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On the Choquet integral for Riesz space valued measures

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Abstract

The Choquet integral is defined for a real function with respect to a fuzzy measure taking values in a complete Riesz space. As applications there are presented: constructions of belief and plausibility measures, the formulation of an extension principle, and the Möbius transform for vector values measures.
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1 Introduction

It is very well known that the *Choquet integral* [5] of a non-negative measurable function can be taken with respect to a very general set function. Indeed, recall that a *fuzzy measure* is a mapping $\mu : \mathcal{A} \to R$ defined on a ring $\mathcal{A}$ of subsets of X and such that

(i) $\mu(\emptyset) = 0$,

(ii) $A, B \in \mathcal{A}, A \subset B$ implies $\mu(A) \leq \mu(B)$.

We say that $f : X \to [0, \infty]$ is measurable with respect to $\mathcal{A}$ if

$$f^{-1}([t, \infty)) = \{x \in X; f(x) \geq t\} \in \mathcal{A}$$

for any $t \in [0, \infty)$. If we define

$$g(t) = \mu(\{x \in X; f(x) \geq t\}), t \in [0, \infty),$$

then g is clearly decreasing and nonnegative function. Hence there exists the *Riemann improper integral*

$$\int_0^\infty g(t)\ dt$$

and then the Choquet integral $(C) \int_X f\ d\mu$ can be defined by the formula

$$(C) \int_X f\ d\mu = \int_0^\infty g(t)\ dt.$$

If $\mathcal{A}$ is a σ-algebra, it is not difficult to see that

$$(C) \int_X f\ d\mu = \int_0^\infty h(t)\ dt,$$

where

$$h(t) = \mu(\{x \in X; f(x) > t\}), t \in [0, \infty).$$

Denote by C the set of all points $t \in [0, \infty)$ such that g is continuous at t. If $t \in C$, then $g(t) = h(t)$. Evidently, $h(t) \leq g(t)$. On the other hand, for every $\varepsilon > 0$, there exists $s > t$ such that

$$g(t) - \varepsilon < g(s) = \mu(\{x \in X; f(x) \geq s\}) \leq \mu(\{x \in X; f(x) > t\}) = h(t).$$

Since $g(t) - \varepsilon < h(t)$ for any $\varepsilon > 0$, we obtain $g(t) \leq h(t)$. Now,

$$\begin{aligned} \int_0^\infty g(t)\ dt &= \int_C g(t)\ dt + \int_{[0,\infty)\setminus C} g(t)\ dt \\ &= \int_C h(t)\ dt + \Sigma_{x \notin C} \int_{\{x\}} g(t)\ dt \\ &= \int_C h(t)\ dt \\ &= \int_0^\infty h(t)\ dt. \end{aligned}$$

The main aim of the paper is the following: to define the Choquet integral in the case that $\mu : \mathcal{A} \to Y$ has values in a Riesz space.

In the following a *Riesz space* is a real vector space Y together with a *partial ordering* $\leq$ satisfying the following conditions:

(i) $(Y, \leq)$ is a lattice;

(ii) if $x, y, z, \in Y$ and $x \leq y$, then $x + z \leq y + z$;

(iii) if $x, y \in Y, \lambda \in R^+$ and $x \leq y$, then $\lambda x \leq \lambda y$.

A Riesz space Y is called to be *σ-complete* if every bounded sequence in Y has supremum. To define $(C) \int_X f(t)\ d\mu$ for $f : X \to [0, \infty), \mu : \mathcal{A} \to Y$, we need first reinvited the *Riemann–Stieltjes integral* $\int_a^b g\ dh$ for vector valued function g. It will be realized in Section 2. In Section 3, we define the Choquet integral $\int f\ d\mu$ and Sections 4 – 6 contain some applications.

2 Riemann– Stieltjes integral

For a Riesz space Y, assume that a real function $h : [a, b] \to R$ and a vector function $g : [a, b] \to Y$ are given. If $\mathcal{D} : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ is a partition and $t_i \in [x_{i-1}, x_i]$ $(i = 1, 2, \ldots, n)$, then we define the integral sums

$$S_g(h, \mathcal{D}) = \Sigma_{i=1}^n h(t_i)(g(x_i) - g(x_{i-1}))$$

and

$$S_h(g, \mathcal{D}) = \Sigma_{i=1}^n g(t_i)(h(x_i) - h(x_{i-1})).$$

As usually, $\|\mathcal{D}\| = \max_i(x_i - x_{i-1})$.

Definition. A scalar function h is *strongly integrable with respect to a vector function g* if there exist $c, u \in Y, u > 0$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|c - S_g(h, \mathcal{D})| < \varepsilon u,$$

whenever $\|\mathcal{D}\| < \delta$.

Lemma 1 *If Y is σ-complete, then the element c is defined uniquely.*

Proof. If c_1, c_2 satisfy the assumptions of the preceding Definition, then

$$|c_1 - c_2| \leq |c_1 - S_g(h, \mathcal{D})| + |S_g(h, \mathcal{D})| < 2\varepsilon u.$$

Put $\varepsilon = \frac{1}{2n}$. Then

$$|c_1 - c_2| < \frac{1}{n} u,$$

hence $|c_1 - c_2| = 0$. □

As usually, we denote the uniquely determined element $c \in Y$ as follows:

$$\int_a^b h\ dg = \int_a^b h(t)\ dg(t).$$

Similarly, the integral

$$\int_a^b g\ dh$$

can be defined (for other definitions of Riemann–Stieltjes integral, see [3], [8], [15]).

Theorem 1 *([6], [8], [15]) Let $h : [a,b] \to R, g : [a,b] \to Y$, where Y is a σ-complete Riesz space. Then $\int_a^b h\ dg$ exists if and only if $\int_a^b g\ dh$ exists. Moreover,*

$$\int_a^b h\ dg = h(b)g(b) - h(a)g(a) - \int_a^b g\ dh.$$

Proof. If we put $t_0 = a, t_{n+1} = b$, then

$\Sigma_{i=1}^n h(t_i)(g(x_i) - g(x_{i-1})) = h(b)g(b) - h(a)g(a) - \Sigma_{j=1}^n g(x_j)(h(t_j) - h(t_{j-1}))$.

By this equality, the formula of integration by parts follows. □

Theorem 2 *Let Y be a complete Riesz space, $g : [a,b] \to Y$ be an increasing mapping. Then the integral $\int_a^b h\ dg$ exists if and only if the following Cauchy – Bolzano condition holds: There exists $u \in Y, u > 0$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|S_g(h, \mathcal{D}_1) - S_g(h, \mathcal{D}_2)| < \varepsilon u,$$

whenever $\|\mathcal{D}_1\| < \delta$ and $\|\mathcal{D}_2\| < \delta$.

Proof. The necessity is clear, we prove the sufficiency.

Let the Cauchy – Bolzano condition holds. For the $\delta > 0$, put

$$a_\delta = \bigwedge_{\|\mathcal{D}\|<\delta} S_g(h, \mathcal{D}), b_\delta = \bigvee_{\|\mathcal{D}\|<\delta} S_g(h, \mathcal{D}).$$

Evidently, $0 \le a_\delta \le b_\delta$. Moreover, if $\delta(\varepsilon)$ corresponds to the given ε, then

$$b_{\delta(\varepsilon)} \le a_{\delta(\varepsilon)} + \varepsilon u. \quad (1)$$

If $\delta_1 < \delta_2$, then $a_{\delta_2} \le a_{\delta_1} \le b_{\delta_1} \le b_{\delta_2}$. It follows that

$$\bigvee_{\delta>0} a_\delta \le b_{\delta_2}.$$

Since the least inequality holds for every $\delta_2 > 0$, we obtain

$$\bigvee_{\delta>0} a_\delta \le \bigwedge_{\delta>0} b_\delta.$$

Let c be any element of Y satisfying the relation

$$\bigvee_{\delta>0} a_\delta \le c \le \bigwedge_{\delta>0} b_\delta. \quad (2)$$

Let $\delta = \delta(\varepsilon) > 0$ correspond to the given $\varepsilon > 0$. Let $\|\mathcal{D}\| < \delta$. Then by (1) and (2),

$$S_g(h, \mathcal{D}) - c \le b_\delta - c \le b_\delta - a_\delta \le \varepsilon u,$$

$$c - S_g(h, \mathcal{D}) \le c - a_\delta \le b_\delta - a_\delta \le \varepsilon u,$$

hence $|c - S_g(h, \mathcal{D})| \le \varepsilon u$. □

Theorem 3 *Let Y be a complete Riesz space, $h : [a,b] \to R$ be a continuous function, $g : [a,b] \to Y$ be an increasing (or decreasing) function. Then $\int_a^b h\ dg$ exists.*

Proof. We use Theorem 1. Assume that g is increasing. Then we can put $u = g(b) - g(a) > 0$ (the case $g(a) = g(b)$ is trivial). Let ε be an arbitrary positive number. Since h is uniformly continuous, there exists $\delta > 0$ such that $|h(t) - h(s)| < \varepsilon$ whenever $|t - s| < \delta$. Let $\mathcal{D}, \mathcal{D}'$ be partitions of $[a,b]$ with $\|\mathcal{D}\| < \delta, \|\mathcal{D}'\| < \delta$. Assume first that $\mathcal{D}'$ is a refinement of $\mathcal{D}$ and denote $x_{i-1} = x_i^0 < x_i^1 < x_i^{k_i} = x_i$ new dividing points, hence

$$g(x_i) - g(x_{i-1}) = \Sigma_{j=1}^{k_i}(g(x_i^j) - g(x_i^{j-1})). \tag{3}$$

Choose $t_i \in [x_{i-1}, x_i], t_i^j \in [x_{i-1}^{j-1}, x_i^j]$ such that

$$S_g(h, \mathcal{D}) = \Sigma_{i=1}^n h(t_i)(g(x_i) - g(x_{i-1})), \tag{4}$$

$$S_g(h, \mathcal{D}') = \Sigma_{i=1}^n \Sigma_{j=1}^{k_i} h(t_i^j)(g(x_i^j) - g(x_{i-1}^j)). \tag{5}$$

Then by (3), (4), and (5), we obtain

$$|S_g(h, \mathcal{D}) - S_g(h, \mathcal{D}')| \leq \Sigma_{i=1}^n \Sigma_{j=1}^{k_i} |h(t_i) - h(t_i^j|(g(x_i^j) - g(x_i^j)).$$

Since $t_i^j \in [x_{i-1}^{j-1}, x_i^j] \subset [x_{i-1}, x_i]$, we have $|t_i - t_i^j| < \delta$, hence $|h(t_i) - h(t_i^j)| < \varepsilon$. Therefore

$$\begin{aligned} |S_g(h, \mathcal{D}) - S_g(h, \mathcal{D}')| &\leq \varepsilon \Sigma_{i=1}^n \Sigma_{j=1}^{k_i}(g(x_i^j) - g(x_i^{j-1})) \\ &= \varepsilon \Sigma_{i=1}^n (g(x_i) - g(x_{i-1})) \\ &= \varepsilon(g(b) - g(a)) = \varepsilon u. \end{aligned}$$

If $\mathcal{D}_1, \mathcal{D}_2$ are arbitrary partitions with $\|\mathcal{D}_1\| < \delta, \|\mathcal{D}_2\| < \delta$, then we can construct the common refinement $\mathcal{D}'$ od $\mathcal{D}_1$ and $\mathcal{D}_2$ by preceding

$$\begin{aligned} |S_g(h, \mathcal{D}_1) - &\ S_g(h, \mathcal{D}_2)| \\ &\leq |S_g(h, \mathcal{D}_1) - S_g(h, \mathcal{D}')| + |S_g(h, \mathcal{D}') - S_g(h, \mathcal{D}_2)| \\ &\leq \varepsilon u + \varepsilon u = 2\varepsilon u. \end{aligned}$$

The theorem is proved. □

Corollary 1 *If Y is a complete Riesz space, $h : [a,b] \to R$ a continuous function and $g : [a,b] \to Y$ a monotone function, then $\int_a^b g\ dh$ exists.*

Proof. It follows by Theorem 1 and Theorem 3. □

3 Choquet integral

Assume again that Y is a complete Riesz space, $\mu : \mathcal{A} \to Y$ an Y-valued fuzzy measure defined on an algebra $\mathcal{A}$ of subsets of X and $f : X \to R$ a non-negative real function measurable with respect to $\mathcal{A}$ in the sense that $\{x \in X; f(x) > t\} \in \mathcal{A}$ for every $t \in R$.

Let $A \in \mathcal{A}$. Define $g_A : [0, \infty) \to Y$ by the formula

$$g_A(t) = \mu(\{x \in A; f(x) > t\}).$$

The mapping g_A is non-increasing. If we define $h : [0, \infty) \to R$ by the formula $h(t) = t$, then h is continuous on any interval $[0, c]$, hence by Corollary 1 there exists

$$\int_0^c g_A\ dh = \int_0^c g_A(t)\ dt = \int_0^c \mu(A \cap \{x; f(x) > t\})\ dt.$$

Put $\varphi(c) = \int_0^c g_A(t)\ dt \in Y$. Clearly $\varphi \geq 0$ and φ is increasing. Now there are two possibilities. If the function φ is bounded, then there exists

$$\int_0^\infty g_A(t)\ dt = \bigvee_{c>0} \varphi(c) = \bigvee_{c>0} \int_0^c g_A(t)\ dt.$$

In the opposite case we define

$$\int_0^\infty g_A(t)\ dt = \infty.$$

Definition. If Y is a complete Riesz space, f is a non-negative measurable function $f : (X, \mathcal{A}) \to R$, $\mathcal{A}$ is an algebra, $\mu : \mathcal{A} \to Y$ is a fuzzy measure and $A \in \mathcal{A}$, then we define the Choquet integral

$$(C) \int_A f\ d\mu$$

by the formula

$$(C) \int_A f\ d\mu = \int_0^\infty g_A(t)\ dt = \bigvee c > 0 \int_0^c \mu(A \cap \{x; f(x) > t\})\ dt.$$

Theorem 4 *If $\mu : \mathcal{A} \to Y$ is a σ-additive measure defined on a σ-algebra, then*

$$(C) \int_X f\ d\mu = \int_X f\ d\mu,$$

where $\int_X f\ d\mu$ is the Lebesgue integral ([1], [10], [15], [18], [19]).

Proof. First, let f be simple, $f = \Sigma_{i=1}^n \alpha_i \chi_{A_i}$, $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n$, and the sets A_i be disjoint. Then

$$g(t) = \begin{cases} \mu(A_1 \cup \ldots \cup A_n), & t < \alpha, \\ \mu(A_{i+1} \cup \ldots \cup A_n), & \alpha_i \leq t < \alpha_{i+1}, i = 1, \ldots, n-1 \\ 0, & t \geq \alpha_n \end{cases}$$

Therefore,

$$\begin{aligned} \int_0^\infty g(t)\ dt &= \alpha_1 \mu(A_1 \cup \ldots \cup A_n) + \Sigma_{i=1}^{n-1} (\alpha_{i+1} - \alpha_i) \mu(A_{i+1} \cup \ldots \cup A_n) \\ &= \Sigma_{j=1}^n \alpha_j \mu(A_j) = \int_X f\ d\mu. \end{aligned}$$

If f is an arbitrary non-negative measurable function, then there exists a sequence $(f_n)_n$ of simple measurable functions such that $f_n \nearrow f$. Then ([15])

$$\begin{aligned} \int_X f\ d\mu &= \bigvee_{n=1}^\infty \int_X f_n\ d\mu \\ &= \bigvee_{n=1}^\infty (C) \int_X f_n\ d\mu \\ &= \bigvee_{n=1}^\infty \int_0^\infty \mu(\{x \in X; f_n(x) > t\})\ dt \\ &= \int_0^\infty \bigvee_{n=1}^\infty \mu(\{x \in X; f_n(x) > t\})\ dt. \end{aligned}$$

But

$$\begin{aligned} \bigvee_{n=1}^\infty \mu(\{x \in X; f_n(x) > t\}) &= \mu(\bigcup_{n=1}^\infty \{x \in X; f_n(x) > t\}) \\ &= \mu(\{x \in X; f(x) > t\}). \end{aligned}$$

Therefore,

$$\int_X f\ d\mu = \int_0^\infty \mu(\{x \in X; f(x) > t\})\ dt = (C) \int_X f\ d\mu.$$

The theorem is proved. □

4 Belief and plausibility measures

A fuzzy measure $\mu : \mathcal{A} \to Y$ is called *lover continuous* if

$$A_n \nearrow A, (A_n) \subset A, A \in \mathcal{A} \Rightarrow \mu(A_n) \nearrow \mu(A).$$

A fuzzy measure $\mu : A \to Y$ is called a belief measure if for any $n \in \mathcal{N}$ and any $A_1, \ldots, A_n \in \mathcal{A}$ there holds

$$\mu(\bigcup_{i=1}^{n} A_i) \leq \Sigma_I (-1)^{|I|+1} \mu(\bigcap_{i\in I} A_i),$$

where the summation is taken over all non-empty subsets I of $\{1, 2, \ldots, n\}$ and $|I|$ denotes the cardinal number of I.

A fuzzy measure $\mu : \mathcal{A} \to Y$ is called a plausibility measure if for every $n \in \mathcal{N}$ and every $A_1, \ldots, A_n \in \mathcal{A}$ there holds

$$\mu(\bigcap_{i=1}^{n} A_i) \leq \Sigma_I (-1)^{|I|+1} \mu(\bigcup_{i\in I} A_i).$$

Theorem 5 ([11]) *Let Y be a complete Riesz space, f be a non-negative real function measurable with respect to a measurable space $(X, \mathcal{A}), X \in \mathcal{A}, \mu : \mathcal{A} \to Y$ be a fuzzy measure. Define $\nu : \mathcal{A} \to Y$ by the formula*

$$\nu(A) = (C) \int_A f \ d\mu.$$

Then ν is a fuzzy measure, ν is lower continuous, whenever μ is lower continuous. If μ is lower continuous and belief (plausibility) measure, then ν is belief (plausibility) measure, too.

Proof. Evidently, $\nu(\emptyset) = \int_0^\infty \mu(\emptyset)\ dt = 0$. If $A \subset B$, then

$$\begin{aligned} \nu(A) &= \int_0^\infty \mu(A \cap \{x \in X; f(x) > t\})\ dt \\ &\leq \int_0^\infty \mu(B \cap \{x \in X; f(x) > t\})\ dt = \nu(B). \end{aligned}$$

Let now μ be lower continuous and let $A_n \nearrow A$. Then

$$\begin{aligned} \nu(A) &= \int_0^\infty \mu(A \cap \{x \in X; f(x) > t\})\ dt \\ &= \int_0^\infty \bigvee_{n=1}^\infty \mu(A_n \cap \{x \in X; f(x) > t\})\ dt \\ &= \bigvee_{n=1}^\infty \int_0^\infty \mu(A_n \cap \{x \in X; f(x) > t\})\ dt \\ &= \bigvee_{n=1}^\infty \nu(A_n). \end{aligned}$$

Now, let μ be lower continuous and plausibility measure. Take simple functions f_n such that $f_n \nearrow f$. Fix n and put $f_n = \sum_{i=1}^m \alpha_i \chi_{A_i}$, $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_m$, A_i disjoint. Denote

$$\begin{aligned} \nu_n(A) &= (C) \int_A f_n \ d\mu \\ &= \int_0^\infty \mu(A_n \cap \{x \in X; f(x) > t\})\ dt \\ &= \sum_{i=1}^m (\alpha_i - \alpha_{i-1}) \mu(A \cap (A_i \cup A_{i+1} \ldots \cup A_m)) \\ &= \sum_{i=1}^m (\alpha_i - \alpha_{i-1}) \mu(A \cap B_i), \end{aligned}$$

where $B_i = A_i \cup A_{i+1} \ldots \cup A_m$. Then

$$\begin{aligned} \nu_n(\bigcap_{j=1}^k C_j) &= \sum_{i=1}^m (\alpha_i - \alpha_{i-1}) \mu(\bigcap_{j=1}^k C_j \cap B_i) \\ &\leq \sum_{i=1}^m (\alpha_i - \alpha_{i-1}) \sum_I (-1)^{|I|+1} \mu(\bigcap_{j=1}^k ((\bigcup_{j\in I} C_j) \cap B_i) \\ &= \sum_I (-1)^{|I|+1} \sum_{i=1}^m (\alpha_i - \alpha_{i-1}) \mu(\bigcap_{j=1}^k ((\bigcup_{j\in I} C_j) \cap B_i) \\ &= \sum_I (-1)^{|I|+1} \nu_n(\bigcup_{j\in I} C_j), \end{aligned}$$

hence

$$\nu_n(\bigcap_{j=1}^k C_j) \leq \sum_I (-1)^{|I|+1} \nu_n(\bigcup_{j\in I} C_j) \tag{6}$$

Now

$$\begin{aligned}\nu(A) &= \int_0^\infty \mu(A\cap\{x\in X; f(x)>t\})\ dt\\ &= \int_0^\infty \mu(A\cap\bigcup_{n=1}^\infty\{x\in X; f_n(x)>t\})\ dt\\ &= \int_0^\infty \bigvee_{n=1}^\infty \mu(A\cap\{x\in X; f_n(x)>t\})\ dt\\ &= \bigvee_{n=1}^\infty \int_0^\infty \mu(A\cap\{x\in X; f_n(x)>t\})\ dt\\ &= \bigvee_{n=1}^\infty (C)\int_A f_n\ d\mu\\ &= \bigvee_{n=1}^\infty \nu_n(A).\end{aligned}$$

We have proved that $\nu_n(A)\nearrow\nu(A)$ for every $A\in\mathcal{A}$, hence $\nu_n(A)$ o-converges to $\nu(A)$. By this fact and by (6) we obtain

$$\nu(\bigcap_{j=1}^k C_j) \le \sum_I (-1)^{|I|+1}\nu(\bigcup_{j\in I} C_j).$$

The assertion concerning belief measures can be proved similarly. □

5 An extension principle

In [17] M. J. Wierman considered a fuzzy measure μ defined on a family of all subsets of a set and by the help of the Choquet integral he extended μ to the family of all fuzzy subsets of X. In this section we shall show that the Wierman principle can be applied also for Riesz space valued fuzzy measures.

Let $\mathcal{A}$ be an algebra of subsets of X. Let $f : X \to R$ be a non-negative function. As before, we will say that f is measurable with respect to $\mathcal{A}$ if $\{x\in X; f(x)>t\}\in\mathcal{A}$ for every $t\in R$.

Let $\mu:\mathcal{A}\to Y$ be a fuzzy measure, Y being a complete Riesz space. Let $M(\mathcal{A})$ be the set of all $\mathcal{A}$-measurable fuzzy subsets of X. Then we define $\overline{\mu}: M(\mathcal{A})\to Y$ by the formula

$$\overline{\mu}(f) = (C)\int_X f\ d\mu.$$

Theorem 6 *The mapping $\overline{\mu}$ is an extension of μ. If $\mu$is lower continuous, then $\overline{\mu}$ is lower continuous, too.*

Proof. Evidently,

$$\begin{aligned}(C)\int_X f\ d\mu &= \int_0^\infty \mu(\{x\in X; f(x)>t\})\ dt\\ &= \int_0^1 f\ d\mu = \int_0^1 \mu(\{x\in X; f(x)>t\})\ dt.\end{aligned}$$

If $f=\chi_A$, then

$$\{x\in X; f(x)>t\} = \begin{cases} A, & t<1\\ \emptyset, & t\ge 1.\end{cases}$$

Therefore

$$\overline{\mu}(\chi_A) = (C)\int_X \chi_A\ d\mu = \int_0^1 \mu(A)\ dt = \mu(A).$$

Let μ be lower continuous and $f_n\in M(\mathcal{A}), f_n\nearrow f$. Then

$$\begin{aligned}\overline{\mu}(\mu)(f) &= \int_0^1 \mu(\{x\in X; f(x)>t\})\ dt\\ &= \int_0^1 \mu(\bigcup_{n=1}^\infty\{x\in X; f_n(x)>t\})\ dt\\ &= \int_0^1 \bigvee_{n=1}^\infty \mu(\{x\in X; f_n(x)>t\})\ dt\\ &= \bigvee_{n=1}^\infty \int_0^1 \mu(\{x\in X; f_n(x)>t\})\ dt\\ &= \bigvee_{n=1}^\infty \overline{\mu}(f_n).\end{aligned}$$

The theorem is proved. □

The Wierman extension principle can be extended to the case of relations ρ in the Cartesian product $2^U\times 2^V$, i.e. for $\rho\subset 2^U\times 2^V$. Namely, if ρ is a relation between sets, then $\overline{\rho}$ will be a relation between fuzzy subsets of U or V, respectively. If $f : U \to [0;1]$ is a fuzzy subset of U, then denote as usually

$$f^t = \{x \in U; f(x) > t\}.$$

Let χ_ρ be the characteristic function of ρ, i.e.

$$\chi_\rho(A,B) = \begin{cases} 1, & A\rho B \\ 0, & otherwise. \end{cases}$$

Then the membership function $\chi_{\overline{\rho}}$ of the fuzzy relation $\overline{\rho}$ is defined by the formula

$$\chi_{\overline{\rho}}(f,g) = \int_0^1 \chi_\rho(f^t, g^t)\ dt.$$

6 Möbius transform

Assume now that X is a finite set $\mu : 2^X \to Y$ be a fuzzy measure. Then the *Möbius transform* $M_\mu : 2^X \to Y$ is defined by the formula

$$M_\mu(A) = \sum_{B\subset A} (-1)^{|A\setminus B|}\mu(B).$$

By the same method as in the real-valued case ([4], [9]) it can be proved that

$$\mu(A) = \sum_{B\subset A} M_\mu(B) \tag{7}$$

for every $A \subset X$.

Now we generalize a theorem by Mesiar ([12]).

Theorem 7 *Let X be a finite set, Y be a complete Riesz space, $f : X \to R$ be a non-negative function, $f_A = \bigwedge_{x\in A} f(x)$ for every $A \subset X$. Then*

$$(C)\int_X f\ d\mu = \sum_{A\subset X} f_A M_\mu(A).$$

Proof. Let $\{a_1, a_2, \ldots, a_n\}$ be the range of f, $a_1 \le a_2 \le \ldots \le a_n$. Therefore $a_i = f(x_{n_i})$ for some $n_i \in N$. Then

$$(C)\int_X f\ d\mu = a_1\mu(X) + \sum_{i=2}^n (a_i - a_{i-1})\mu(X \setminus \{x_{n_1}, \ldots x_{n_{i-1}}\}).$$

Therefore, by (7),

$$\begin{aligned} (C)\textstyle\int_X f\ d\mu &= a_1 \textstyle\sum\{M_\mu(A); A \subset X\} + \\ &+ \textstyle\sum_{i=2}^n (a_i - a_{i-1}) \sum\{M_\mu(A); A \subset X \setminus \{x_{n_1}, \ldots x_{n_{i-1}}\}\} \\ &= a_1 \textstyle\sum\{M_\mu(A); x_{n_i} \in A \subset X\} + \\ &+ \textstyle\sum_{i=2}^n a_i \sum\{M_\mu(A); x_{n_i} \in A \subset X \setminus \{x_{n_1}, \ldots x_{n_{i-1}}\}\} \\ &= \textstyle\sum_{A\subset X} f_A M_\mu(A). \end{aligned}$$

The theorem is proved. □

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