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ON A DIOPHANTINE EQUATION DERIVED FROM THE THEORY OF MEANTONE SYSTEMS

Ján Haluška[1] - Ondrej Hutník[2] - Tomáš Mihály[3]

1. Introduction

The overview of the general mathematical theory of tone systems we can find in [1]. For the excellent historical retrospective of meantone systems from the viewpoint of the present musicology, c.f. [3]. In this preliminary section, we recall some facts and formalize the problem which we solve in the rest sections.

Denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ and by $\mathbb{Q}, \mathbb{P}$ the sets of all natural, rational, and prime numbers, respectively.

Rational and irrational numbers have their counterparts in music, *harmonic* and *inharmonic* music intervals, respectively. It is known that Pythagorean System can be constructed using only two rational numbers $X = 256/243$ (the minor Pythagorean semitone, *limma*) and $Y = (9/8) : X = 2\,187/2\,048$ (the major Pythagorean semitone, *apotome*). It is also well known that this couple (X, Y) is unique. Another tone system, Equal Temperament, is based on one number, the *equal tempered semitone* $X = \sqrt[12]{2}$, which is irrational number (i.e., it is *inharmonic* music interval) and it is unique. The only rational numbers (harmonic intervals) in this tone system are numbers $\sqrt[12]{2}^{12i}$, $i \in \mathbb{N}$, i.e., octaves in music. The third famous meantone system is the Praetorius tone system, also known as the 1/4-comma meantone. It contains both harmonic (octaves, major thirds, augmented fifths) and inharmonic music intervals.

Note that words "number" and "(relative) musical interval" (or simply – interval) are used as synonyms in the tone systems theory and the mathematical theory of music. In this paper, we will do this as well. This will not lead to any misunderstanding because the notion of interval is only used in this sense. More precisely, if we denote by $\mathcal{L} = ((0, \infty), \cdot, 1, \leq)$ the usual multiplicative group on real line with the usual order, then b/a is called the $\mathcal{L}$-length of (a, b), $0 < a \leq b < \infty$ is called the interval.

Having a motivation the meantone algorithm ("the spiral of generalized fifths"), c.f. [1], we can formulate the problem we solve in this paper as follows.

Problem. Let p, q be given prime numbers, let a, b be given natural relatively prime numbers. To find all couples (X, Y) in rational, such that there exist natural numbers α, β, γ, δ such that

$$X^\alpha Y^\beta = p, \quad X^\gamma Y^\delta = q,$$

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[1]Ján Haluška, Mathematical Institute, Slovak Academy of Science, Grešákova 6, 040 01 Košice, Slovakia and Dept. of Math. Analysis and Applied Mathematics, University of Žilina, Hurbanova 15, 010 26 Žilina, Slovakia, e-mail: `jhaluska@saske.sk`

[2]Ondrej Hutník, Dept. of Mathematical Analysis and Applied Mathematics, University of Žilina, Hurbanova 15, 010 26 Žilina, e-mail: `ondrej.hutnik@fpv.utc.sk`

[3]Tomáš Mihály, Dept. of Mathematical Analysis and Applied Mathematics, University of Žilina, Hurbanova 15, 010 26 Žilina, e-mail: `tomas.mihaly@fpv.utc.sk`

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and

$$\alpha + \beta = a, \qquad \gamma + \delta = b.$$

Such numbers p, q, a, b, X, Y exist. For instance, for Pythagorean system we have: $p = 2$, $q = 3$, $a = 12$, $b = 19$, $X = 256/243$, $Y = 2\,187/2\,048$.

2. Results

For the sake of simplicity and without loss of generality, we will consider $p < q$ in this paper on.

Theorem 1 *Let $a \in \mathbb{N}$ and $b \in \mathbb{N}$ be two fixed relatively prime positive integer numbers. Then there exists unique positive integer solution of the following Diophantine equation*

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = 1, \begin{cases} \alpha + \beta = a, \\ \gamma + \delta = b. \end{cases} \tag{1}$$

on the rectangle $I = \langle 0, b\rangle \times \langle 0, a\rangle$.

Proof. We split the proof into two steps:

Step 1 (Existence). The Diophantine equation (1) can be rewritten as $a\delta - b\beta = 1$, where $\delta \in \{0, 1, \ldots, b\}$, and $\beta \in \{0, 1, \ldots, a\}$. Since a, b are relatively prime numbers, there exists a solution of Diophantine equation, c.f. [2].

Step 2 (Uniqueness). Let (δ_0, β_0) and (δ_1, β_1) be two positive integer solutions of Diophantine equation $a\delta - b\beta = 1$ on the rectangle $I = \langle 0, b\rangle \times \langle 0, a\rangle$. Then

$$a\delta_0 - b\beta_0 = 1, \quad \delta_0 \in \{0, 1, \ldots, b\}, \beta_0 \in \{0, 1, \ldots, a\}$$

and

$$a\delta_1 - b\beta_1 = 1, \quad \delta_1 \in \{0, 1, \ldots, b\}, \beta_1 \in \{0, 1, \ldots, a\}.$$

So

$$a(\delta_0 - \delta_1) = b(\beta_0 - \beta_1).$$

That implies that a divides $b(\beta_0 - \beta_1)$ and $(a, b) = 1$ (the greatest common divisor), so a divides $(\beta_0 - \beta_1)$. Since $\beta_0 - \beta_1 < a$, this relation holds if and only if $\beta_0 = \beta_1$.

Analogously for δ_0 and δ_1. $\square$

We use Theorem 1 for proving of the following main theorem.

Theorem 2 *Let $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be a matrix of nonnegative integers with* $\det A \neq 0$. *Then for all relatively prime $a, b \in \mathbb{N}$ such that $\alpha + \beta = a, \gamma + \delta = b$, there exist a unique and positive $(X, Y) \in \mathcal{L}^2$, such that*

$$X^\alpha Y^\beta = p \tag{2}$$

$$X^\gamma Y^\delta = q \tag{3}$$

for $p \in \mathbb{P}$ and $q \in \mathbb{P}$ and the following statements are equivalent:

$$\text{(a) } (X, Y) \in \mathbb{Q}^2, \text{ (b) } \det A = 1.$$

The values are as follows:

$$X = p^\delta q^{-\beta}, Y = p^{-\gamma} q^\alpha. \tag{4}$$

Proof. If $X, Y \in \mathbb{Q}$, then there exist $r \in \mathbb{P}$ and $E_{2,X}, \dots, E_{r,Y} \in \mathbb{Z}$ such that

$$\begin{aligned} X &= 2^{E_{2,X}} \dots p^{E_{p,X}} \dots q^{E_{q,X}} \dots r^{E_{r,X}}, \\ Y &= 2^{E_{2,Y}} \dots p^{E_{p,Y}} \dots q^{E_{q,Y}} \dots r^{E_{r,Y}}. \end{aligned}$$

Combining (2) and (3),

$$\begin{pmatrix} E_{2,X} & E_{2,Y} \\ \vdots & \vdots \\ E_{p,X} & E_{p,Y} \\ \vdots & \vdots \\ E_{q,X} & E_{q,Y} \\ \vdots & \vdots \\ E_{r,X} & E_{r,Y} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ \vdots & \vdots \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}.$$

Since $\det A = 1$,

$$\begin{pmatrix} E_{2,X} & E_{2,Y} \\ \vdots & \vdots \\ E_{p,X} & E_{p,Y} \\ \vdots & \vdots \\ E_{q,X} & E_{q,Y} \\ \vdots & \vdots \\ E_{r,X} & E_{r,Y} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ \vdots & \vdots \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ \delta & -\gamma \\ \vdots & \vdots \\ -\beta & \alpha \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$$

$\square$

Equivalently we can obtain values of X, Y as follows: system (2) and (3) could be rewritten into

$$\begin{aligned} \alpha + \beta \log_X Y = \log_X p \\ \gamma + \delta \log_X Y = \log_X q. \end{aligned}$$

Multiplying both sides of the first equation by δ, multiplying both sides of the second equation by $-\beta$ and adding the resulting equations one side by side we get

$$\alpha\delta - \beta\gamma = \log_X \frac{p^\delta}{q^\beta}.$$

Since $\det A \neq 0$, we have $X = (p^\delta q^{-\beta})^{\frac{1}{\alpha\delta-\beta\gamma}}$ and $Y = (p^{-\gamma} q^{\alpha})^{\frac{1}{\alpha\delta-\beta\gamma}}$. Then X, Y are rational if and only if $\alpha\delta - \beta\gamma = 1$, i.e. $\det A = 1$.

Remark 1 If $p = q$ then $X = p^{\delta-\beta}, Y = p^{\alpha-\gamma}$.

The proof of the following lemma is easy and therefore can be omitted.

Lemma 1 *Let $p, q \neq 1$. Then $X \neq Y$ in the equation (4).*

3. Applications

We bring examples and some corollaries of the Problem solution.

b	13	17	19	23	29	31
$X_{2,3}$	4096/177147	1024/2187	256/243	4/3	131072/2187	8192/243
$Y_{2,3}$	3/2	243/128	2187/2048	177147/2097152	243/4096	2187/262144
$X_{2,5}$	4096/48828125	1024/78125	256/3125	4/5	131072/78125	8192/3125
$Y_{2,5}$	5/2	3125/128	78125/2048	48828125/2097152	3125/4096	78125/262144

Table 1: The values of $X_{p,q}, Y_{p,q}$ for $a = 12$ and different p, q, b

Example 1 Consider $p = 2$ and $q = 3$. Let $\alpha+\beta = 12, \gamma+\delta = b$, such that $\det A = 1$. The rational semitones for some appropriate values of $b \in \mathbb{N}$ $(12 < b < 32)$ are collected in Table 1. As we can see, the bigger natural b is, the bigger difference between rational interval X, Y we can observe. When we establish some other conditions for intervals X, Y, e.g. upper and lower bound for them, we obtain the following well-known results.

Corollary 1 *The unique (up to symmetry) pair of rational intervals X, Y satisfying $X^\alpha Y^\beta = 2$ and $X^\gamma Y^\delta = 3$ for $\alpha + \beta = 12$, $\gamma + \delta = b$ and $\alpha\delta - \gamma\beta = 1$, such that $1 < X < Y \leq 9/8$ are $X = 2^8 3^{-5}$ and $Y = 2^{-11} 3^7$ for $b = 19$.*

Corollary 2 *For Praetorius tone system, $p = 2$, $q = 5$ and $a = 12$ there is no b such that there exist rational solutions X, Y of our problem such that $1 < X < Y \leq 16/15$ (musically, there are no rational semitones for this tone system).*

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Zusammenfassung

Wir stellen und lösen das folgende Problem, das mit der mathematischen Theorie des mittentönigen Temperatursystems inspiriert war.

Sei p, q zwei gegebene Primzahlen und a, b zwei gegebene natürliche teilerfremde Zahlen. Wir suchen alle rationalen Zahlenpaare (X, Y), für die solche natürliche Zahlen α, β, γ, δ existieren, dass

$$X^\alpha Y^\beta = p, \quad X^\gamma Y^\delta = q, \quad \alpha + \beta = a, \quad \gamma + \delta = b$$

gilt. Dieses Gleichungsystem ist lösbar und wir werden zeigen, dass die Lösung eindeutig ist. Wir führen einige Verwendungen in der Mittentönigkeitstheorie ein.