective function. Then the aggregation operator given by the following binary operator $A(\varphi', \varphi'') = \beta^{-1}\{\beta(\varphi') \otimes \beta(\varphi'')\}$ is an associative and symmetrical operator. Indeed,

$$\begin{array}{rcl} A(A(\varphi', \varphi''), \varphi''') & = & \beta^{-1}\{\beta(A(\varphi', \varphi'')) \otimes \beta(\varphi''')\} \\ & = & \beta^{-1}\{\beta[\beta^{-1}(\beta(\varphi') \otimes \beta(\varphi''))] \otimes \beta(\varphi''')\} \\ & = & \beta^{-1}\{\beta(\varphi') \otimes \beta(\varphi'') \otimes \beta(\varphi''')\} \end{array}.$$

For a fixed $\tau \in \mathbb{T}$ (cf. Definition 2), we will apply the various associative symmetrical aggregation operators A_k, resp. the operations $\otimes_k$, $k = 1, \dots, K$, to vectors $\varphi(\tau) = (\varphi_1, \dots, \varphi_n)(\tau) = (\varphi_1(\tau), \dots, \varphi_n(\tau))$. It leads us to the notion of an *n-dimensional algebra* over the set $\mathbb{T}$.

Definition 3. Let $\mathcal{T} \neq \emptyset$ be a set. Let $\mathbb{T} \subset \mathcal{T}$ be a subset. Let $n \in \mathbb{N}$, $K \in \mathbb{N}$. The $(K+1)$-tuple $\mathfrak{R}^{\mathbb{T},n} = \langle \mathcal{R}^{\mathbb{T},n}; \quad \otimes_1, \dots, \otimes_K\rangle$ is said to be an *n-dimensional algebra* over the set $\mathbb{T}$ if the binary operations $\otimes_1, \dots, \otimes_K$ are commutative, associative, and closed in $\mathfrak{R}^{\mathbb{T},n}$, i.e.,

$\forall \tau \in \mathbb{T}, \forall k = 1, \dots, K$, $\forall \varphi'(\tau) \in \mathcal{R}_1^{\mathbb{T},n} \in \mathfrak{R}^{\mathbb{T},n}$, $\forall \varphi''(\tau) \in \mathcal{R}_2^{\mathbb{T},n} \in \mathfrak{R}^{\mathbb{T},n}$,
$\exists \mathcal{R}_3^{\mathbb{T},n} \in \mathfrak{R}^{\mathbb{T},n}$, $\exists \varphi'''(\tau) \in \mathcal{R}_3^{\mathbb{T},n}$: $\quad \varphi'(\tau) \otimes_k \varphi''(\tau) = \varphi'''(\tau)$.

Remark 1. Concerning the notion of algebra in Definition 3 and to make clear why *we acquire the aggregation operators generated by binary operations* $\otimes_1, \ldots, \otimes_K$ *to be "mean-like", symmetrical, and associative*, let us mention the physical sound aggregation in a violin orchestra playing in unison. If nobody plays, we have zero loudness or pitch (cf. (3)). The higher the pitches/frequences of individual players in a mean are, the higher the resulting pitch/frequence of a sound in a mean is (cf. (4)). The question is how to aggregate the various pitches and loudness at the same time when individual players can join or disjoin the sound sequentially and in a various order. Which kind of a mean is appropriate, which one weighs more – loudness/amplitude? Tones aggregate into new resulting tones with respect their nearness. The "closer" they are, the "better" the aggregation is. See the practical outputs: some orchestras result to play in Pythagorean Tone System, some in 12-tone Equal Temperament, and some in Just Intonation. If we join or disjoin the aggregated tones of an orchestra sound in arbitrary orders, the final result will be the same.

4. Introducing operations

For the sake of simplicity, let us consider $\Lambda = \mathbb{F}_\Delta$, $\Gamma = \mathbb{F}_\Delta$ (the set of all triangular fuzzy numbers) in this paper farther. The sets Λ, Γ may be equipped with many different operations and orders.

If $A : \bigcup_{n\in\mathbb{N}} \Lambda^n \to \Lambda$ and $B : \bigcup_{m\in\mathbb{N}} \Gamma^m \to \Gamma$ are two (symmetrical and associative) aggregation operators, then it is clear that $(A, B) : \bigcup_{n\in\mathbb{N}} \Lambda^n \times \bigcup_{m\in\mathbb{N}} \Gamma^m \to \Lambda \times \Gamma$ is also a 2-dimensional (symmetrical and associative) aggregation operator it the order and the operations are defined coordinate-wisely. But it is not our case (cf. Remark 1) because the coordinates, e.g., pitch and loudness, are not independent. A louder tone system prevails in a pitch, too. Moreover, many seemingly appropriate aggregation operators are not apt for one dimensional variant of the pitch function. For instance, aggregation operators called means are non-symmetric and non-associative. The idea in Section 4 is analogous to an idea of how to define the operation of a multiplication for complex numbers: the solution of an equation $x^2 + 1 = 0$ can be obtained by mixing both coordinates into a two complex numbers multiplication. This is not possible to use a "coordinate-wise" approach. So, in our case, we will treat the aggregation operators. We will find out that some 2-dimensional (n-dimensional, $n \geq 2$) algebras of aggregation operators with the specially defined operations are symmetrical and associative although the coordinates considered separately are not.

Definition 4. Let $\zeta : \Lambda \to \Lambda$ be a bijective function. Let $\mathfrak{C}^{\mathbb{T}}$ be a family of all two-dimensional algebras over the set $\mathbb{T} \neq \emptyset$, cf. Definition 3. Let $(\Gamma; 0, \oplus, \circledast, \leq)$ be a commutative l-ring and $(\Lambda; 0, \boxplus, \leq)$ be a free l-modul over Γ such that $\boxdot : \Lambda \times \Gamma \to \Lambda$ be a binary operation distributive with respect to $\boxplus$. Let $\mathcal{C}_1^{\mathbb{T}} = \{(\varphi_1(\tau), \psi_1(\tau)); \varphi_1(\tau) \in \Lambda, \psi_1(\tau) \in \Gamma, \tau \in \mathbb{T}\} \in \mathfrak{C}^{\mathbb{T}}$, $\mathcal{C}_2^{\mathbb{T}} = \{(\varphi_2(\tau), \psi_2(\tau)); \varphi_2(\tau) \in \Lambda, \psi_2(\tau) \in \Gamma, \tau \in \mathbb{T}\} \in \mathfrak{C}^{\mathbb{T}}$. Let us generate an aggregation operator by a binary operator $A : \mathfrak{C}^{\mathbb{T}} \times \mathfrak{C}^{\mathbb{T}} \to \mathfrak{C}^{\mathbb{T}}$ as follows

$$\begin{gathered}(\varphi_1(\tau), \psi_1(\tau)) \otimes_A (\varphi_2(\tau), \psi_2(\tau)) = \\ = [\zeta^{-1}(\psi_1(\tau) \circledast \{\psi_1(\tau) \oplus \psi_2(\tau)\}^{-1} \boxdot \zeta(\varphi_1(\tau)) \\ \boxplus \psi_2(\tau) \circledast \{\psi_1(\tau) \oplus \psi_2(\tau)\}^{-1} \boxdot \zeta(\varphi_2(\tau))), \\ \psi_1(\tau) \oplus \psi_2(\tau))].\end{gathered} \tag{5}$$

In a trivial case, we put $0 \circledast 0^{-1} = 0$, [1].

The following theorem constructively describes a class of 2-dimensional associative and symmetric aggregation operators generated by formula (5).

Theorem 1. *A 2-dimensional aggregation operator defined by (5) is associative and symmetrical.*

Proof. It is obvious that the operator is symmetrical. To show associativity, we have:

$A(A(\mathcal{C}_1^{\mathbb{T}}, \mathcal{C}_2^{\mathbb{T}}), \mathcal{C}_3^{\mathbb{T}}) = A\{[\zeta^{-1}(\psi_1(\tau) \circledast \{\psi_1(\tau) \oplus \psi_2(\tau)\}^{-1} \boxdot \zeta(\varphi_1(\tau)) \boxplus \psi_2(\tau) \circledast \{\psi_1(\tau) \oplus \psi_2(\tau)\}^{-1} \boxdot \zeta(\varphi_2(\tau))), \psi_1(\tau) \oplus \psi_1(\tau)), (\varphi_3(\tau), \psi_3(\tau)\} = \{\zeta^{-1}(\psi_1(\tau) \oplus \psi_2(\tau) \circledast \{\psi_1(\tau) \oplus \psi_2(\tau) \oplus \psi_3(\tau)\}_{-1} \boxdot \zeta[\zeta^{-1}(\psi_1(\tau) \circledast \{\psi_1(\tau) \oplus \psi_2(\tau)\}^{-1} \boxdot \zeta(\varphi_1(\tau)) \boxplus \psi_2(\tau) \circledast \{\psi_1(\tau) \oplus \psi_2(\tau)\}^{-1} \boxdot \zeta(\varphi_2(\tau))]) \boxplus \psi_3(\tau) \circledast \{\psi_1(\tau) \oplus \psi_2(\tau) \oplus \psi_3(\tau)\}^{-1} \boxdot \zeta(\varphi_3(\tau)),$
$\psi_1(\tau) \oplus \psi_2(\tau) \oplus \psi_3(\tau)\} =$
$\{\zeta^{-1}(\psi_1(\tau) \circledast \{\psi_1(\tau) \oplus \psi_2(\tau) \oplus \psi_3(\tau)\}^{-1} \boxdot \zeta(\varphi_1(\tau)) \boxplus \psi_2(\tau) \circledast \{\psi_1(\tau) \oplus \psi_2(\tau) \oplus \psi_3(\tau)\}^{-1} \boxdot \zeta(\varphi_2(\tau)) \boxplus \psi_3(\tau) \circledast \{\psi_1(\tau) \oplus \psi_2(\tau) \oplus \psi_3(\tau)\}^{-1} \boxdot \zeta(\varphi_3(\tau))) \psi_1(\tau) \oplus \psi_2(\tau) \oplus \psi_3(\tau)\},$
where $(\varphi_1(\tau), \psi_1(\tau)) \in \mathcal{C}_1^{\mathbb{T}}$, $(\varphi_2(\tau), \psi_2(\tau)) \in \mathcal{C}_2^{\mathbb{T}}$ and $(\varphi_3(\tau), \psi_3(\tau)) \in \mathcal{C}_3^{\mathbb{T}}$, $\tau \in \mathbb{T}$. □

Remark 2. The construction of an aggregation operator (5) and the assertion of Theorem 1 can be easily extended from 2 to every finite number of coordinates.

Example 6. Let $\psi_1 \oplus \psi_2 = \psi_1 + \psi_2$ (a usual addition of scalars ψ_1, ψ_2), $\psi \boxdot \varphi(\tau) = \psi \cdot \varphi(\tau)$ [a multiplication of vector $\varphi(\tau)$ (pitch) by scalar ψ (loudness)], and let $\varphi_1(\tau) \boxplus \varphi_2(\tau) = \varphi_1(\tau) + \varphi_2(\tau)$ (an addition of vectors $\varphi_1(\tau), \varphi_2(\tau)$).

(a) Let $\zeta^{-1}(\cdot) = \exp(\cdot)$. Then an operation $\otimes_A$ over $\mathcal{C}_1^{\mathbb{T}}, \mathcal{C}_2^{\mathbb{T}}$ corresponds with a weighed quasi-arithmetic mean (letter G – Geometric) in the first coordinate.

Indeed,

$$\begin{aligned} G_{(2)}(\mathcal{C}_1^{\mathbb{T}}, \mathcal{C}_2^{\mathbb{T}}) &= [\exp\left(\tfrac{\psi_1}{\psi_1+\psi_2} \cdot \ln \varphi_1(\tau) + \tfrac{\psi_2}{\psi_1+\psi_2} \cdot \ln \varphi_2(\tau)\right), \psi_1 + \psi_2] \\ &= [\varphi_1(\tau)^{\frac{\psi_1}{\psi_1+\psi_2}} \cdot \varphi_2(\tau)^{\frac{\psi_2}{\psi_1+\psi_2}}, \psi_1 + \psi_2] \\ &= [(\varphi_1(\tau)^{\psi_1} \cdot \varphi_2(\tau)^{\psi_2})^{\frac{1}{\psi_1+\psi_2}}, \psi_1 + \psi_2]. \end{aligned}$$

(b) Let $\zeta^{-1}(\cdot) = (\cdot)^r$. The case $r = 1$ corresponds with a weighed quasi-arithmetic mean (letter A – Arithmetic) in the first coordinate.

(c) For $\zeta^{-1}(\cdot) = (\cdot)^r$, the case $r = -1$ corresponds with a weighed quasi-arithmetic mean (letter H – Harmonic) in the first coordinate.

Remark 3. For the real functions $\varphi(\tau)$, $\psi(\tau)$ of real variables (in particular, fuzzy sets), many types of operations $\oplus, \boxdot, \circledast, \boxplus$ (pseudo addition, pseudo multiplication, the g-Calculus) can be chosen (cf. book [8]). In particular, we will use this fact in following Section 5 where the operations depend on parameter α (α-compatibility of tones and tone systems).

5. Compatibility of tones

V. Liern [7] introduced the notion of a so-called α-compatibility of fuzzy (one-dimensional) tones and fuzzy tone systems. Although the view to consider the tones as triangular fuzzy numbers is very simplified, we will use this imagination. The idea of an α-compatibility of fuzzy tones and operations over tone systems is glimpsed: to define the "mean-like" symmetrical associative aggregation operators

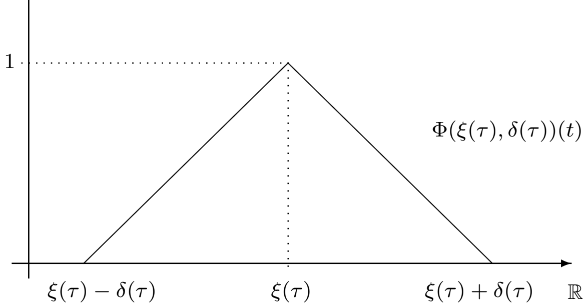

FIGURE 1. Fuzzy pitch $\varphi(\tau)(t) = \Phi(\xi(\tau), \delta(\tau))(t)$ of a tone $\tau \in \mathbb{T}$

over tone systems such that the aggregation will happen only if the operands are mutually close in some sense (α-compatible or compatible in any generalized sense).

Let $\mathbb{F}_\Delta$ be the set of all triangular fuzzy numbers. Consider the 2-dimensional tone system, $\mathcal{C}^{\mathbb{T}} = \mathcal{R}^{\mathbb{T},2} = \{(\varphi(\tau), \psi(\tau)); \quad \tau \in \mathbb{T}\}$ such that the first coordinate, pitch of a tone, $\varphi(\tau) \in \Lambda = \mathbb{F}_\Delta$ is a triangular fuzzy number. Let the refining physical attributes of a given tone $\tau \in \mathbb{T}$ be the frequency $\xi(\tau) \in \mathbb{R}$ and accuracy of its measurement $\delta(\tau) > 0$. So, the first coordinate of our tone system has the form $\varphi(\tau) = \Phi(\xi(\tau), \delta(\tau))$. For the shape of this fuzzy imagination of the pitch see Figure 1. For the sake of simplicity, we put the second coordinate representing the loudness to be constant for every tone $\tau \in \mathbb{T}$, i.e., it is not depending on tone frequency or accuracy of measurement, $\psi(\tau) = \Psi(\xi(\tau), \delta(\tau)) = \text{const}$ for every $\tau \in \mathbb{T}$.

The triangular symmetrical fuzzy number $\varphi(\tau)$ (we identify the fuzzy number with the membership function of it)

$$\varphi(\tau)(t) = \Phi(\xi(\tau), \delta(\tau))(t) = \begin{cases} 1 - \frac{|\xi(\tau)-t|}{\delta(\tau)} & \text{if } |\xi(\tau) - t| < \delta(\tau) \\ 0 & \text{otherwise} \end{cases}$$

can also be called a *fuzzy tone pitch*.

The following definition is a straightforward generalization of Liern's α-compatibility of two fuzzy tones (*fuzzy notes* in his terminology). Two fuzzy tones corresponding with one (real) tone $\tau \in \mathbb{T}$ are taken from various tone systems which belong to a tone systems algebra.

Definition 5. Let $\alpha \in [0, 1]$. Two fuzzy tones $(\varphi_1(\tau), \psi_1(\tau))$ and $(\varphi_2(\tau), \psi_2(\tau))$ are α-compatible if

$$\kappa = \sup_{t\in\mathbb{R}}\{\kappa_\alpha[(\varphi_1(\tau), \psi_1(\tau)), (\varphi_2(\tau), \psi_2(\tau))](t)\} \geq \alpha,$$

where

$$\begin{aligned} &\kappa_\alpha[(\varphi_1(\tau), \psi_1(\tau)), (\varphi_2(\tau), \psi_2(\tau))](t) = \\ &= \{[\varphi_1(\tau) \otimes \psi_1(\tau)] \oplus [\varphi_2(\tau) \otimes \psi_2(\tau)]\}(t), \end{aligned} \tag{6}$$

where $\oplus, \otimes$ are two commutative associative binary operations.

Remark 4. The notion of α-compatibility can be generalized in various directions, e.g. by considering other aggregation operators instead of sup, min and multiplication as in Definition 5. We can rewrite formula (6) from Definition 5 and by induction to generalize it to n dimensions as follows:

$$(\varphi_1(\tau), \psi_1(\tau), \chi_1(\tau)) \otimes (\varphi_2(\tau), \psi_2(\tau), \chi_2(\tau)) =$$

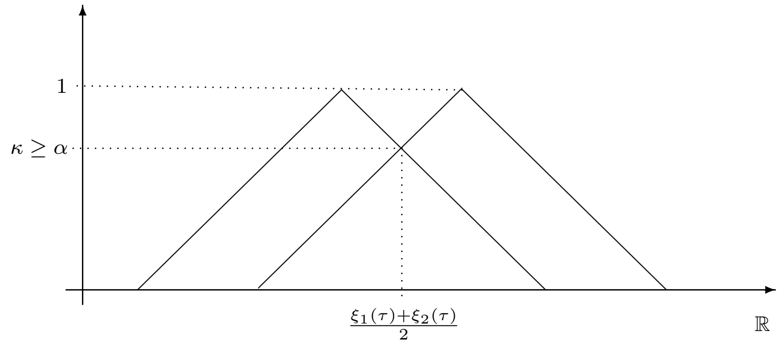

FIGURE 2. α-compatibility of two fuzzy tones

$$([(\varphi_1(\tau), \psi_1(\tau), \chi_1(\tau)) \otimes_{\alpha''} (\varphi_2(\tau), \psi_2(\tau), \chi_2(\tau))] \oplus_2 \kappa_\alpha^{-1},$$
$$\kappa_\alpha \oplus_1 [\chi_1(\tau) \otimes_{\alpha'} \chi_2(\tau)]^{-1},$$
$$[\chi_1(\tau) \otimes_{\alpha'} \chi_2(\tau)]),$$

etc. for $n \in \mathbb{N}$. $\oplus_1, \oplus_2, \otimes_{\alpha'}, \otimes_{\alpha''}$ are the associative and symmetrical aggregation operators here.

Note that the impreciseness of a triangular fuzzy number depends on accuracy of hearing $\delta(\tau)$ the given tone $\tau \in \mathbb{T}$. For the sake of simplicity we consider the same δ for all tones and for every considered tone systems, so $\delta_1(\tau) = \delta_2(\tau) = \delta = \text{const}$. In the sequel to this paper we suppose that $\psi_1(\tau) = \psi_2(\tau) = \text{const}$ and we take min instead of $\oplus$ and the usual multiplication of functions instead of $\otimes$. This situation is shown in Figure 2.

The next lemma allows us to ensure the α-compatibility and is a necessary and sufficient condition for compatibility level α, cf. [7].

Lemma 1. Let $\alpha \in [0, 1]$ and $\tau \in \mathbb{T}$. Two fuzzy tones $(\varphi_1(\tau), \psi_1(\tau))$, and $(\varphi_2(\tau), \psi_2(\tau))$ such that $\varphi_1(\tau) = \Phi_1(\xi_1(\tau), \delta), \varphi_2(\tau) = \Phi_2(\xi_2(\tau), \delta) \in \mathbb{F}_\Delta, \delta > 0$ and $\psi_1(\tau) = \psi_2(\tau) = \text{const}$, are α-compatible if and only if $|\xi_1(\tau) - \xi_2(\tau)| \leq 2\delta(1 - \alpha)$.

Proof. We can assume that $\xi_1(\tau) < \xi_2(\tau)$ without loss of generality. According to Definition 5 if the intersection between $\varphi_1(\tau) \cdot \psi_1(\tau)$ and $\varphi_2(\tau) \cdot \psi_2(\tau)$ is non-empty then $\varphi_1(\tau) \cdot \psi_1(\tau) \wedge \varphi_2(\tau) \cdot \psi_2(\tau)$ is a triangular non-normalized fuzzy number. Therefore from Definition 5 we have

$$\begin{aligned}
\kappa &= \sup_{t \in \mathbb{R}} \{\kappa_\alpha[(\varphi_1(\tau), \psi_1(\tau)), (\varphi_2(\tau), \psi_2(\tau))](t)\} = \\
&= \sup_{t \in \mathbb{R}} \{[\varphi_1(\tau) \cdot \psi_1(\tau) \wedge \varphi_2(\tau) \cdot \psi_2(\tau)](t)\} = \\
&= \{[\varphi_1(\tau) \cdot \psi_1(\tau) \wedge \varphi_2(\tau) \cdot \psi_2(\tau)] \left(\frac{\xi_1(\tau) + \xi_2(\tau)}{2}\right)\} = \\
&= \max\left\{0, 1 - \frac{|\xi_1(\tau) - \xi_2(\tau)|}{2\delta}\right\}.
\end{aligned}$$

Then $\kappa \geq \alpha$ if and only if $1 - \frac{|\xi_1(\tau) - \xi_2(\tau)|}{2\delta} \geq \alpha$, i.e. $|\xi_1(\tau) - \xi_2(\tau)| \leq 2\delta(1 - \alpha)$. $\square$

Remark 5. The individual tones of tone systems are supposed to be hierarchically ordered with respect to more criterions depending on musical style, mood, etc. Therefore the Garbuzov zones representing the accuracy of measurement of frequency (cf. [4]) are of various size. If we need to take an information view (unambiguity of tones) into account, at some level $\alpha > 0$, tones = codes should be distinguished. Turkish music is based on 53 step dividing the octave. So, when considering E_{53}, one step is approximately $22,6$ cents $\approx$ syntonic comma which need not be distinguishable by a non-musically trained ear. Less dramatic situation exists in Arabic and Indian music where less steps within octave (22 in Indian music, Arabic: $17, 19, 24, 31$ or 53) are used. European music based on using the harmony uses 12–tone (or less) systems per octave mostly. Since there are valuable changes in a pitch for one note in orchestra play, it yields the interpretation of a sound. Formally we can say that every instrument plays its own tone system and we aggregate these tone systems into a whole.

6. Compatibility of tone systems

The following definition gives us an α-compatibility of two (or more) tone systems. Note that the set $\mathbb{T}$ may be of various cardinality (12 in the case of E_{12}, but of continuum in glissando, too).

Definition 6. Let $\mathbb{T} \subset \mathcal{T} \neq \emptyset$ be a set. Let $\mathcal{C}_1^{\mathbb{T}} = \{(\varphi_1(\tau), \psi_1(\tau)); \tau \in \mathbb{T}\}$, $\mathcal{C}_2^{\mathbb{T}} = \{(\varphi_2(\tau), \psi_2(\tau)); \tau \in \mathbb{T}\}$ be two fuzzy tone systems such that

$$\varphi_1(\tau) = \Phi_1(\xi_1(\tau), \delta), \varphi_2(\tau) = \Phi_2(\xi_2(\tau), \delta) \in \mathbb{F}_\Delta$$

and $\psi_1(\tau) = \psi_2(\tau) = \text{const}$. Let $\alpha \in [0, 1]$. We say that $\mathcal{C}_1^{\mathbb{T}}$ and $\mathcal{C}_2^{\mathbb{T}}$ are α-compatible if

$$\min_{\tau \in \mathbb{T}} \sup_{t \in \mathbb{R}} \kappa_\alpha[(\varphi_1(\tau), \psi_1(\tau)), (\varphi_2(\tau), \psi_2(\tau))](t) \geq \alpha,$$

where κ_α is defined by (5).

Remark 6. Here $\tau \in \mathbb{T}$ is represented by one fuzzy number for one fuzzy tone. However, tone systems can also involve other types of uncertainty (fuzziness, strike, principal impreciseness) there. For example, Pythagorean $c_\sharp$ and $d_\flat$ differ from each other, i.e., $\xi_{c_\sharp}$(minor second) $\neq \xi_{d_\flat}$(minor second), ascending versus discending scales, etc. So, we can see that the set of triangular fuzzy numbers for Pythagorean Tone System is an insufficient tool for abstraction. For the sake of simplicity, in this paper we do not study the other types of uncertainty.

Lemma 2 (Upper bound for α)**.** *Let $\mathbb{T} \neq \emptyset$ be a set of tones and let $\mathcal{C}_1^{\mathbb{T}}, \mathcal{C}_2^{\mathbb{T}}$ be two fuzzy tone systems α-compatible, $\alpha \in [0, 1]$, as in Definition 6. Then*

$$\alpha \leq \min_{\tau \in \mathbb{T}} \left\{ 1 - \frac{|\xi_1(\tau) - \xi_2(\tau)|}{2\delta} \right\}.$$

The proof is a direct consequence of Lemma 1 and Definition 6.

Theorem 2. *If two fuzzy tone systems $\mathcal{C}_1^{\mathbb{T}}$ and $\mathcal{C}_2^{\mathbb{T}}$ (*cf. Definition 6*), are α-compatible, then there exists $\tau^* \in \mathbb{T}$ such that for all $\tau \in \mathbb{T}$*

$$|\xi_1(\tau^*) - \xi_2(\tau^*)| \geq |\xi_1(\tau) - \xi_2(\tau)|$$

holds.

Proof. According to Definition 6 two fuzzy tone systems are α-compatible, if

$$\min_{\tau\in\mathbb{T}}\sup_{t\in\mathbb{R}}\kappa_\alpha[(\varphi_1(\tau),\psi_1(\tau)),(\varphi_2(\tau),\psi_2(\tau))](t)\geq\alpha,$$

for every $\tau\in\mathbb{T}$. Let $\tau^*\in\mathbb{T}$ for all $\tau\in\mathbb{T}$. Then

$$\begin{aligned}&\sup_{t\in\mathbb{R}}\kappa_\alpha[(\varphi_1(\tau),\psi_1(\tau)),(\varphi_2(\tau),\psi_2(\tau))](t)\geq\\ &\geq\sup_{t\in\mathbb{R}}\kappa_\alpha[(\varphi_1(\tau^*),\psi_1(\tau^*)),(\varphi_2(\tau^*),\psi_2(\tau^*))](t)\end{aligned}$$

holds for all pairs of fuzzy tones such that $(\varphi_1(\tau),\psi_1(\tau))\in\mathcal{C}_1^{\mathbb{T}}$ and $(\varphi_2(\tau),\psi_2(\tau))\in\mathcal{C}_2^{\mathbb{T}}$, and for all $\tau\in\mathbb{T}$, i.e.,

$$\max\left\{0,1-\frac{|\xi_1(\tau)-\xi_2(\tau)|}{2\delta(\tau)}\right\}\geq\max\left\{0,1-\frac{|\xi_1(\tau^*)-\xi_2(\tau^*)|}{2\delta(\tau)}\right\},$$

i.e. $|\xi_1(\tau^*)-\xi_2(\tau^*)|\geq|\xi_1(\tau)-\xi_2(\tau)|$. □

Now, we will show that the aggregation does not worsen the compatibility of tone systems. We could consider a general situation of an operation over tone systems similarly as in Theorem 1, however, we will only prove the theorem for a harmonic mean, which is a special case of 2–dimensional aggregation operator defined in (4).

Theorem 3. *Let $\mathcal{C}_1^{\mathbb{T}}$, $\mathcal{C}_2^{\mathbb{T}}$ be two α-compatible fuzzy tone systems defined in Definition 6. Then $H_{(2)}(\mathcal{C}_1^{\mathbb{T}},\mathcal{C}_2^{\mathbb{T}})$ and $\mathcal{C}_2^{\mathbb{T}}$ are β-compatible, where $\beta\geq\alpha$.*

Remark 7. This theorem holds analogously for $H_{(2)}(\mathcal{C}_1^{\mathbb{T}},\mathcal{C}_2^{\mathbb{T}})$ and $\mathcal{C}_1^{\mathbb{T}}$.

Proof. Denote by

$$\begin{aligned}\mathcal{C}^{\mathbb{T}}&=&&H_{(2)}(\mathcal{C}_1^{\mathbb{T}},\mathcal{C}_2^{\mathbb{T}})=\{(\varphi(\tau),\psi(\tau));\tau\in\mathbb{T}\}=\\ &=&&\left\{\left(\frac{\psi_1(\tau)+\psi_2(\tau)}{\frac{\psi_1(\tau)}{\varphi_1(\tau)}+\frac{\psi_2(\tau)}{\varphi_2(\tau)}},\psi_1(\tau)+\psi_2(\tau)\right);\tau\in\mathbb{T}\right\}.\end{aligned}$$

Prove that $\mathcal{C}^{\mathbb{T}}$ is β-compatible with $\mathcal{C}_2^{\mathbb{T}}$, i.e.

$$\min_{\tau\in\mathbb{T}}\sup_{t\in\mathbb{R}}\kappa_\alpha\left[\left(\frac{\psi_1(\tau)+\psi_2(\tau)}{\frac{\psi_1(\tau)}{\varphi_1(\tau)}+\frac{\psi_2(\tau)}{\varphi_2(\tau)}},\psi_1(\tau)+\psi_2(\tau)\right),(\varphi_2(\tau),\psi_2(\tau))\right](t)=\beta\geq\alpha.$$

Without loss of generality consider $\varphi_1(\tau)\leq\varphi_2(\tau)$ for all $\tau\in\mathbb{T}$. Then

$$\frac{\psi_1(\tau)+\psi_2(\tau)}{\frac{\psi_1(\tau)}{\varphi_1(\tau)}+\frac{\psi_2(\tau)}{\varphi_2(\tau)}}=\varphi_1(\tau)\cdot\eta(\tau)\geq\varphi_1(\tau),$$

where

$$\eta(\tau)=\frac{(\psi_1(\tau)+\psi_2(\tau))\cdot\varphi_2(\tau)}{\psi_1(\tau)\cdot\varphi_2(\tau)+\psi_2(\tau)\cdot\varphi_1(\tau)}\geq 1.$$

For the second coordinate we obtain $\psi_1(\tau)+\psi_2(\tau)\geq\psi_1(\tau)$. Now, we have

$$\begin{aligned}\beta&=&&\min_{\tau\in\mathbb{T}}\sup_{t\in\mathbb{R}}\kappa_\alpha\left[\left(\frac{\psi_1(\tau)+\psi_2(\tau)}{\frac{\psi_1(\tau)}{\varphi_1(\tau)}+\frac{\psi_2(\tau)}{\varphi_2(\tau)}},\psi_1(\tau)+\psi_2(\tau)\right),(\varphi_2(\tau),\psi_2(\tau))\right](t)\\ &\geq&&\min_{\tau\in\mathbb{T}}\sup_{t\in\mathbb{R}}\kappa_\alpha[(\varphi_1(\tau),\psi_1(\tau)),(\varphi_2(\tau),\psi_2(\tau))](t)\geq\alpha.\end{aligned}$$

Thus $\mathcal{C}^{\mathbb{T}}=H_{(2)}(\mathcal{C}_1^{\mathbb{T}},\mathcal{C}_2^{\mathbb{T}})$ is β-compatible with $\mathcal{C}_2^{\mathbb{T}}$, where $\beta\geq\alpha$. Analogously for $\mathcal{C}^{\mathbb{T}}$ and $\mathcal{C}_1^{\mathbb{T}}$.

Note	Pythagorean (P) (in cents)	Equal (E) (in cents)	Garbuzov zone of tone (in cents)	Compatibility of P and E tones
$\mathcal{C}$	0	0	$(-12, 12)$	1
$\mathcal{D}_\flat$	90.225	100	$(48, 124)$	0.492276
$\mathcal{C}_\sharp$	113.685			0.694821
$\mathcal{D}$	203.91	200	$(160, 230)$	0.894196
$\mathcal{E}_\flat$	294.135	300	$(272, 330)$	0.816331
$\mathcal{D}_\sharp$	317.595			0.534493
$\mathcal{E}$	407.820	400	$(372, 430)$	0.762382
$\mathcal{F}$	498.045	500	$(472, 530)$	0.934784
$\mathcal{G}_\flat$	588.270	600	$(566, 630)$	0.690215
$\mathcal{F}_\sharp$	611.730			0.688895
$\mathcal{G}$	701.955	700	$(672, 730)$	0.934784
$\mathcal{A}_\flat$	792.180	800	$(766, 830)$	0.782233
$\mathcal{G}_\sharp$	815.904			0.601922
$\mathcal{A}$	905.865	900	$(866, 930)$	0.832105
$\mathcal{B}_\flat$	996.090	1000	$(966, 1024)$	0.873688
$\mathcal{A}_\sharp$	1019.550			0.495809
$\mathcal{B}$	1109.775	1100	$(1066, 1136)$	0.754936
$\mathcal{C}'$	1200	1200	$(1188, 1212)$	1

TABLE 1. α-compatibility of Pythagorean and Equal tone systems

Note that the case $\alpha = 0$ and $\beta > 0$ is also possible. □

Remark 8. The consideration the α-compatibility of fuzzy tone systems could yield the following interesting situations: If $\mathcal{C}_1^{\mathbb{T}}$ is α-compatible with $\mathcal{C}_2^{\mathbb{T}}$ and $\mathcal{C}_2^{\mathbb{T}}$ is α-compatible with $\mathcal{C}_3^{\mathbb{T}}$, then it may happen that $\mathcal{C}_1^{\mathbb{T}}$ is not α-compatible with $\mathcal{C}_3^{\mathbb{T}}$. And vice versa, if $\mathcal{C}_1^{\mathbb{T}}$ is not α-compatible with $\mathcal{C}_2^{\mathbb{T}}$ and $\mathcal{C}_2^{\mathbb{T}}$ is not α-compatible with $\mathcal{C}_3^{\mathbb{T}}$, then it may happen that $\mathcal{C}_1^{\mathbb{T}}$ is α-compatible with $\mathcal{C}_3^{\mathbb{T}}$.

Example 7. Let $\varphi_i : \mathbb{T} \to \mathbb{R}$, $i \in \{P, E\}$ be a fuzzy pitch function of Pythagorean (P) and Equal Tempered (E) 12–tone systems, respectively. Consider the fuzzy tones $(\varphi_i(\tau), \psi_i(\tau))$ where $\varphi_i(\tau) = \Phi_i(\xi_i(\tau), \delta(\tau)) \in \mathbb{F}_\Delta$, $\psi_i(\tau) = \text{const}$ for all $\tau \in \mathbb{T}$, i.e. for the tone $\tau = \mathcal{F}$ is $\xi_P(\mathcal{F}) = 498.045C$ and $\xi_E(\mathcal{F}) = 500C$, cf. Table 1, Columns 2 and 3. We calculate the compatibility between fuzzy tones (see column 5) of the two fuzzy tone systems considering the accuracy of hearing $\delta(\tau) = 25$ cents. It is obvious, the smaller (the bigger) $\delta(\tau) > 0$ is chosen, the smaller (the bigger) α-compatibility of fuzzy tones and fuzzy tone systems we obtain, and vice versa, respectively. Using so-called Garbuzov zones (cf. [4]) we can refine our considerations. In that case we have non-symmetrical triangular fuzzy numbers (a fuzzy pitch of tones) $\varphi_i(\tau) = \Phi_i(\xi_i(\tau), \delta) = (n \cdot 100, \delta_\tau^+, \delta_\tau^-)$, where $\delta_\tau^+, \delta_\tau^-$ are right and left interval deviations from the points $n \cdot 100$ cents, $n \in \mathbb{Z}$ (cf. Table 1, Column 4).

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