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Uncertainty and tuning in music

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Abstract

The tuning in music is an excellent example of that the human perceptive mechanism uses various multivalued systems of information coding. The ambiguous intervals in the Just Intonation Set are the second and the minor seventh and the tritone (the relative frequencies 10/9, 9/8, 8/7 and 7/4, 16/9, 18/10 and 45/32, 64/45, respectively). Pythagorean Tuning is also 17-valued. In the present paper we find a 17-valued diatonic tone system which is a consequence of the unique solution of a Diophantine equation describing the basic acoustic relations among octave, perfect fifth and major (minor) third. This system has properties of the Just Intonation Set (it involves octave, perfect fifth, perfect fourth, major third, minor third, major whole tone, minor whole tone, diatonic semitone and chromatic semitone) and also of Pythagorean Tuning. We bring applications of our theory to superparticular ratios and partial monounary algebras.
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1 Introduction

Based on the postulate that only the numbers 1,2, and 3 are perfect, *Pythagorean Tuning* was created as a sequence of 17 numbers of the form $2^a 3^b$, where a, b are integers. This tuning was established about five hundred years B. C. and used in European music up to the 14th century, cf. Table 1.

The gradual development of polyphony led to the introduction of thirds (5/4 and 6/5) and sixths (8/5 and 5/3). Targeting at a strict mathematical definition of consonance, the problem of ratios of integers was studied by G i o s e f f o Z a r l i n o (1517 – 1590), S i m o n S t e v i n (1548 – 1620), and J o h a n n K e p l e r (1571 – 1630). One practical result of this period was the creation of the 17-valued *Just Intonation Set*, cf. [4], cf. Table 2, on which the *Equal Tempered Scale* was later based (A n d r e a s W e r c k m e i s t e r, 1698) (it is represented by the sequence $W = \{1, \sqrt[12]{2}, (\sqrt[12]{2})^2, \ldots, (\sqrt[12]{2})^{12} = 2\}$).

The further development was connected with mathematical physics. J o s e p h F o u r i e r (1768 – 1830) demonstrated that the oscillations of a string can be represented as the superposition of elementary sinusoid oscillations whose frequency ratios are related to each other as integers. H e r m a n n v o n H e l m h o l t z (1821 – 1894) assumed that the aesthetic characteristics of intervals were connected with the beating produced by overtones.

In the present time, some of musicologists and psycho-acousticians assert that the twelve-degree musical scales appeared not as merely artificial convention, but rather the natural development of musical culture led to aural selection of qualitative intervals (to be more precisely, interval "zones") with a particular degree of individuality. They are: the unison, minor second, major second, minor third, major third, fourth, fifth, tritone, fifth, minor sixth, major sixth, minor seventh, major seventh, and octave which is qualitative identified with unison. Thus, the diatonic scales appeared as the result of categorizing certain intervals with specific psycho-acoustical properties. There have been also many theoretical and experimental works concerned – both *pro* and *contra* – with the assumed universality of diatonic scales. For a review of the literature, cf. [4], [5].

Our aim is not to judge such non-mathematical problem. We are interested is the fact that the tuning (which should be strict and exact, otherwise it is not a tuning!) in music is an excellent example of that the human perceptive mechanism uses various multivalued systems of information coding.

Particularly, two 17-tone tuning systems for the 12 qualitative different music intervals (Pythagorean Tuning and Just Intonation) are well-known. The main requirement is that the subject's perceptive system be able to encode the informational content unambiguously: this factor distinguishes a musical interval from any other pair of sounds.

In the present paper we find a 17-valued diatonic tuning system which is a consequence of the unique solution of a Diophantine equation describing the basic acoustic relations among octave, perfect fifth and major (minor) third. This system has properties of the

C	$2^0 3^0$	$1/1$	1.0	$x^0 y^0$
$C_\sharp$	$2^8 3^{-5}$	$256/243$	1.053497942	$x^1 y^0$
$D_\flat$	$2^{-11} 3^7$	$2187/2048$	1.067871094	$x^0 y^1$
D	$2^{-3} 3^2$	$9/8$	1.125	$x^1 y^1$
$D_\sharp$	$2^5 3^{-3}$	$32/27$	1.185185185	$x^2 y^1$
$E_\flat$	$2^{-14} 3^9$	$19683/16384$	1.201354981	$x^1 y^2$
E	$2^{-6} 3^4$	$81/64$	1.265625	$x^2 y^2$
F	$2^2 3^{-1}$	$4/3$	1.333333333	$x^3 y^2$
$F_\sharp$	$2^{10} 3^{-6}$	$1024/729$	1.404663923	$x^4 y^2$
$G_\flat$	$2^{-9} 3^6$	$729/512$	1.423828125	$x^3 y^3$
G	$2^{-1} 3^1$	$3/2$	1.5	$x^4 y^3$
$G_\sharp$	$2^7 3^{-4}$	$128/81$	1.580246914	$x^5 y^3$
$A_\flat$	$2^{-12} 3^8$	$6561/4096$	1.601806641	$x^4 y^4$
A	$2^{-4} 3^3$	$27/16$	1.6875	$x^5 y^4$
$A_\sharp$	$2^4 3^{-2}$	$16/9$	1.777777777	$x^6 y^4$
$B_\flat$	$2^{-15} 3^{10}$	$59049/32768$	1.802032473	$x^5 y^5$
B	$2^{-7} 3^5$	$243/128$	1.898437528	$x^6 y^5$
C'	2	$2/1$	2.0	$x^7 y^5$

Table 1: **Pythagorean Tuning**

$\mathcal{C}$	$2^0 3^0 5^0$	$1/1$	1.0
$\mathcal{C}_\sharp, \mathcal{D}_\flat$	$2^4 3^{-1} 5^{-1}$	$16/15$	1.066666666
	$2^1 3^{-2} 5^1$	$10/9$	1.111111111
$\mathcal{D}$	$2^{-3} 3^2$	$9/8$	1.125
	$2^3 7^{-1}$	$8/7$	1.14285714
$\mathcal{D}_\sharp, \mathcal{E}_\flat$	$2^1 3^1 5^{-1}$	$6/5$	1.2
$\mathcal{E}$	$2^{-2} 5^1$	$5/4$	1.25
$\mathcal{F}$	$2^2 3^{-1}$	$4/3$	1.333333333
$\mathcal{F}_\sharp$	$2^{-5} 3^2 5^1$	$45/32$	1.40625
$\mathcal{G}_\flat$	$2^6 3^{-2} 5^{-1}$	$64/45$	1.422222222
$\mathcal{G}$	$2^{-1} 3^1$	$3/2$	1.5
$\mathcal{G}_\sharp, \mathcal{A}_\flat$	$2^3 5^{-1}$	$8/5$	1.6
$\mathcal{A}$	$3^{-1} 5^1$	$5/3$	1.666666666
	$2^{-2} 7^1$	$7/4$	1.75
$\mathcal{A}_\sharp, \mathcal{B}_\flat$	$2^4 3^{-2}$	$16/9$	1.777777777
	$3^2 5^{-1}$	$9/5$	1.8
$\mathcal{B}$	$2^{-3} 3^1 5^1$	$15/8$	1.875
$\mathcal{C}'$	2^1	$2/1$	2.0

Table 2: **Just Intonation**

Just Intonation Set (it involves octave, perfect fifth, perfect fourth, major third, minor third, major whole tone, minor whole tone, diatonic semitone and chromatic semitone) and also of Pythagorean Tuning. We bring applications of our theory to superparticular ratios and partial monounary algebras.

2 Preliminaries

Denote by $\mathcal{R}, \mathcal{N}, \mathcal{Z}, \mathcal{Q}$ the sets of all real, natural, integer, and rational numbers, respectively. Denote by $L = ((0, \infty), \cdot, 1, \leq)$ the usual multiplicative group with the usual order on $\mathcal{R}$. So, if $a \leq b, a, b \in (0, \infty)$, then b/a is an L-length of the interval (a, b). Since this terminology is not obvious, we borrow the usual musical terminology, i.e. we simply say that b/a is an interval. This inaccurateness does not lead to any misunderstanding because the therm "interval" is used only in this sense in this paper.

Definition 1 Let $\{x, \dots, y \in \mathcal{R}; 1 < x \leq \dots \leq y < 10/9\}$ be a set of N reals. Let $m_0, m_1, \dots, m_M; \dots; n_0, n_1, \dots, n_M$ be $M \times N$ nonnegative integers, such that

$$0 = m_0 \leq m_1 \leq \dots \leq m_M, \dots, 0 = n_0 \leq n_1 \leq \dots \leq n_M$$

and

$$m_j + \dots + n_j = j, j = 0, 1, \dots, M.$$

Then the set $S = \{x^{m_0} \dots y^{n_0}, x^{m_1} \dots y^{n_1}, \dots, x^{m_M} \dots y^{n_M}\}$ is said to be an *M-degree N-interval scale*.

A 12-degree N-interval scale S is a *12-degree N-interval* (2/1, 3/2)-*scale* [(2/1, 3/2, 5/4)-*scale*] if

$$x^{m_{12}} \cdots y^{n_{12}} = 2/1, x^{m_7} \cdots y^{n_7} = 3/2, [x^{m_4} \cdots y^{n_4} = 5/4].$$

We say that $\mathcal{S}$ is an *M-degree N-interval system ((2/1, 3/2)-system)*, [(2/1, 3/2, 5/4)-*system*] if $\mathcal{S} = \bigcup S$, where S are M-degree N-interval scales ((2/1, 3/2)-scales) [(2/1, 3/2, 5/4)-scales].

Example 1 Equal Tempered Scale is a 12-degree 1-interval scale which is neither a 12-degree N-interval (2/1, 3/2)-system nor a 12-degree N-interval (2/1, 3/2, 5/4)-system for every $N \in \mathcal{N}$ ($\sqrt[12]{2}$ is an irrational number). □

Example 2 Pythagorean Tuning is a 12-degree 2-interval (2/1, 3/2)-system, cf. Table 1, [6], [2], where these two intervals are $x =$

$2^8/3^5, y = 3^7/2^{11}$, known in the literature as *diesis* (the Pythagorean minor semitone) and *apotome* (the Pythagorean major semitone), respectively. This tone system contains no major triad (the relatively frequency ratio intervals $1:\ 5/4:\ 3/2 = 4:\ 5:\ 6$). Pythagorean Tuning is not a 12-degree N-interval (2/1, 3/2, 5/4)-system for every $N \in \mathcal{N}$. $\square$

Definition 2 We say that a matrix

$$A = \begin{pmatrix} m_{12} & n_{12} & r_{12} \\ m_7 & n_7 & r_7 \\ m_4 & n_4 & r_4 \end{pmatrix}$$

of nonnegative integers, is a (12, 7, 4)-*matrix*, if

$$0 \le m_4 \le m_7 \le m_{12}, 0 \le n_4 \le n_7 \le n_{12}, 0 \le r_4 \le r_7 \le r_{12}$$

and

$$m_j + n_j + r_j = j, j = 4, 7, 12.$$

Definition 3 If to a given (12, 7, 4)-matrix A there exists an M-degree N-interval scale S (system $\mathcal{S}$), then we say that the scale S (system $\mathcal{S}$) is *generated* by A.

In the present paper we find a (12, 7, 4)-matrix and then construct and consider the generated 12-degree 3-interval (2/1, 3/2, 5/4)-system.

3 (12, 7, 4)-matrices

Theorem 1 *Let*

$$A = \begin{pmatrix} m_{12} & n_{12} & r_{12} \\ m_7 & n_7 & r_7 \\ m_4 & n_4 & r_4 \end{pmatrix}$$

be a matrix of nonnegative integers with $\det A \neq 0$.

Then there exist unique and positive $x, y, z \in \mathcal{R}$, *such that*

$$x^{m_{12}} y^{n_{12}} z^{r_{12}} = 2/1, x^{m_7} y^{n_7} z^{r_7} = 3/2, x^{m_4} y^{n_4} z^{r_4} = 5/4, \qquad (1)$$

and the following statements are equivalent:

(a) $x, y, z \in \mathcal{Q}$, *(b)* $\det A = 1$.

The values of x, y, z are as follows:

$$x = \sqrt[\det A]{2^{D_{2,x}} 3^{D_{3,x}} 5^{D_{5,x}}}, y = \sqrt[\det A]{2^{D_{2,y}} 3^{D_{3,y}} 5^{D_{5,y}}}, z = \sqrt[\det A]{2^{D_{2,z}} 3^{D_{3,z}} 5^{D_{5,z}}},$$

where

$$D_{2,x} = \begin{array}{ccc} 1 & n_{12} & r_{12} \\ -1 & n_7 & r_7 \\ -2 & n_4 & r_4 \end{array}, D_{2,y} = \begin{array}{ccc} m_{12} & 1 & r_{12} \\ m_7 & -1 & r_7 \\ m_4 & -2 & r_4 \end{array}, D_{2,z} = \begin{array}{ccc} m_{12} & n_{12} & 1 \\ m_7 & n_7 & -1 \\ m_4 & n_4 & -2 \end{array},$$

$$D_{3,x} = \begin{array}{ccc} 0 & n_{12} & r_{12} \\ 1 & n_7 & r_7 \\ 0 & n_4 & r_4 \end{array}, D_{3,y} = \begin{array}{ccc} m_{12} & 0 & r_{12} \\ m_7 & 1 & r_7 \\ m_4 & 0 & r_4 \end{array}, D_{3,z} = \begin{array}{ccc} m_{12} & n_{12} & 0 \\ m_7 & n_7 & 1 \\ m_4 & n_4 & 0 \end{array},$$

$$D_{5,x} = \begin{array}{ccc} 0 & n_{12} & r_{12} \\ 0 & n_7 & r_7 \\ 1 & n_4 & r_4 \end{array}, D_{5,y} = \begin{array}{ccc} m_{12} & 0 & r_{12} \\ m_7 & 0 & r_7 \\ m_4 & 1 & r_4 \end{array}, D_{5,z} = \begin{array}{ccc} m_{12} & n_{12} & 0 \\ m_7 & n_7 & 0 \\ m_4 & n_4 & 1 \end{array}.$$

Proof If $x, y, z \in \mathcal{Q}$, then there exist $p \in \mathcal{P}$ and $E_{2,x}, \ldots E_{p,z} \in \mathcal{Z}$ such that

$$x = 2^{E_{2,x}} 3^{E_{3,x}} \ldots p^{E_{p,x}}, y = 2^{E_{2,y}} 3^{E_{3,y}} \ldots p^{E_{p,y}}, z = 2^{E_{2,z}} 3^{E_{3,z}} \ldots p^{E_{p,z}}, \tag{2}$$

Combining (1) and (2),

$$\begin{pmatrix} E_{2,x} & E_{2,y} & E_{2,z} \\ E_{3,x} & E_{3,y} & E_{3,z} \\ E_{5,x} & E_{5,y} & E_{5,z} \\ E_{7,x} & E_{7,y} & E_{7,z} \\ & \cdots & \\ E_{p,x} & E_{p,y} & E_{p,z} \end{pmatrix} \begin{pmatrix} m_{12} & m_7 & m_4 \\ n_{12} & n_7 & n_4 \\ r_{12} & r_7 & r_4 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ & \cdots & \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\det A = 1$,

$$\begin{pmatrix} E_{2,x} & E_{2,y} & E_{2,z} \\ E_{3,x} & E_{3,y} & E_{3,z} \\ E_{5,x} & E_{5,y} & E_{5,z} \\ E_{7x} & E_{7,y} & E_{7,z} \\ & \cdots & \\ E_{p,x} & E_{p,y} & E_{p,z} \end{pmatrix} = \begin{pmatrix} D_{2,x} & D_{2,y} & D_{2,z} \\ D_{3,x} & D_{3,y} & D_{3,z} \\ D_{5,x} & D_{5,y} & D_{5,z} \\ 0 & 0 & 0 \\ & \cdots & \\ 0 & 0 & 0 \end{pmatrix}.$$

□

Theorem 2 *No 12-degree 1- or 2- rational interval system is generated by any (12, 7, 4)-matrix.*

Proof The analysis of all (12, 7, 4)-matrices with $\det A = 1$ contains no case $D_{2,x} = D_{3,x} = D_{5,x} = 0$. So, $D_{2,x}^2 + D_{3,x}^2 + D_{5,x}^2 > 0$. Analogously, $D_{2,y}^2 + D_{3,y}^2 + D_{5,y}^2 > 0, D_{2,z}^2 + D_{3,z}^2 + D_{5,z}^2 > 0$. The assertion for 1-interval system is trivial. $\square$

Corollary 1 *Equal Temperament and Pythagorean Tuning are not generated by any (12, 7, 4)-matrix.*

Theorem 3 *Let A be a (12, 7, 4)-matrix. Then*

$$D_{2,x}^2 + D_{2,y}^2 + D_{2,z}^2 > 0, D_{3,x}^2 + D_{3,y}^2 + D_{3,z}^2 > 0, D_{5,x}^2 + D_{5,y}^2 + D_{5,z}^2 > 0,$$

where $D_{2,x}, D_{2,y}, D_{2,z}, D_{3,x}, D_{3,y}, D_{3,z}, D_{5,x}, D_{5,y}, D_{5,z}$ are as in Theorem 1.

Proof At first, note that a (12, 7, 4)-matrix exists, e.g. the matrix A_* in Theorem 5. If $D_{5,x}^2 + D_{5,y}^2 + D_{5,z}^2 = 0$, then there exist no m_4, n_4, r_4 in A such that $x^{m_4} y^{n_4} z^{r_4} = 5/4$. A contradiction. Similarly for the numbers 2 and 3. $\square$

Theorem 4 *Let A be a (12, 7, 4)-matrix with* $\det A = 1$. *Then* $x \neq y \neq z \neq x$.

Proof If $1 < x = y = z$, then $x \notin \mathcal{Q}$. A contradiction.

Suppose $x = y \neq z$ (the cases $y = z \neq x$, $x = z \neq y$ are symmetric). By Definition 1 and 2,

$$x^{4-r_4} z^{r_4} = 5/4, x^{7-r_7} z^{r_7} = 3/2, x^{12-r_{12}} z^{r_{12}} = 2/1. \tag{3}$$

By Theorem 1, $x = 2^\alpha 3^\beta 5^\gamma, z = 2^\delta 3^\epsilon 5^\theta$, for some $\alpha, \beta, \gamma, \delta, \epsilon, \theta \in \mathcal{Z}$. Then (3) implies

$$\alpha(12 - r_{12}) + r_{12}\delta = 1, \beta(12 - r_{12}) + r_{12}\epsilon = 0, \gamma(12 - r_{12}) + r_{12}\theta = 0, \tag{4}$$

$$\alpha(7 - r_7) + r_7\delta = -1, \beta(7 - r_7) + r_7\epsilon = 1, \gamma(7 - r_7) + r_7\theta = 0, \tag{5}$$

$$\alpha(4 - r_4) + r_4\delta = -2, \beta(4 - r_4) + r_4\epsilon = 0, \gamma(4 - r_4) + r_4\theta = 1. \tag{6}$$

If $\epsilon = \beta$, then (5) implies $\beta = 1/7 \notin \mathcal{Z}$. If $\gamma = \theta$, then (6) implies $\gamma = 1/4 \notin \mathcal{Z}$. So, $\epsilon \neq \beta, \theta \neq \gamma$. Then (4), and (6) imply

$$r_{12} = \frac{-12\beta}{\epsilon - \beta} = \frac{-12\gamma}{\theta - \gamma}, r_4 = \frac{-4\beta}{\epsilon - \beta} = \frac{1 - 4\gamma}{\theta - \gamma}. \tag{7}$$

If $\beta \neq 0$, then (7) implies

$$\frac{-12\beta}{-4\beta} = \frac{-12\gamma}{1-4\gamma},$$

which implies $0 = 1$. If $\beta = 0$ then (7) implies $0 = \gamma = 1/4$. A contradiction. $\square$

Corollary 2 *If S ($\mathcal{S}$) is a 12-degree 3-interval (2/1, 3/2, 5/4)-scale (system) with $x, y, z \in \mathcal{Q}$, then we can redenote x, y, z, such that $1 < x < y < z < 10/9$.*

The analysis of all (12,7,4)-matrices A with $\det A = 1$ yields the following surprising statement.

Theorem 5 *According to the symmetry,*

$$A_* = \begin{pmatrix} 5 & 4 & 3 \\ 3 & 2 & 2 \\ 2 & 1 & 1 \end{pmatrix},$$

is the unique solution of the Diophantine equation

$$\begin{vmatrix} m_{12} & n_{12} & r_{12} \\ m_7 & n_7 & r_7 \\ m_4 & n_4 & r_4 \end{vmatrix} = 1, \begin{cases} 0 & < m_4 & < m_7 & < m_{12} \\ 0 & < n_4 & < n_7 & < n_{12} \\ 0 & < r_4 & < r_7 & < r_{12} \end{cases}, \begin{cases} & m_i + n_i + r_i = i, \\ & i = 4, 7, 12. \end{cases}$$

Corollary 3 *By Corollary 2, let $1 < x < y < z < 10/9$. By Theorem 1, for A_* we have:*

$$\begin{pmatrix} D_{2,x} & D_{2,y} & D_{2,z} \\ D_{3,x} & D_{3,y} & D_{3,z} \\ D_{5,x} & D_{5,y} & D_{5,z} \end{pmatrix} = \begin{pmatrix} -3 & 4 & 0 \\ -1 & -1 & 3 \\ 2 & -1 & -2 \end{pmatrix},$$

and denote by

$$X_* = (x, y, z) = (25/24, 16/15, 27/25) = (1.041666..., 1.0666..., 1, 08).$$

4 Construction of 12-degree scales generated by A_*

Theorem 6 *Let A be a (12, 7, 4)-matrix and corresponding $x, y, z \in \mathcal{R}$ as in Theorem 1 (not necessary rational).*

Put $m_2^* = 2m_7 - m_{12}, n_2^* = 2n_7 - n_{12}, r_2^* = 2r_7 - r_{12}, m_5^* = m_{12} - m_7, n_5^* = n_{12} - n_7, r_5^* = r_{12} - r_7, m_9^* = m_{12} - m_7 + m_4, n_9^* = n_{12} - n_7 + n_4, r_9^* = r_{12} - r_7 + r_4, m_{11}^* = m_7 + m_4, n_{11}^* = n_7 + n_4, r_{11}^* = r_7 + r_4$.

Put $c = x^0y^0z^0, d = x^{m_2^*}y^{n_2^*}z^{r_2^*}, e = x^{m_4}y^{n_4}z^{r_4}, f = x^{m_5^*}y^{n_5^*}z^{r_5^*}, g = x^{m_7}y^{n_7}z^{r_7}, a = x^{m_9^*}y^{n_9^*}z^{r_9^*}, b = x^{m_{11}^*}y^{n_{11}^*}z^{r_{11}^*}, c' = x^{m_{12}}y^{n_{12}}z^{r_{12}}, d' = 2d$.

Then

1^0 $c : e : g = g : b : d' = f : a : c' = 1 : 5/4 : 3/2$,

2^0 $m_2^* + n_2^* + r_2^* = 2, m_5^* + n_5^* + r_5^* = 5, m_9^* + n_9^* + r_9^* = 9, m_{11}^* + n_{11}^* + r_{11}^* = 11$.

Proof 1^0 We have: $1 : (5/4) : (3/2) = c : e : g = 3/2 : ((3/2)(5/4)) : (3/2)^2 = g : b : d' = ((2)(2/3)) : ((2)(2/3)(5/4)) : 2 = f : a : c'$.

2^0 For 5, $(m_{12} - m_7) + (n_{12} - n_7) + (r_{12} - r_7) = (m_{12} + n_{12} + r_{12}) - (m_7 + n_7 + r_7) = 12 - 7 = 5$, analogously for 2, 9, 11. □

Corollary 4 *If*

$$\begin{array}{llllllll}
0 = m_0 & \le m_2^* & \le m_4 & \le m_5^* & \le m_7 & \le m_9^* & \le m_{11}^* & \le m_{12}, \\
0 = n_0 & \le n_2^* & \le n_4 & \le n_5^* & \le n_7 & \le n_9^* & \le n_{11}^* & \le n_{12}, \\
0 = r_0 & \le r_2^* & \le r_4 & \le r_5^* & \le r_7 & \le r_9^* & \le r_{11}^* & \le r_{12},
\end{array}$$

then numbers d, f, a, b *can be taken as the 3rd, 6th, 10th, and 12th coordinates of the 12-degree scale vector and put* $m_2^* = m_2, n_2^* = n_2, r_2^* = r_2, m_5^* = m_5, n_5^* = n_5, r_5^* = r_5, m_9^* = m_9, n_9^* = n_9, r_9^* = r_9, m_{11}^* = m_{11}, n_{11}^* = n_{11}, r_{11}^* = r_{11}$.

Theorem 7 *Let* $(x, yz) = (25/24, 16/15, 27/25)$. *Let* $c = x^0y^0z^0, c_\sharp = x^1y^0z^0, d_\flat = x^0y^0z^1, d = x^1y^0z^1, d_\sharp = x^2y^0z^1, e_\flat = x^1y^1z^1, e = x^2y^1z^1, f = x^2y^2z^1, f_\sharp = x^2y^2z^2, g_\flat = x^3y^2z^1, g = x^3y^2z^2, g_\sharp = x^4y^2z^2, a_\flat = x^3y^3z^2, a = x^4y^3z^2, a_\sharp = x^5y^3z^2, b_\flat = x^4y^3z^3, b = x^5y^3z^3, c' = x^5y^4z^3, d' = 2d$.

Then the all 12-degree scales generated by A_* *satisfying the condition* $c : e : g = g : b : d' = f : a : c' = 1 : 5/4 : 3/2$ *are the next:* $(c, i, d, j, e, f, k, g, l, a, m, b, c')$, *where* $i = c_\sharp, d_\flat; j = d_\sharp, e_\flat; k = f_\sharp, g_\flat; l = g_\sharp, a_\flat; m = a_\sharp, b_\flat$.

Proof Combine Corollary 3, Theorem 6, and Corollary 4. It is easy to verify, cf. Table 3, that $i = c_\sharp, d_\flat; j = d_\sharp, e_\flat; k = f_\sharp, g_\flat; l =$

$g_\sharp, a_\flat; m = a_\sharp, b_\flat$, are the all possibilities how to complete $\{c, d, e, f, g, a, b, c'\}$ to the 12-degree scales. $\Box$

Corollary 5 *The tuning* $\mathcal{S}_* = \{c, c_\sharp, d_\flat, d, d_\sharp, e_\flat, e, f, f_\sharp, g_\flat, g, g_\sharp, a_\flat, a, a_\sharp, b_\flat, b, c'\}$, *cf. Table 3, is a 17-valued 12-degree 3-interval (2/1, 3/2, 5/4)-system.*

We see that the structure of $\mathcal{S}_*$ is similar to Pythagorean Tuning.

5 Application to superparticular ratios in music

In this section we show that the system $\mathcal{S}_*$ is very near also to Just Intonation.

The only pairs of naturals $(n+1, n)$, for which $(n+1)$ and $n, n \in \mathcal{N}$, are divisible only by 2,3, or 5, are

$$(2,1),(3,2),(4,3),(5,4),(6,5),(9,8),(10,9),(16,15),(25,24),(81,80).$$

The following *superparticular ratios*, cf. [1],

$$2/1, 3/2, 4/3, 5/4, 6/5, 9/8, 10/9, 16/15, 25/24, 81/80$$

account for common music intervals (they denominate the relative acoustic frequency or, inversely, the length of the pipe or the string) and correspond to *octave* $(= x^5y^4z^3)$, *perfect fifth* $(= x^3y^2z^2)$, *perfect fourth* $(= x^2y^2z^1)$, *major third* $(= x^2y^1z^1)$, *minor third* $(= xyz)$, *major whole tone* $(= xz)$, *minor whole tone* $(= xy)$, *diatonic semitone* $(= y)$, *chromatic semitone* $(= x)$, and *comma of Dydimus* $(= y^{-1}z)$, respectively, where $(x, y, z) = (25/24, 16/15, 27/25)$, cf. Corollary 3.

In the next section we consider some partial monounary algebras on the system $\mathcal{S}_*$ (representing corresponding transpositions in music).

6 Application to partial monounary algebras

Let X be a nonempty set and $Y \subset X$. Let $\Omega : Y \to X$ be a mapping of Y into X. Then the couple (X, Ω) is said to be a *partial monounary algebra*. More about this notion we can find e.g. in [3].

Consider the partial functions Ω defined on some subsets of the set $\mathcal{S}_* = \{c, c_\sharp, d_\flat, d, d_\sharp, e_\flat, e, f, f_\sharp, g_\flat, g, g_\sharp, a_\flat, a, a_\sharp, b_\flat, b, c'\} \subset [1, 2]$ as follows:

for some $u \in \mathcal{S}_*$, $\Omega_\omega(u) = u \cdot \omega$ if $1 \leq u \cdot \omega < 2$ or $\Omega_\omega(u) = u\omega/2$ otherwise, where

$\omega = \alpha_{1,11}, \beta_{1,11}, \gamma_{1,11}$; $\alpha_{2,10}, \beta_{2,10}, \gamma_{2,10}, \delta_{2,10}, \epsilon_{2,10}$; $\alpha_{3,9}, \beta_{3,9}, \gamma_{3,9}, \delta_{3,9}, \epsilon_{3,9}, \zeta_{3,9}$; $\alpha_{4,8}, \beta_{4,8}, \gamma_{4,8}, \delta_{4,8}, \epsilon_{4,8}$; $\alpha_{5,7}, \beta_{5,7}, \gamma_{5,7}, \delta_{5,7}$; $\alpha_{6,6}, \beta_{6,6}, \gamma_{6,6}, \delta_{6,6}$, respectively, where

$\alpha_{1,11} = x$, $\beta_{1,11} = y$, $\gamma_{1,11} = z$;

$\alpha_{2,10} = xz$, $\beta_{2,10} = xy$, $\gamma_{2,10} = x^2$, $\delta_{2,10} = yz, \epsilon_{2,10} = y^2$;

$\alpha_{3,9} = x^1y^1z^1, \beta_{3,9} = x^2y^1z^0, \gamma_{3,9} = x^1y^2z^0,\ \delta_{3,9} = x^2y^0z^1,\ \epsilon_{3,9} = x^0y^2z^1, \zeta_{3,9} = x^0y^1z^2$;

$\alpha_{4,8} = x^2y^1z^1, \beta_{4,8} = x^1y^2z^1, \gamma_{4,8} = x^2y^0z^2, \delta_{4,8} = x^2y^2z^0, \epsilon_{4,8} = x^3y^1z^0$;

$\alpha_{5,7} = x^3y^2z^2, \beta_{5,7} = x^3y^3z^1, \gamma_{5,7} = x^2y^3z^2, \delta_{5,7} = x^4y^2z^1$;

$\alpha_{6,6} = x^2y^2z^2, \beta_{6,6} = x^3y^2z^1, \gamma_{6,6} = x^2y^3z^1, \delta_{6,6} = x^3y^1z^2$;

where $(x, y, z) = (25/24, 16/15, 27/25)$.

We define the following partial monounary algebras:

$(\mathcal{S}_*, \Omega_{\alpha_{1,11}})$, $(\mathcal{S}_*, \Omega_{\beta_{1,11}})$, $(\mathcal{S}_*, \Omega_{\gamma_{1,11}})$;

$(\mathcal{S}_*, \Omega_{\alpha_{2,10}})$, $(\mathcal{S}_*, \Omega_{\beta_{2,10}})$, $(\mathcal{S}_*, \Omega_{\gamma_{2,10}})$, $(\mathcal{S}_*, \Omega_{\delta_{2,10}})$, $(\mathcal{S}_*, \Omega_{\epsilon_{2,10}})$;

$(\mathcal{S}_*, \Omega_{\alpha_{3,9}})$, $(\mathcal{S}_*, \Omega_{\beta_{3,9}})$, $(\mathcal{S}_*, \Omega_{\gamma_{3,9}})$, $(\mathcal{S}_*, \Omega_{\delta_{3,9}})$, $(\mathcal{S}_*, \Omega_{\epsilon_{3,9}})$, $(\mathcal{S}_*, \Omega_{\zeta_{3,9}})$;

$(\mathcal{S}_*, \Omega_{\alpha_{4,8}})$, $(\mathcal{S}_*, \Omega_{\beta_{4,8}})$, $(\mathcal{S}_*, \Omega_{\gamma_{4,8}})$, $(\mathcal{S}_*, \Omega_{\delta_{4,8}})$, $(\mathcal{S}_*, \Omega_{\epsilon_{4,8}})$;

$(\mathcal{S}_*, \Omega_{\alpha_{5,7}})$, $(\mathcal{S}_*, \Omega_{\beta_{5,7}})$, $(\mathcal{S}_*, \Omega_{\gamma_{5,7}})$, $(\mathcal{S}_*, \Omega_{\delta_{5,7}})$;

$(\mathcal{S}_*, \Omega_{\alpha_{6,6}})$, $(\mathcal{S}_*, \Omega_{\beta_{6,6}})$, $(\mathcal{S}_*, \Omega_{\gamma_{6,6}})$, $(\mathcal{S}_*, \Omega_{\delta_{6,6}})$,

cf. Figures 1, 2, 3, 4, 5, and 6.

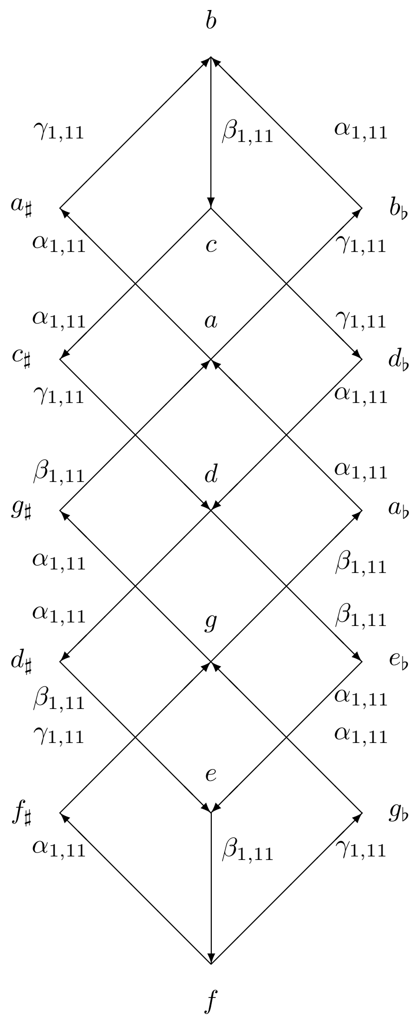

Figure 1: **The minor seconds (the major sevenths)**

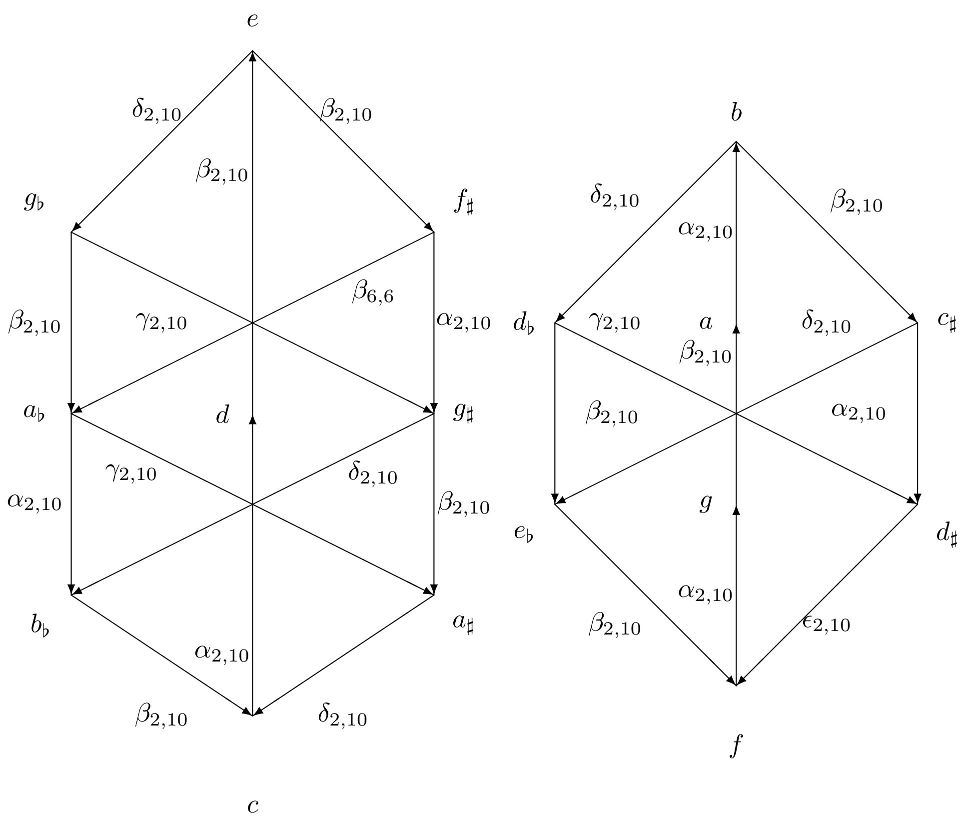

Figure 2: **The major seconds (the minor sevenths)**

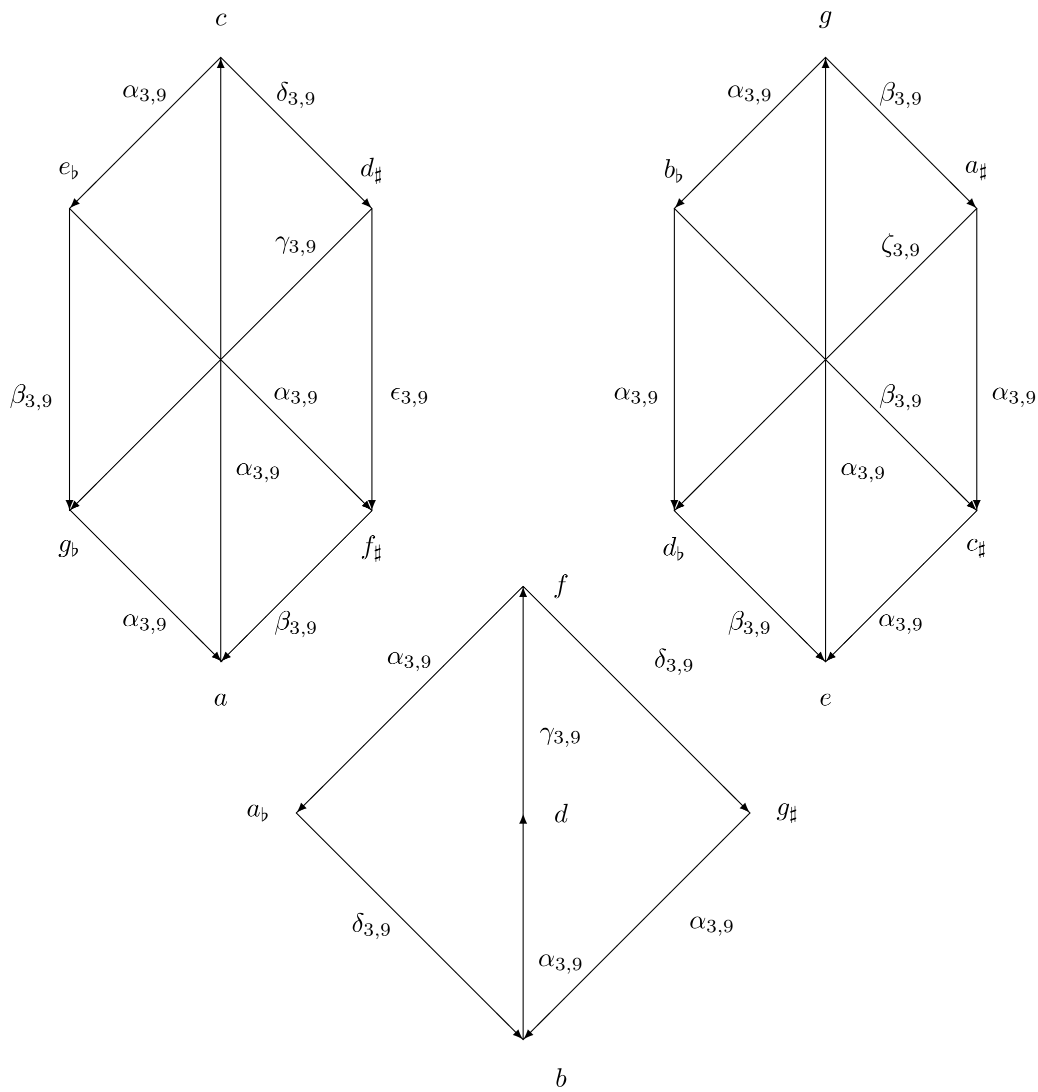

Figure 3: **The minor thirds (the major sixths)**

If we consider the inverse partial functions Ω_ω^{-1} (they exist; all arrows switch the direction in Figures 1, 2, 3, 4, 5, 6), then Figures 1, 2, 3, 4, 5, 6 define also the partial monounary algebras
$(\mathcal{S}_*, \Omega^{-1}_{\alpha_{6,6}})$, $(\mathcal{S}_*, \Omega^{-1}_{\beta_{6,6}})$, $(\mathcal{S}_*, \Omega^{-1}_{\gamma_{6,6}})$, $(\mathcal{S}_*, \Omega^{-1}_{\delta_{6,6}})$;
$(\mathcal{S}_*, \Omega^{-1}_{\alpha_{5,7}})$, $(\mathcal{S}_*, \Omega^{-1}_{\beta_{5,7}})$, $(\mathcal{S}_*, \omega^{-1}_{\gamma_{5,7}})$, $(\mathcal{S}_*, \Omega^{-1}_{\delta_{5,7}})$;
$(\mathcal{S}_*, \Omega^{-1}_{\alpha_{4,8}})$, $(\mathcal{S}_*, \Omega^{-1}_{\beta_{4,8}})$, $(\mathcal{S}_*, \Omega^{-1}_{\gamma_{4,8}})$, $(\mathcal{S}_*, \Omega^{-1}_{\delta_{4,8}})$, $(\mathcal{S}_*, \Omega^{-1}_{\epsilon_{4,8}})$;
$(\mathcal{S}_*, \Omega^{-1}_{\alpha_{3,9}})$, $(\mathcal{S}_*, \Omega^{-1}_{\beta_{3,9}})$, $(\mathcal{S}_*, \Omega^{-1}_{\gamma_{3,9}})$, $(\mathcal{S}_*, \Omega^{-1}_{\delta_{3,9}})$, $(\mathcal{S}_*, \Omega^{-1}_{\epsilon_{3,9}})$, $(\mathcal{S}_*, \Omega^{-1}_{\zeta_{3,9}})$;
$(\mathcal{S}_*, \Omega^{-1}_{\alpha_{2,10}})$, $(\mathcal{S}_*, \Omega^{-1}_{\beta_{2,10}})$, $(\mathcal{S}_*, \Omega^{-1}_{\gamma_{2,10}})$, $(\mathcal{S}_*, \Omega^{-1}_{\delta_{2,10}})$, $(\mathcal{S}_*, \Omega^{-1}_{\epsilon_{2,10}})$;
$(\mathcal{S}_*, \Omega^{-1}_{\alpha_{1,11}})$, $(\mathcal{S}_*, \Omega^{-1}_{\beta_{1,11}})$, $(\mathcal{S}_*, \omega^{-1}_{\gamma_{1,11}})$.
The proof of the following theorem can be obtained easy and omitted.

Theorem 8 *The following partial monounary algebras are isomorphic:*
(i) $(\mathcal{S}_*, \Omega_{\alpha_{5,7}}), (\mathcal{S}_*, \Omega^{-1}_{\alpha_{5,7}}), (\mathcal{S}_*, \Omega_{\alpha_{4,8}}), (\mathcal{S}_*, \Omega^{-1}_{\alpha_{4,8}})$,
(ii) $(\mathcal{S}_*, \Omega_\omega)$ *and* $(\mathcal{S}_*, \Omega_\omega^{-1})$,
where $\omega = \alpha_{1,11}, \beta_{1,11}, \gamma_{1,11}$; $\alpha_{2,10}, \beta_{2,10}, \gamma_{2,10}, \delta_{2,10}, \epsilon_{2,10}$; $\alpha_{3,9}, \beta_{3,9}, \gamma_{3,9}, \delta_{3,9}, \epsilon_{3,9}, \zeta_{3,9}$; $\alpha_{4,8}, \beta_{4,8}, \gamma_{4,8}, \delta_{4,8}, \epsilon_{4,8}$; $\alpha_{5,7}, \beta_{5,7}, \gamma_{5,7}, \delta_{5,7}$; $\alpha_{6,6}, \beta_{6,6}, \gamma_{6,6}, \delta_{6,6}$.

Remark 1 $(\mathcal{S}_*, \Omega_{\alpha_{1,11}}$, $(\mathcal{S}_*, \Omega_{\beta_{1,11}})$, $(\mathcal{S}_*, \Omega_{\gamma_{1,11}})$ *are partial monounary algebras of the minor seconds (the chromatic (25/24), diatonic (16/15) semitones and the complement of the chromatic semitone to the major whole tone (27/25 = 9/8 : 25/24), respectively),*
$(\mathcal{S}_*, \Omega_{\alpha_{2,10}}$, $(\mathcal{S}_*, \Omega_{\beta_{2,10}})$ *are partial monounary algebras of the major seconds (the major (9/8) and minor (10/9) whole tones, respectively),*
$(\mathcal{S}_*, \Omega_{\alpha_{3,9}})$ *is a partial monounary algebra of the minor thirds (6/5),*
$(\mathcal{S}_*, \Omega_{\alpha_{4,8}})$ *is a partial monounary algebra of the major thirds (5/4),*
$(\mathcal{S}_*, \Omega^{-1}_{\alpha_{5,7}})$ *is a partial monounary algebra of the perfect fourths (4/3),*
$(\mathcal{S}_*, \Omega_{\alpha_{6,6}})$ *is a partial monounary algebra of the tritones - the augmented fourths (25/18),*
$(\mathcal{S}_*, \Omega^{-1}_{\alpha_{6,6}})$ *is a partial monounary algebra of the tritones - the diminished fifths (36/25),*
$(\mathcal{S}_*, \Omega_{\alpha_{5,7}})$ *is a partial monounary algebra of the perfect fifths (3/2),*

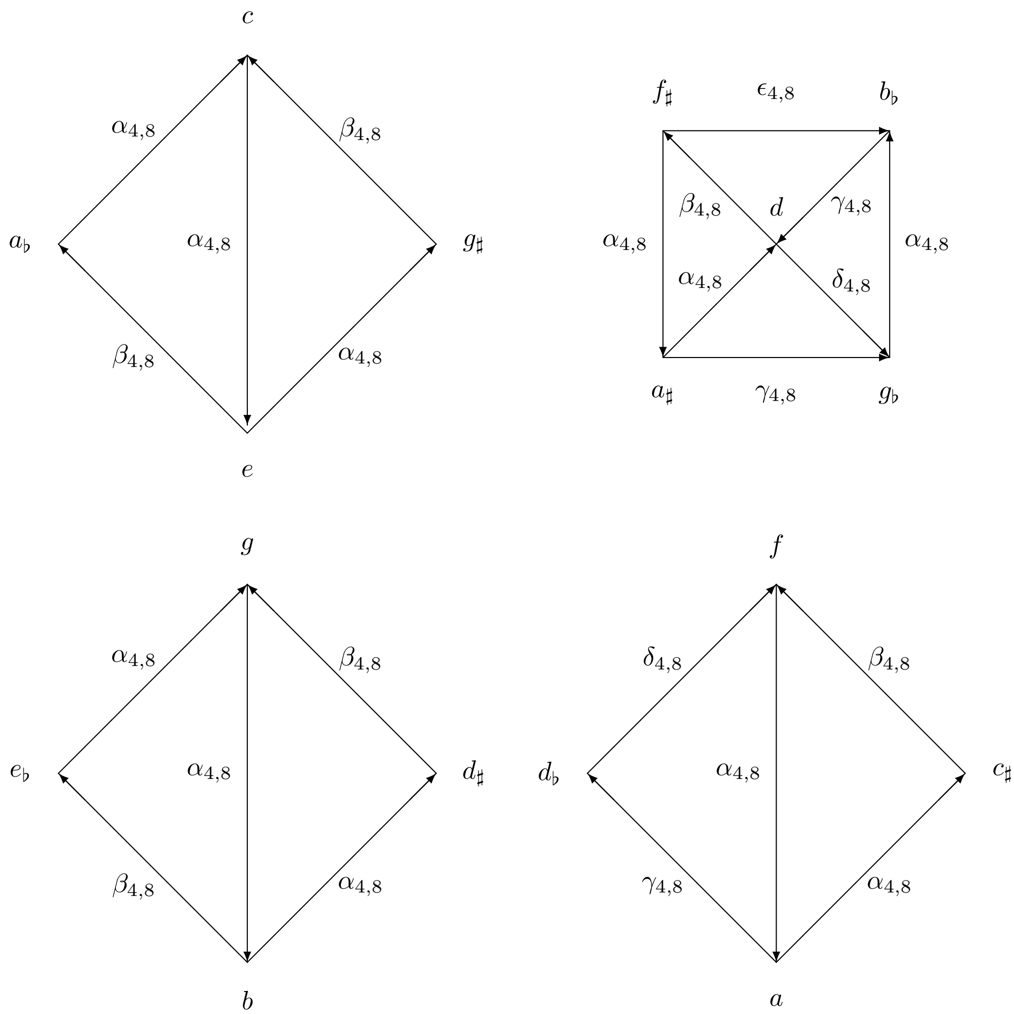

Figure 4: **The major thirds (the minor sixths)**

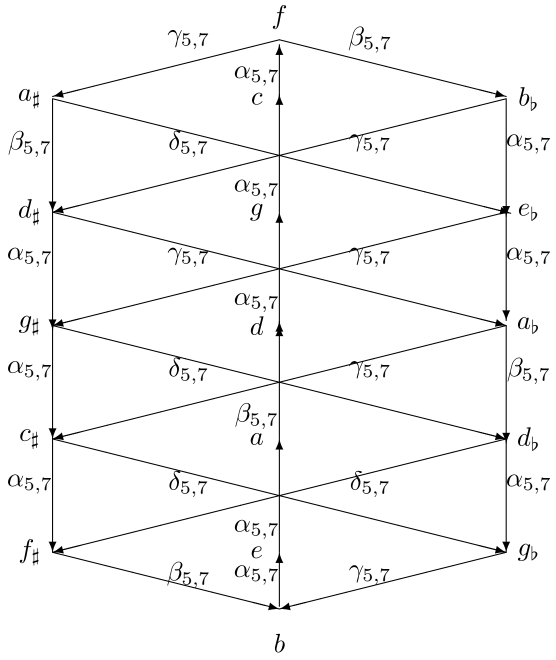

Figure 5: **The fourths (the fifths)**

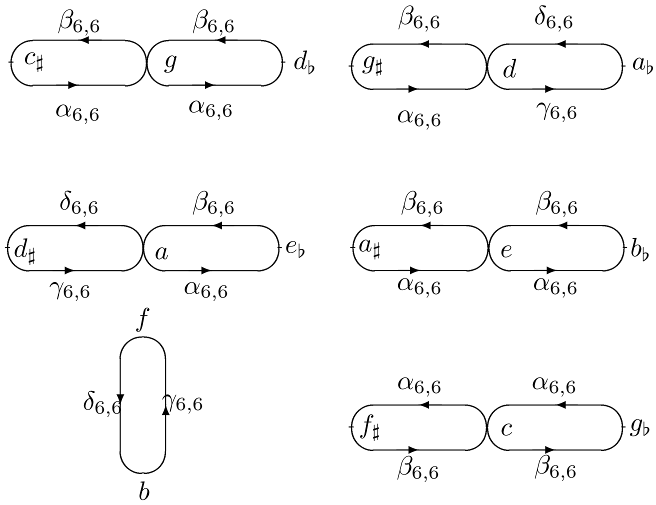

Figure 6: **The tritones - the augmented fourths (the diminished fifths)**

$(\mathcal{S}_*, \Omega^{-1}_{\alpha_{4,8}})$ *is a partial monounary algebra of the minor sixths (8/5),*

$(\mathcal{S}_*, \Omega^{-1}_{\alpha_{3,9}})$ *is a partial monounary algebra of the major sixths (5/3),*

$(\mathcal{S}_*, \Omega^{-1}_{\alpha_{2,10}})$*,* $(\mathcal{S}_*, \Omega^{-1}_{\beta_{2,10}})$ *are some partial monounary algebras of the minor sevenths,*

$(\mathcal{S}_*, \Omega_{\alpha^{-1}_{1,11}}$*,* $(\mathcal{S}_*, \Omega^{-1}_{\beta_{1,11}})$*,* $(\mathcal{S}_*, \Omega^{-1}_{\gamma_{1,11}})$ *are partial monounary algebras of the major sevenths.*

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c	$2^0 3^0 5^0$	1/1	1.0	$x^0 y^0 z^0$
$c_\sharp$	$2^{-3} 3^{-1} 5^2$	25/24	1.041666666	$x^1 y^0 z^0$
$d_\flat$	$2^0 3^3 5^{-2}$	27/25	1.08	$x^0 y^0 z^1$
d	$2^{-3} 3^2 5^0$	9/8	1.125	$x^1 y^0 z^1$
$d_\sharp$	$2^{-6} 3^1 5^2$	75/64	1.171875	$x^2 y^0 z^1$
$e_\flat$	$2^1 3^1 5^{-1}$	6/5	1.2	$x^1 y^1 z^1$
e	$2^{-2} 3^0 5^1$	5/4	1.25	$x^2 y^1 z^1$
f	$2^2 3^{-1} 5^0$	4/3	1.333333333	$x^2 y^2 z^1$
$f_\sharp$	$2^{-1} 3^{-2} 5^2$	25/18	1.388888888	$x^3 y^2 z^1$
$g_\flat$	$2^2 3^2 5^{-2}$	36/25	1.44	$x^2 y^2 z^2$
g	$2^{-1} 3^1 5^0$	3/2	1.5	$x^3 y^2 z^2$
$g_\sharp$	$2^{-4} 3^0 5^2$	25/16	1.5625	$x^4 y^2 z^2$
$a_\flat$	$2^3 3^0 5^{-1}$	8/5	1.6	$x^3 y^3 z^2$
a	$2^0 3^{-1} 5^1$	5/3	1.666666666	$x^4 y^3 z^2$
$a_\sharp$	$2^{-3} 3^{-2} 5^3$	125/72	1.736111111	$x^5 y^3 z^2$
$b_\flat$	$2^0 3^2 5^{-1}$	9/5	1.8	$x^4 y^3 z^3$
b	$2^{-3} 3^1 5^1$	15/8	1.875	$x^5 y^3 z^3$
c'	$2^1 3^0 5^0$	2/1	2.0	$x^5 y^4 z^3$

Table 3: $x = 25/24, y = 16/15, z = 27/25$