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SEARCHING THE FRONTIER OF THE PYTHAGOREAN SYSTEM

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Abstract

It is known that the perfect fifth and octave can be expressed as X^4Y^3 and X^7Y^5 respectively, where $X = 256/243 = 2^8/3^5$ is the minor and $Y = 9/8 : X = 3^7/2^11$ the major Pythagorean semitone. In this paper, Pythagorean system is studied when couples of semitones need not be considered in rational numbers; we deal with semitone couples (X, Y) in algebraic numbers expressed with dth roots, $d = 1, 2, 3, \ldots$.
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1 Anatomy of the whole Pythagorean tone

The Pythagorean system was developed in antiquity and satisfactorily formalized by B o e t h i u s (480 – 524). Musically, this system is based on the perfect fifths (or, equivalently, on the perfect fourths) and octaves.

Examples. Usually, under Pythagorean Tuning (Scale) (cf. e.g. [1], [5], [2]), we mean one of the following number sequences (physically: the relative frequencies of tones within the octave [1/1; 2/1)): $P_2, P_3, P_5, P_7, P_{12}, P_{17}, P_{22}, P_{27}, P_{31} \subset \mathcal{P} = \{2^\alpha 3^\beta; \alpha, \beta \in \mathcal{Z}\}$, $\mathcal{Z} = \{\cdots, \text{-2, -1, 0, 1, 2, } \cdots\}$.

$P_2 = \langle$1/1, 3/2$\rangle$,
$P_3 = \langle$1/1, 4/3, 3/2$\rangle$,
$P_5 = \langle$1/1, 9/8, 81/64, 3/2, 27/16$\rangle$,
$P_7 = \langle$1/1, 9/8, 81/64, 4/3, 3/2, 27/16, 16/9$\rangle$,
$P_{12} = \langle$1/1, 2 187/2 048, 9/8, 32/27, 81/64, 4/3, 729/512, 3/2, 6 561/4 096, 27/16, 16/9, 243/128$\rangle$,
$P_{17} = \langle$1/1, 256/243, 2 187/2 048, 9/8, 32/27, 19 683/16 384, 81/64, 4/3, 1 024/729, 729/512, 3/2, 128/81, 6 561/4 096, 27/16, 16/9, 59 049/32 768, 243/128$\rangle$,
$P_{22} = \langle$1/1, 256/243, 2 187/2 048, 65 536/59 049, 9/8, 32/27, 19 683/16 384, 8 192/6 561, 81/64, 4/3, 177 147/131 072, 1 024/729, 729/512, 3/2, 128/81, 6 561/4 096, 32 768/19 683, 27/16, 16/9, 59 049/32 768, 4 096/2 187, 243/128$\rangle$,
$P_{27} = \langle$1/1, 531 441/524 288, 256/243, 2 187/2 048, 65 536/59 049, 9/8, 4 782 969/4 194 304, 32/27, 19 683/16 384, 8 192/6 561, 81/64, 4/3, 177 147/131 072, 1 024/729, 729/512, 262 144/177 147,

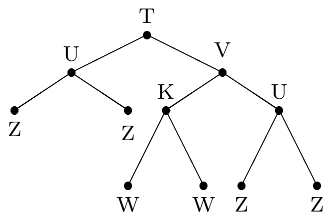

Figure 1: Boethius scheme of the Pythagorean whole tone

3/2, 1 594 323/1 048 576, 128/81, 6 561/4 096, 32 768/19 683, 27/16, 14 348 907/8 388 608, 16/9, 59 049/32 768, 4 096/2 187, 243/128⟩,

P_{31} = ⟨1/1, 531 441/524 288, 256/243, 2 187/2 048, 1 162 261 467/1 073 741 824, 9/8, 4 782 969/4 194 304, 32/27, 19 683/16 384, $3^{21}/2^{33}$, 8 192/6 561, 81/64, 43 046 721/33 554 432, 4/3, 177 147/131 072, 1 024/729, 729/512, 387 420 489/268 435 456, 3/2, 1 594 323/1 048 576, 128/81, 6 561/4 096, $3^{20}/2^{31}$, 27/16, 14 348 907/8 388 608, 16/9, 59 049/32 768, $3^{22}/2^{34}$, 4 096/2 187, 243/128, 129 140 163/67 108 864⟩.

We will deal with the usual multiplicative group $\mathcal{L}$, $((0,\infty),\cdot,1,\leq)$, equipped with the usual order. In Figure 1, we see the Boethius scheme, skeleton of the Pythagorean whole tone, cf. [5]. If we denote by $Q = 4/3$ the *perfect fourth* and by $O = 2$ the *octave*, then $R = O/Q = 3/2$ is the *perfect fifth*, $T = R/Q = 9/8$ the *Pythagorean whole tone, tonos*; $U = Q/T^2 = 256/243$ the *minor Pythagorean semitone, limma*; $V = T/U = 2\ 187/2\ 048$ the *major Pythagorean semitone, apotome*; $Z = \sqrt{U} = 3\sqrt[4]{2}/4$ *diaschizma*, $K = V/U = 531\ 441/524\ 288$ *comat*; $W = \sqrt{K} = \sqrt{531\ 441/524\ 288}$ *schizma*. **We see that the Boethius music theory does not avoid irrational numbers and it can be considered as a microtonal theory in the present modern sense.** Recall that one octave is of the length 1 200 cents ($x \mapsto 1\ 200 \log_2 x, 1 \leq x \leq 2$). So, $Z \approx 1.026400479 \approx 45.11$ cents, $K \approx 1.013643265 \approx 23.46$ cents; $W \approx 1.006798522 \approx 11.73$ cents. For comparison: $81/80 = 1.0125 \approx 21.51$ cents.

By a generalized sequence (net) with values in $\mathcal{L}$ we mean a function from I to $\mathcal{L}$, where I is a directed partially ordered set. For the generalized sequences in general, cf. [3].

The semitones U and V are unique rational numbers which enables to express Pythagorean Tuning P_{31} (and also. $P_{12}, P_{17}, P_{22}, P_{28}$; for P_{17}, cf. [1], [2], [4]) via the **geometrical generalized sequence** (the abbreviation: **GGS**) **in rational numbers** with some natural conditions. More precisely, in the form

$$\{\Gamma_i = \alpha \cdot X^{m_i} Y^{n_i}; i \in \mathcal{Z}\} \tag{1}$$

where X, Y are positive rational, α positive real, and (possible not unique for every $i \in \mathcal{Z}$) $(m_i, n_i) \in I \subset \mathcal{Z}^2$ such that

$$m_i + n_i = i,\ m_i \leq m_{i+1}, n_i \leq n_{i+1},\ i \in \mathcal{Z}. \tag{2}$$

The additional conditions are

$$\alpha = 1, \tag{3}$$

$$X^{m_{12}} Y^{n_{12}} = 2, \tag{4}$$

$$X^{m_7}Y^{n_7} = 3/2, \quad (5)$$

$$\Gamma_{j+12k} = 2^k \cdot \Gamma_j, \ j = 0, 1, 2, \ldots, 11, \ k \in \mathcal{Z}. \quad (6)$$

The condition (3) says that we consider **no shift of the fundamental tone**. The condition (4) shows that **the 12th degree is the octave**. The condition (5) asserts that the seventh degree is the **perfect fifth (3/2)**. The condition (6) describes the so called **octave equivalency**.

Twelve qualitative music degrees are the characteristic attribute of the European tone systems and culture. They are: the unison (octave), minor second, major second, minor third, major third, fourth, tritone, fifth, minor sixth, major sixth, minor seventh, major seventh. Each degree may contain one or more tuning values The GGSs $P_{17}, P_{22}, P_{28}, P_{31}$ have 17, 22, 28, 31 values for 12 degrees within the octave $[0, 2)$, respectively. There are different I for $P_{12}, P_{17}, P_{22}, P_{28}, P_{31}$. In this paper, we will consider GGSs only for pairs (X, Y) (two different semitones, not three or more). Examples of index sets I for GGGs we can see also in Figure 2 (Table 3; a sequence) and in Figure 2 (Table 4). We deal with index sets directed both forward and backward (the two side GGS).

Pythagorean Tunings $P_{12}, P_{17}, P_{22}, P_{28}, P_{31}$ are GGSs, the solutions of the equation system (1), (2), (3), (4), (5) with rational (X, Y). Therefore, there arises a question: **does there exist any GGS, the solution of the equation system (1), (2), (3), (4), (5) with irrational** (X, Y)**?** In this paper, we solve this problem in the positive for semitone couples (X, Y) expressed with algebraic numbers.

The Pythagorean tunings $P_{17}, P_{22}, P_{28}, P_{31}$ **are many valued (fuzzy) coding systems of information in music.** For more explanations about the philosophy of fuzziness of tunings in music see [2]. The important and obvious fact is that the theoretical tunings (and operations among them) never occur in the real live music because there are ever at least two different tone systems in the sound interaction during the performance: the tone subset $\mathbf{A}$ of the tuned system itself, and physically, the systems of overtones $\mathbf{B}$ of tones in $\mathbf{A}$. Clearly, always $\mathbf{A} \neq \mathbf{B}$.

2 The second level: algebraic numbers

Observe, [1], that P_i, $i = 2, 3, 5, 7, 12, 17, 22, 28, 31$, can be obtained as the solution of the following Problem 1 and using the enlargement algorithm based on the fifth $X^{m_7}Y^{n_7} = 3/2$ and the octave $X^{m_{12}}Y^{n_{12}} = 2$, i.e.:

$$Q_{[k_1,k_2]} = \{(3/2)^k \ (\mathrm{mod}\ 2); k \in [k_1, k_2], \ k_1 < k_2, \ k_1, k_2 \in \mathcal{Z}\}.$$

For instance, $k = 1, 2, 3, 4, 5$ for P_5.
Problem 1.
To find X, Y rational numbers such that (3), (4), (5) for some m_i, n_i nonnegative integer numbers such that

$$m_i + n_i = i, \ i = 0, 7, 12, \ 0 \leq m_7 \leq m_{12}, 0 \leq n_7 \leq n_{12}.$$

Problem 2. To find all triplets (m_{12}, m_7, d) in nonnegative integers such that

$$7m_{12} - 12m_7 = d, 0 \leq 7 - m_7 \leq 12 - m_{12}, 0 \leq m_7 \leq m_{12}.$$

m_7	d	
0	7, 14,21,28,35,	42,49,56,63,70,77
1	2,9,16,23,30,	37,44,51,58,65
2	4, 11,18,25,	32,39,46,53
3	6,13,20,	27,34,41
4	1,8,15	22, 29
5	3,10,	17
6	5	

Table 1: Solutions of $7m_{12} - 12m_7 = d$

The solution of Problem 1 is given by the solution of Problem 2. Indeed, for $d = 0$, we have $m_{12} = 12, m_7 = 7$, i.e. Equal Temperament ($X = Y = \sqrt[12]{2}$) which does not contain any perfect fifth 3/2, (analogously for $d = 12$). If $d \neq 0$ and (m_{12}, m_7, d) is a solution of Problem 2, then the solution of (3), (4), (5) we obtain as follows:

$$X = 2^{E_{2,X}} 3^{E_{3,X}}, Y = 2^{E_{2,Y}} 3^{E_{3,Y}}, \tag{7}$$

where $E_{2,X} = (19 - m_7 - m_{12})/d$, $E_{3,X} = (m_{12} - 12)/d$, $E_{2,Y} = (-m_{12} - m_7)/d$, $E_{3,Y} = m_{12}/d$. The numbers X, Y are rational solutions of Problem 1 if and only if $d = 1$ (or $d = -1$, the symmetrical case). For $d = 1$, the values are $X = U, Y = V$.

Now it is possible to extend Pythagorean Tuning (in any form: P_2 or ... or P_{31}) to algebraic numbers: to solve Problem 1a, where

Problem 1a

To find X, Y algebraic numbers such that (4), (5) for some m_i, n_i nonnegative integer numbers such that

$$m_i + n_i = i, \ i = 0, 7, 12, \ 0 \leq m_7 \leq m_{12}, 0 \leq n_7 \leq n_{12}.$$

Theorem 1 *For every $d \in \mathcal{Z}$, there exists unique or none solution (X_d, Y_d) of Problem 1a such that X_d, Y_d are expressed via dth roots, multiplication, and/or dividing of numbers 2 and 3.*

Proof The solutions of the Diophantine equation

$$7m_{12} - 12m_7 = d, \ 0 \leq m_7 \leq m_{12},$$

where $d, m_7, m_{12} = 0, 1, 2, 3, 4 \ldots$ are collected in Table 1. By (2), we obtained values n_{12}, n_7. Take only the solutions such that $n_{12} \geq n_7$. Then the solutions of Problem 2 are collected in Table 2. By (7) we obtain the solutions of Problem 1a ($d < 0$ are the symmetrical cases). □

Theorem 2 *Denote by $\mathcal{G}_d$ the union of all GGSs as sets such that they are generated by the solution (X_d, Y_d) of Problem 1a, d in Table 2. Then*

$$\begin{array}{lcl} \mathcal{G}_1 & = & \mathcal{P} = \{2^\alpha 3^\beta; \alpha, \beta \in \mathcal{Z}\}, \\ \mathcal{G}_2 & = & \mathcal{P} \cup (\sqrt{2}) \cdot \mathcal{P} \\ \mathcal{G}_3 & = & \mathcal{P} \cup (\sqrt[3]{4}) \cdot \mathcal{P} \cup (3\sqrt[3]{2}) \cdot \mathcal{P} \\ & \ldots & \\ \mathcal{G}_d & = & \bigcup_{i=1}^{d} \beta_{i,d} \cdot \mathcal{P} \\ & \ldots & \\ \mathcal{G}_{35} & = & \bigcup_{i=1}^{35} \beta_{i,35} \cdot \mathcal{P}, \end{array}$$

d	1	2	3	4	5	6	7	8	9	10	11	
m_{12}	7	2	9	4	11	6	1	8	3	10	5	
n_{12}	5	10	3	8	1	6	11	4	9	2	7	
m_7	4	1	5	2	6	3	0	4	1	5	2	
n_7	3	6	2	5	1	4	7	3	6	2	5	
d	13	14	15	16	18	20	21	23	25	28	30	35
m_{12}	7	2	9	4	6	8	3	5	7	4	6	5
n_{12}	5	10	3	8	6	4	9	7	5	8	6	7
m_7	3	0	4	1	2	3	0	1	2	0	1	0
n_7	4	7	3	6	5	4	7	6	5	7	6	7

Table 2: Solutions of Problem 2

C	$X_2^0Y_2^0$	1	0.00	U^0K^0	◯
$C_\sharp$	$X_2^0Y_2^1$	$3/\sqrt{8}$	101.96	$U^1K^{1/2}$	•
D	$X_2^0Y_2^2$	$9/8$	203.91	U^2K^1	◯
$E_\flat$	$X_2^0Y_2^3$	$27/\sqrt{512}$	305.86	$U^3K^{3/2}$	•
E	$X_2^0Y_2^4$	$81/64$	407.82	U^4K^2	◯
F	$X_2^1Y_2^4$	$4/3$	498.04	U^5K^2	◯
$F_\sharp$	$X_2^1Y_2^5$	$\sqrt{2}$	600.00	$U^6K^{5/2}$	•
G	$X_2^1Y_2^6$	$3/2$	701.96	U^7K^3	◯
$G_\sharp$	$X_2^1Y_2^7$	$9/\sqrt{32}$	803.91	$U^8K^{7/2}$	•
A	$X_2^1Y_2^8$	$27/16$	905.86	U^9K^4	◯
$B_\flat$	$X_2^1Y_2^9$	$81/\sqrt{2048}$	1 007.82	$U^{10}K^{9/2}$	•
B	$X_2^1Y_2^{10}$	$243/128$	1 109.78	$U^{11}K^5$	◯
C'	$X_2^2Y_2^{10}$	2	1 200.00	$U^{12}K^5$	

Table 3: $d = 2$, P_7 (◯), $(\sqrt{2}) \cdot P_5$ (•)

where $\beta_{1,d}, \ldots, \beta_{d,d}$ are pairwise incommensurable algebraic numbers for every d.

Proof

Consider the set

$$Q_{\mathcal{Z}} = \{(3/2)^k \pmod 2; k \in \mathcal{Z}\} = \{(X^{m_7}Y^{n_7})^k \pmod{X^{m_{12}}Y^{n_{12}}}; k \in \mathcal{Z}\}.$$

Clearly, $Q_{\mathcal{Z}} = \mathcal{P}$ and $Q_{\mathcal{Z}} \supset Q_{[k_1,k_2]}$ for every $k_1, k_2 \in \mathcal{Z}, k_1 < k_2$.

The assertion $\mathcal{G}_1 = \mathcal{P}$ is obvious for $d = 1$, $X_1 = X = 256/243, Y_1 = Y = (9/8) : X_1$.

For $d = 2$, $X_2 = 256/243, Y_2 = 3/\sqrt{8}$. We have (for illustration, cf. Figure 2):

$$Q_{\mathcal{Z}} = \mathcal{P} = \bigcup_{p \in \mathcal{Z}} \bigcup_{q \in \mathcal{Z}} X_2^p Y_2^{2q}.$$

Then

$$\mathcal{G}_2 = \mathcal{P} \cup (Y_2) \cdot \mathcal{P} = \mathcal{P} \cup (\sqrt{2}) \cdot \mathcal{P}.$$

For $d = 3$, $X_3 = \sqrt[3]{32}/3, Y_3 = 27/\sqrt[3]{16384}$. We have (for illustration, cf. Figure 3):

$$Q_{\mathcal{Z}} = \mathcal{P} = \bigcup_{p \in \mathcal{Z}} \bigcup_{q \in \mathcal{Z}} X_3^p Y_3^p \cdot X_3^{3q}.$$

Then

$$\mathcal{G}_3 = \mathcal{P} \cup (X_3) \cdot \mathcal{P} \cup (X_3^2) \cdot \mathcal{P} = \mathcal{P} \cup (\sqrt[3]{4})\mathcal{P} \cup (3\sqrt[3]{2}) \cdot \mathcal{P}.$$

C	$X_3^0 Y_3^0$	1	0.00	$U^0 K^0$	◯
$D_\flat$	$X_3^1 Y_3^0$	$\sqrt[3]{32}/3$	98.04	$U^1 K^{1/3}$	•
$C_\sharp$	$X_3^0 Y_3^1$	$27/\sqrt[3]{16384}$	105.86	$U^1 K^{2/3}$	■
D	$X_3^1 Y_3^1$	$9/8$	203.91	$U^2 K^1$	◯
$E_\flat$	$X_3^2 Y_3^1$	$3/\sqrt[3]{16}$	301.96	$U^3 K^{4/3}$	•
$D_\sharp$	$X_3^1 Y_3^2$	$243/\sqrt[3]{8388608}$	309.78	$U^3 K^{5/3}$	■
E	$X_3^2 Y_3^2$	$81/64$	407.82	$U^4 K^2$	◯
F	$X_3^3 Y_3^2$	$27/\sqrt[3]{8192}$	505.86	$U^5 K^{7/3}$	•
$F_\sharp$	$X_3^4 Y_3^2$	$9/\sqrt[3]{256}$	603.91	$U^6 K^{8/3}$	■
G	$X_3^5 Y_3^2$	$3/2$	701.96	$U^7 K^3$	◯
$A_\flat$	$X_3^6 Y_3^2$	$\sqrt[3]{4}$	800.00	$U^8 K^{10/3}$	•
$G_\sharp$	$X_3^5 Y_3^3$	$81/\sqrt[3]{931072}$	807.82	$U^8 K^{11/3}$	■
A	$X_3^6 Y_3^3$	$27/16$	905.86	$U^9 K^4$	◯
$B_\flat$	$X_3^7 Y_3^3$	$9/\sqrt[3]{128}$	1 003.91	$U^{10} K^{13/3}$	•
B	$X_3^8 Y_3^3$	$3/\sqrt[3]{4}$	1 101.96	$U^{11} K^{14/3}$	■
C'	$X_3^9 Y_3^3$	2	1 200.00	$U^{12} K^5$	

Table 4: $d = 3$, P_5 (◯), $(\sqrt[3]{32}/3) \cdot P_5$ (•), $(3/\sqrt[3]{4}) \cdot P_5$ (■)

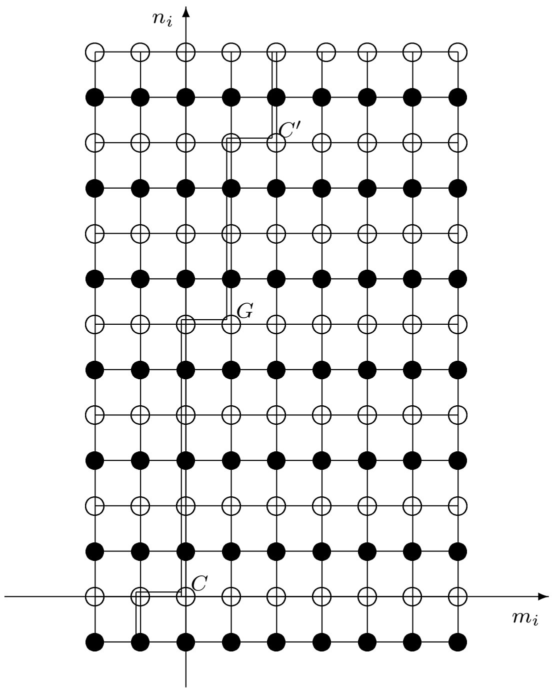

Figure 2: $\mathcal{G}_2 = \mathcal{P} \cup (\sqrt{2}) \cdot \mathcal{P}$

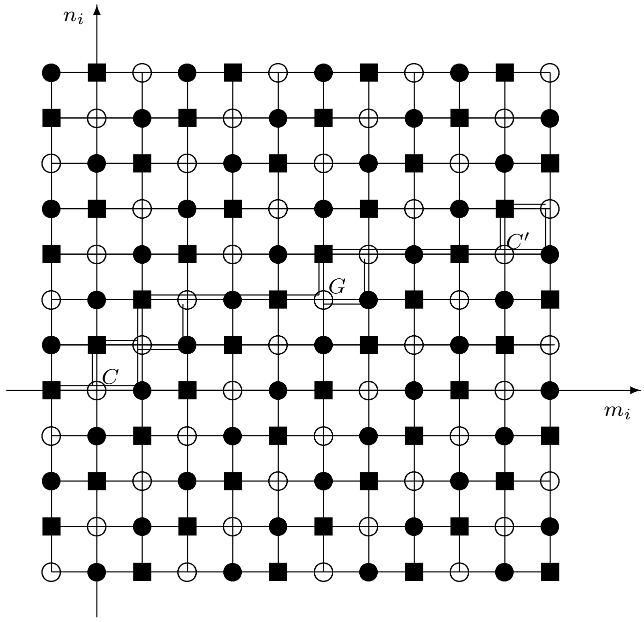

Figure 3: $\mathcal{G}_3 = \mathcal{P} \cup (\sqrt[3]{4}) \cdot \mathcal{P} \cup (3\sqrt[3]{2}) \cdot \mathcal{P}$

Analogously for every $d = 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 18, 20, 21, 23, 25, 28, 30$, and 35. □

3 Semitone metric space

Now we show that Pythagorean tone systems expressed via GGSs are the closer in some sense to Equal Temperament the greater the positive integer d is.

Theorem 3 *For $K = V/U$ (comat), denote by $\rho(X,Y) = |\log_K(X) - \log_K(Y)|$, $X, Y \in \mathcal{L}$. Let $(X_{d_1}, Y_{d_1}), (X_{d_2}, Y_{d_2})$ are solutions of Problem 1a corresponding d_1, d_2 and $d_1 < d_2$, $d_1, d_2 = 1, 2, \ldots$. Then $\rho(X_1, Y_1) > \rho(X_2, Y_2)$.*

Proof. It can be verified directly that $(\mathcal{L}, \rho)$ is a metric space.

If (X_d, Y_d) is a couple of numbers which is solution of Problem 1a corresponding d, $d = 0, 1, 2, 3, \ldots$, then by (7),

$$|\log_K(Y_d) - \log_K(X_d)| = \log_K 2^{\frac{-m_{12}-m_7}{d}} 3^{\frac{m_{12}}{d}} - \log_K 2^{\frac{19-m_{12}-m_7}{d}} 3^{\frac{m_{12}-12}{d}}$$

$$= \frac{12}{d}\log_K 3 - \frac{19}{d}\log_K 2 = \frac{1}{d}\log_K \frac{3^{12}}{2^{19}} = \frac{1}{d}\log_K K = \frac{1}{d}.$$

If $d_1 < d_2$, then

$$\rho(X_{d_1}, Y_{d_1}) = \frac{1}{d_1} > \frac{1}{d_2} = \rho(X_{d_2}, Y_{d_2}).$$

Corollary 1 *If $Y_d > X_d$, then $Y_d/X_d = \sqrt[d]{K}$.*

Definition Denote by $\mathcal{S}$ the set of all numbers $X \in \mathcal{L}$ such that (X, Y) is a solution of Problem 1a for some integer m_7, m_{12}, d and $Y \in \mathcal{L}$. We will call the metric space $(\mathcal{S}, \rho)$ the *metric space of semitones generating the perfect fifths.*

4 Applications

The sensitivity of human perception apparat is 5 – 6 cents. The psycho-acoustical boundary (to what the listener or player tent to) is 2 – 3 cents. This is the reason why there is no practical sense to use GGSs for $d > 12$, we hear no difference between Equal Temperament and the constructed GGS (we do not suppose too many values within 12 qualitative degrees). From the acoustical point of view, the interesting are cases of small values of $d = 1, 2, 3, \ldots, 11$ for which $Y_d/X_d \approx$ 23.46, 11.73, 7.82, 5.86, 4.69, 3.91, 3.35, …, 2.13 cents, respectively. From the viewpoint of modulations, the ordering is reverse: $d = 11, 10, \ldots 3, 2, 1$. So, we have to choose an appropriate compromise of d depending on the kind of modulations within a given musical composition (set of compositions). Some of GGSs in this paper, new from the musicology viewpoint, may be considered for tuning of musical instruments.

For $d = 2$, cf. Table 3 (Figure 2). The set P_7 is the Pythagorean heptatonic $F \to C \to G \to D \to A \to E \to B$, the "white keys"). The set $(\sqrt{2}) \cdot P_5$, is a Pythagorean pentatonic $F_\sharp \to C_\sharp \to G_\sharp \to D_\sharp \to A_\sharp$, (the "black keys").

For $d = 3$, cf. Table 4 (Figure 3, many valued tuning). The sets P_5 (i.e. $C \to G \to D \to A \to E$), $(3/\sqrt[3]{4}) \cdot P_5$, (i.e. $B \to F_\sharp \to C_\sharp \to G_\sharp \to D_\sharp$), and $(\sqrt[3]{32}/3) \cdot P_5$ (i.e. $D_\flat \to A_\flat \to E_\flat \to B_\flat \to F$) are three Pythagorean pentatonics.

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