Tatra Mountains Mathematical Publications 14(1998), 145 – 151.

On numbers $256/243, 25/24, 16/15$

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Abstract

The ratios $256/243, 25/24, 16/15$ are known as the minor Pythagorean, chromatic, and diatonic semitone, respectively. The main result of this paper is the following statement which has a valuable consequence for the music acoustic theory:

According to the symmetry, all rational triplets (X_1, X_2, X_3) *TDS-generating generalized geometrical progressions*

$$\langle \Gamma_i \rangle = \left\langle X_1^{\nu_{i,1}} X_2^{\nu_{i,2}} X_3^{\nu_{i,3}}; \nu_{i,1} + \nu_{i,2} + \nu_{i,3} = i, 0 \le \nu_{0,\cdot} \le \nu_{1,\cdot} \le \ldots \le \nu_{i,\cdot} \le \ldots \right\rangle_{\nu_{i,\cdot} \in \mathcal{N}^3}$$

with the subsequences

$$\langle \Gamma_{12l} \rangle = \left\langle 2^l \right\rangle, \langle \Gamma_{12l+7} \rangle = \left\langle 3 \cdot 2^{l-1} \right\rangle, \langle \Gamma_{12l+4} \rangle = \left\langle 5 \cdot 2^{l-2} \right\rangle$$

are exactly the following:

$$(25/24, 135/128, 16/15), (256/243, 135/128, 16/15), (25/24, 16/15, 27/25).$$

Thus, not only the diatonic and chromatic but also the minor Pythagorean semitone (together with the diatonic semitone and its complement to the major whole tone) can serve as a basis for the construction of 12-degree diatonic scales.

1 Introduction

Every periodic waveform $f(t)$, understood as a function of time $t > 0$, can be represented as follows: $f(t) = \sum_{k=1}^{\infty} a_k \sin(2\pi\omega_k t - \varphi_k)$, where ω_k is the frequency of the k-th partial (which is k times the *fundamental frequency* ω of the tone associated with its *pitch*, i. e. $\omega_k = k\omega$); a_k is the amplitude of the k-th partial (corresponding to its *loudness*); φ_k is the phase of the k-th partial (conventionally interpreted as the *entry delay* of the given partial). According to Fourier's theorem, any tone with a periodic waveform is a sum of *harmonics*, i. e. partials with frequencies ω_k satisfying the *harmonic frequency ratio* $\omega_1 : \omega_2 : \ldots : \omega_k : \ldots = 1 : 2 : \ldots : k : \ldots$ Besides harmonics, there are sounds with no salient pitch (not considered in this paper, cf. [1]) which are of various types, e.g. the ratio of their partial frequencies ω_k is not harmonic but *inharmonic*. For instance, the ratio $1 : \sqrt[12]{2}$ is used in the well-known Equal Tempered Tuning.

AMS subject classification: 42 A 16, 40 B 05, 94 B 60, 94 D 05, 92 J 30, 11 D 25
Key words: Generalized geometrical progression, Fourier's theorem, Diophantine equation, Many valued coding, Music acoustic

The ratios $256/243, 25/24, 16/15$ are known as the *minor Pythagorean, chromatic,* and *diatonic semitone*, respectively. The Just Intonation Set (cf. [3], [4], we avoid the minor seventh (7/4) and the second (8/7)), considered as the most natural tuning from many (physical, psycho-acoustical, polyphonic, etc.) points of view in the present time, is constructed on the basis of the chromatic and diatonic semitones. On the other side, Pythagorean Tuning (cf. [3], [4]) is based on the minor Pythagorean semitone, *diesis* (256/243). The Just Intonation Set and Pythagorean Tuning are still considered by music theoreticians as two fully incompatible tone systems.

The principal result of this paper, new for the music acoustic theory, is the fact that not only the diatonic and chromatic but also **the minor Pythagorean semitone (together with the diatonic semitone and its complement to the major whole tone) can serve as a basis for the construction of 12-degree diatonic scales,** cf. Table 2.

Recall that the *major whole tone* (the Pythagorean whole tone) is derived from the perfect fourth and the perfect fifth, i.e. $9/8 = 3/2 : 4/3$. The *minor whole tone* is obtained from the major third and the major whole tone, i.e. $10/9 = 5/4 : 9/8$. Diesis is derived from the perfect fourth and two major whole tones, i.e. $256/243 = 4/3 : (9/8)^2$. The *diatonic semitone* is obtained from the perfect fourth and major third, i.e. $16/15 = 4/3 : 5/4$. For the review of the literature, see [4], [5].

2 Preliminaries

Denote by $\mathcal{N} = \{0, 1, 2, \ldots\}$ and by $\mathcal{Q}$ the set of all rational numbers. If we denote by $\mathcal{L} = ((0, \infty), \cdot, 1, \leq)$ the usual multiplicative group on reals with the usual order, then b/a is called the $\mathcal{L}$-length of the interval $(a, b), 0 < a \leq b < \infty$.

Suppose (the fundamental frequency) $\omega = 1$. We restrict our considerations to the subsequences $\left\langle 2^l \right\rangle, \left\langle 3 \cdot 2^{l-1} \right\rangle, \left\langle 5 \cdot 2^{l-2} \right\rangle$ of the sequence $\langle \omega_k \rangle = \langle k \rangle, k \in \mathcal{N}$ (corresponding to the first three overtones; these subsequences are known in music as the classes of equivalency of the *octaves, perfect fifths,* and *major thirds*; members of each subsequence are denoted by the same letter in music, e.g. C, G, E, respectively).

We will use the following conventional notation:

$$X = (X_1, X_2, \cdots, X_n) \in \mathcal{L}^n, \nu_{i,\cdot} = (\nu_{i,1}, \nu_{i,2}, \ldots, \nu_{i,n}) \in \mathcal{N}^n,$$

$$\nu_{i,\cdot} \leq \nu_{i+1,\cdot} \Leftrightarrow \nu_{i,k} \leq \nu_{i+1,k} \quad (k = 1, 2, \ldots, n),$$

$$|\nu_{i,\cdot}| = \nu_{i,1} + \nu_{i,2} + \ldots + \nu_{i,n}, X^{\nu_{i,\cdot}} = X_1^{\nu_{i,1}} X_2^{\nu_{i,2}} \ldots X_n^{\nu_{i,n}} \quad (i, n \in \mathcal{N}).$$

Definition 1 For $n \in \mathcal{N}$, we say that a sequence $\langle \Gamma_i \rangle$ is an *n-generalized geometrical progression* if there exist $X \in \mathcal{L}^n$ and $\nu_{i,j} \in \mathcal{N}$ $(i \in \mathcal{N}, j = 1, 2, \ldots, n)$

such that

$$\Gamma_i = X^{\nu_{i,\cdot}}, 0 \le \nu_{0,\cdot} \le \nu_{1,\cdot} \le \ldots \le \nu_{i,\cdot} \le \ldots, |\nu_{i,\cdot}| = i.$$

In this paper, we will consider the case $n = 3$.

We say that a matrix $(\nu_{i,j})_{12,7,4}^{1,2,3} \in \mathcal{N}^3 \times \mathcal{N}^3$ is a (12, 7, 4)-*matrix*, cf. [3], Definition 2, if $0 \le \nu_{4,\cdot} \le \nu_{7,\cdot} \le \nu_{12,\cdot}$ and $|\nu_{i,\cdot}| = i, i = 12, 7, 4$.

Theorem 1 ([3], Theorem 1) *Let* $A = (\nu_{i,j})_{i=12,7,4}^{j=1,2,3} \in \mathcal{N}^3 \times \mathcal{N}^3$ *with* $\det A \neq 0$. *Then there exist a unique* $X \in \mathcal{L}^3$, *such that* $X^{\nu_{12,\cdot}} = 2/1, X^{\nu_{7,\cdot}} = 3/2, X^{\nu_{4,\cdot}} = 5/4$, *and the following statements are equivalent:*

$$\text{(a) } X \in \mathcal{Q}^3, \text{ (b) } \det A = 1.$$

The values are as follows:

$$X_1 = \sqrt[\det A]{2^{D_{2,1}}3^{D_{3,1}}5^{D_{5,1}}}, X_2 = \sqrt[\det A]{2^{D_{2,2}}3^{D_{3,2}}5^{D_{5,2}}}, X_3 = \sqrt[\det A]{2^{D_{2,3}}3^{D_{3,3}}5^{D_{5,3}}},$$

where

$$D_{2,1} = \begin{vmatrix} 1 & \nu_{12,2} & \nu_{12,3} \\ -1 & \nu_{7,2} & \nu_{7,3} \\ -2 & \nu_{4,2} & \nu_{4,3} \end{vmatrix}, D_{2,2} = \begin{vmatrix} \nu_{12,1} & 1 & \nu_{12,3} \\ \nu_{7,1} & -1 & \nu_{7,3} \\ \nu_{4,1} & -2 & \nu_{4,3} \end{vmatrix}, D_{2,3} = \begin{vmatrix} \nu_{12,1} & \nu_{12,2} & 1 \\ \nu_{7,1} & \nu_{7,2} & -1 \\ \nu_{4,1} & \nu_{4,2} & -2 \end{vmatrix},$$

$$D_{3,1} = \begin{vmatrix} 0 & \nu_{12,2} & \nu_{12,3} \\ 1 & \nu_{7,2} & \nu_{7,3} \\ 0 & \nu_{4,2} & \nu_{4,3} \end{vmatrix}, D_{3,2} = \begin{vmatrix} \nu_{12,1} & 0 & \nu_{12,3} \\ \nu_{7,1} & 1 & \nu_{7,3} \\ \nu_{4,1} & 0 & \nu_{4,3} \end{vmatrix}, D_{3,3} = \begin{vmatrix} \nu_{12,1} & \nu_{12,2} & 0 \\ \nu_{7,1} & \nu_{7,2} & 1 \\ \nu_{4,1} & \nu_{4,2} & 0 \end{vmatrix},$$

$$D_{5,1} = \begin{vmatrix} 0 & \nu_{12,2} & \nu_{12,3} \\ 0 & \nu_{7,2} & \nu_{7,3} \\ 1 & \nu_{4,2} & \nu_{4,3} \end{vmatrix}, D_{5,2} = \begin{vmatrix} \nu_{12,1} & 0 & \nu_{12,3} \\ \nu_{7,1} & 0 & \nu_{7,3} \\ \nu_{4,1} & 1 & \nu_{4,3} \end{vmatrix}, D_{5,3} = \begin{vmatrix} \nu_{12,1} & \nu_{12,2} & 0 \\ \nu_{7,1} & \nu_{7,2} & 0 \\ \nu_{4,1} & \nu_{4,2} & 1 \end{vmatrix}.$$

Definition 2 We say that a 3-generalized geometrical progression $\langle \Gamma_i \rangle$ is *TDS-generated by a (12, 7, 4)-matrix A* [or, is *TDS-generated by* $X \in \mathcal{Q}^3$ such that $X^{\nu_{12,\cdot}} = 2/1, X^{\nu_{7,\cdot}} = 3/2, X^{\nu_{4,\cdot}} = 5/4$ $(\nu_{i,j}, i, j \in \mathcal{N})$, cf. Theorem 1] if

$$\nu_{2,\cdot} = 2\nu_{7,\cdot} - \nu_{12,\cdot}, \nu_{5,\cdot} = \nu_{12,\cdot} - \nu_{7,\cdot}, \nu_{9,\cdot} = \nu_{12,\cdot} - \nu_{7,\cdot} + \nu_{4,\cdot}, \nu_{11,\cdot} = \nu_{7,\cdot} + \nu_{4,\cdot},$$

and for $i \ge 12$, there exists $p \in \mathcal{N}, 0 \le p < 12$, and $q \in \mathcal{N}$, such that $\nu_{i,\cdot} = q\nu_{12,\cdot} + \nu_{p,\cdot}$.

Note that the members $\Gamma_i, i = 1, 3, 6, 8, 10$, mentioned in Definition 2, are not uniquely determined.

3 Three sequences

Theorem 2 *According to the symmetry, all* $X \in \mathcal{Q}^3$ *TDS-generating 3-generalized geometrical progressions*

$$\langle \Gamma_i \rangle = \langle X^{\nu_{i,\cdot}}; |\nu_{i,\cdot}| = i, 0 \le \nu_{0,\cdot} \le \nu_{1,\cdot} \le \ldots \le \nu_{i,\cdot} \le \ldots \rangle_{\nu_{i,\cdot} \in \mathcal{N}^3}$$

with the subsequences

$$\langle\Gamma_{12l}\rangle = \left\langle 2^l \right\rangle, \langle\Gamma_{12l+7}\rangle = \left\langle 3\cdot 2^{l-1}\right\rangle, \langle\Gamma_{12l+4}\rangle = \left\langle 5\cdot 2^{l-2}\right\rangle$$

are the following:

$$(25/24, 135/128, 16/15), (256/243, 135/128, 16/15), (25/24, 16/15, 27/25).$$

Proof. The analysis of the Diophantine equation

$$\det[(\nu_{i,j})_{i=12,7,4}^{j=1,2,3}] = 1, 0 \le \nu_{4,\cdot} \le \nu_{7,\cdot} \le \nu_{12,\cdot}, |\nu_{i,\cdot}| = i$$

in $\mathcal{N}^3 \times \mathcal{N}^3$ with the additional (not restricting the solution) condition

$$2\nu_{7,\cdot} - \nu_{12,\cdot} \ge 0$$

yields the following matrices (excluding symmetries, permutations of columns):

$$A_1 = \begin{pmatrix} 2 & 7 & 3 \\ 1 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 2 & 5 & 5 \\ 1 & 3 & 3 \\ 1 & 1 & 2 \end{pmatrix}, A_3 = \begin{pmatrix} 5 & 4 & 3 \\ 3 & 2 & 2 \\ 2 & 1 & 1 \end{pmatrix}, A_4 = \begin{pmatrix} 1 & 2 & 9 \\ 1 & 1 & 5 \\ 0 & 1 & 3 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 1 & 3 & 8 \\ 1 & 2 & 4 \\ 0 & 1 & 3 \end{pmatrix}, A_6 = \begin{pmatrix} 1 & 4 & 7 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix}, A_7 = \begin{pmatrix} 1 & 5 & 6 \\ 1 & 3 & 3 \\ 1 & 2 & 1 \end{pmatrix}, A_8 = \begin{pmatrix} 2 & 3 & 7 \\ 1 & 2 & 4 \\ 1 & 0 & 3 \end{pmatrix}.$$

Apply Theorem 1 and find all sequences by the algorithm in Definition 2. Excluding all such sequences $\langle\Gamma_i\rangle$ which do not satisfy the condition $\nu_{0,\cdot} \le \nu_{1,\cdot} \le \ldots \le \nu_{i,\cdot} \le \ldots$ in Definition 1, we obtain the following three matrices: A_1, A_2, A_3.

In Tables 1 and 2 there are all TDS-generated sequences $\langle\Gamma_i\rangle$ (in the fifth column, there are values in cents, i.e. in the isomorphism $\Gamma_i \mapsto 1200 \cdot \log_2 \Gamma_i$; in the sixths column, there is a musical denotation) corresponding to the matrices A_1 and A_2. The TDS-generated sequences corresponding to A_3 can be found in [3], Table 3. □

In the connection with the previous theorem we mention here the following

Theorem 3 ([3], Theorem 5) *According to the symmetry, A_3 is the unique solution of the Diophantine equation* $\det[(\nu_{i,j})_{i=12,7,4}^{j=1,2,3}] = 1, 0 < \nu_{4,\cdot} < \nu_{7,\cdot} < \nu_{12,\cdot}, |\nu_{i,\cdot}| = i$.

All superparticular ratios for numbers 2,3, and 5, are exactly: 2/1, 3/2, 4/3, 5/4, 6/5, 9/8, 10/9, 16/15, 25/24, and 81/80, cf. [2]. The proof of the following theorem is easy.

Theorem 4 *See Table 3.*

$X_1^0X_2^0X_3^0$	$2^03^05^0$	1/1	1.0	0	C
$X_1^0X_2^0X_3^1$	$2^{-7}3^351$	135/128	1.0546875	92.1787	$C_\sharp$
$X_1^0X_2^1X_3^0$	$2^43^{-1}5^{-1}$	16/15	1.066666666	111.7313	$D_\flat$
$X_1^0X_2^1X_3^1$	$2^{-3}3^25^0$	9/8	1.125	203.9100	D
$X_1^1X_2^1X_3^1$	$2^{-6}3^15^2$	75/64	1.171875	274.5824	$D_\sharp$
$X_1^0X_2^2X_3^1$	$2^13^15^{-1}$	6/5	1.2	315.6413	$E_\flat$
$X_1^1X_2^2X_3^1$	$2^{-2}3^05^1$	5/4	1.25	386.3137	E
$X_1^1X_2^3X_3^1$	$2^23^{-1}5^0$	4/3	1.333333333	498.0450	F
$X_1^1X_2^3X_3^2$	$2^{-5}3^25^1$	45/32	1.40625	590.2237	$F_\sharp$
$X_1^1X_2^4X_3^1$	$2^63^{-2}5^{-1}$	64/45	1.422222222	609.7763	$G_\flat$
$X_1^1X_2^4X_3^2$	$2^{-1}3^15^0$	3/2	1.5	701.9550	G
$X_1^2X_2^4X_3^2$	$2^{-4}3^05^2$	25/16	1.5625	772.6274	$G_\sharp$
$X_1^1X_2^5X_3^2$	$2^33^05^{-1}$	8/5	1.6	813.6863	$A_\flat$
$X_1^2X_2^5X_3^2$	$2^03^{-1}5^1$	5/3	1.666666666	884.3587	A
$X_1^2X_2^5X_3^3$	$2^{-7}3^25^2$	225/128	1.7578125	976.5374	$A_\sharp$
$X_1^2X_2^6X_3^2$	$2^43^{-2}5^0$	16/9	1.777777777	996.0900	$B_\flat$
$X_1^2X_2^6X_3^3$	$2^{-3}3^15^1$	15/8	1.875	1088.2687	B
$X_1^2X_2^7X_3^3$	$2^13^05^0$	2/1	2.0	1200	C'
...	...	...	...	...	...

Table 1: $X_1 = 25/24, X_2 = 16/15, X_3 = 135/128$

$X_1^0X_2^0X_3^0$	$2^03^05^0$	1/1	1.0	0	C
$X_1^0X_2^0X_3^1$	$2^{-7}3^35^1$	135/128	1.0546875	92.1787	$C_\sharp$
$X_1^0X_2^1X_3^0$	$2^43^{-1}5^{-1}$	16/15	1.066666666	111.7313	$D_\flat$
$X_1^0X_2^1X_3^1$	$2^{-3}3^25^0$	9/8	1.125	203.9100	D
$X_1^1X_2^1X_3^1$	$2^53^{-3}5^0$	32/27	1.185185185	294.1350	$D_\sharp$
$X_1^0X_2^1X_3^2$	$2^{-10}3^55^1$	1215/1024	1.186523438	296.0887	$E_\flat$
$X_1^1X_2^1X_3^2$	$2^{-2}3^05^1$	5/4	1.25	386.3137	E
$X_1^1X_2^2X_3^2$	$2^23^{-1}5^0$	4/3	1.333333333	498.0450	F
$X_1^1X_2^2X_3^3$	$2^{-5}3^25^1$	45/32	1.40625	590.2237	$F_\sharp$
$X_1^1X_2^3X_3^2$	$2^63^{-2}5^{-1}$	64/45	1.422222222	609.7763	$G_\flat$
$X_1^1X_2^3X_3^3$	$2^{-1}3^15^0$	3/2	1.5	701.9550	G
$X_1^2X_2^3X_3^3$	$2^73^{-4}5^0$	128/81	1.580246914	792.1800	$G_\sharp$
$X_1^1X_2^3X_3^4$	$2^{-8}3^45^1$	405/256	1.58203125	794.1337	$A_\flat$
$X_1^2X_2^3X_3^4$	$2^03^{-1}5^1$	5/3	1.666666666	884.3587	A
$X_1^2X_2^3X_3^5$	$2^{-7}3^25^2$	225/128	1.7578125	976.5374	$A_\sharp$
$X_1^2X_2^4X_3^4$	$2^43^{-2}5^0$	16/9	1.777777777	996.0900	$B_\flat$
$X_1^2X_2^4X_3^5$	$2^{-3}3^15^1$	15/8	1.875	1088.2687	B
$X_1^2X_2^5X_3^5$	$2^13^05^0$	2/1	2.0	1200	C'
...	...	...	...	...	...

Table 2: $X_1 = 256/243, X_2 = 16/15, X_3 = 135/128$

	(x,y,z)	(x,v,y)	(w,v,y)
2/1	$x^5y^4z^3$	$x^2v^3y^7$	$w^2v^5y^5$
3/2	$x^3y^2z^2$	xv^2y^4	wv^3y^3
4/3	x^2y^2z	xvy^3	wv^2y^2
5/4	x^2yz	xvy^2	wv^2y
6/5	xyz	vy^2	vy^2
9/8	xz	vy	vy
10/9	xy	xy	wv
16/15	y	y	y
25/24	x	x	wvy^{-1}
81/80	$y^{-1}z$	$x^{-1}v$	$w^{-1}y$

Table 3: $x = 25/24, y = 16/15, z = 27/25, v = 135/128, w = 256/243$

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