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On fuzzy coding of information in music

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Abstract

The tuning in music is an excellent example of the fact that the human perceptive mechanism uses various systems of fuzzy coding of information. In this paper, we show a surprising connection between two, perhaps, the most used tunings during the music history: *an image of Equal Tempered Scale (12-valued) is the projection of the image of Pythagorean Tuning (17-valued) to a line in the plane.*

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1 Introduction

In the present time, some of musicologists and psycho-acousticians assert that the twelve-degree musical scales appeared not as merely artificial convention but, rather, the natural development of musical culture led to aural selection of qualitative intervals (to be more precisely, interval "zones"), each of them having a particular degree of individuality. The tuning (which should be strict and exact, otherwise it is not a tuning!) in music is an excellent example of the fact that the human perceptive mechanism uses various systems of fuzzy coding of information. Particularly, 17-tone tuning systems for the 12 qualitative different music intervals (e.g. Pythagorean Tuning and Just Intonation, [2]) are well-known. The main requirement is that the subject's perceptive system be able to encode the informational content unambiguously: this factor distinguishes a musical interval from any other pair of sounds.

Pythagorean Tuning, cf. [1], Table 0.1, was created as a sequence of numbers of the form $2^p 3^q$, where p, q are integers. This tuning was established about five hundred years B. C. and used in the Western music up to the 14th century. In the present time, this tuning is used when interpreting Gregorian chants.

Equal Tempered Scale (known already to Andreas Werckmeister − this is obvious from his book "Erweiterte und verbesserte Orgel-Probe", 1698), widely used since the appearance of the collection of compositions "Das wohltemperierte Klavier" (1721, by Johann Sebastian Bach), and almost commonly used up to present days.

In this paper, we show a surprising connection between these two, perhaps, the most used tunings during the music history: *an image of Equal Tempered Scale is the projection of the image of Pythagorean Tuning to a line in the plane.*

2 Preliminaries

The vector (the ratios of pitch frequencies relatively to the frequency of the first tone in the scale)

$$W = \{1, \sqrt[12]{2}, (\sqrt[12]{2})^2, \ldots, (\sqrt[12]{2})^{12} = 2\}$$

represents *Equal Tempered Scale* (or simply, *Equal Temperament*) in music. Analogously, *Pythagorean Tuning* is represented by the following sequence of 18 rational numbers (ordered increasingly):

$$P = \{1/1, 2^8/3^5, 3^7/2^{11}, 3^2/2^3, 2^5/3^3, 3^9/2^{14}, 3^4/2^6, 2^2/3, 2^{10}/3^6,$$
$$3^6/2^9, 3/2, 2^7/3^4, 3^8/2^{12}, 3^3/2^4, 2^4/3^2, 3^{10}/2^{15}, 3^5/2^7, 2/1\}.$$

Denote by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ the sets of all natural, integer, rational, real, and complex numbers, respectively. Equal Tempered Scale can be generalized as follows:

Definition 1 Let $\{x, \ldots, y \in \mathbb{R}; 1 < x \leq \cdots \leq y < 10/9\}$ be a set of N numbers. Let $m_0, m_1, \ldots, m_M; \ldots; n_0, n_1, \ldots, n_M$ be $M \times N$ nonnegative integers, such that

$$0 = m_0 \leq m_1 \leq \cdots \leq m_M, \ldots, 0 = n_0 \leq n_1 \leq \cdots \leq n_M$$

and

$$m_j + \cdots + n_j = j, j = 0, 1, \ldots, M.$$

Then $S = \{x^{m_0} \ldots y^{n_0}, x^{m_1} \ldots y^{n_1}, \ldots, x^{m_M} \ldots y^{n_M}\}$ is called to be a *M-ton N-interval scale*. Particularly, a 12-ton N-interval scale S is a *12-ton 2-interval* (2/1, 3/2)*-scale* if

$$x^{m_{12}} y^{n_{12}} = 2/1, x^{m_7} y^{n_7} = 3/2.$$

Clearly, Equal Tempered Scale is a 12-ton 1-interval scale which is not a 12-ton (2/1, 3/2)-scale. In this paper we deal with 12-ton 2-interval (2/1, 3/2)-scales where the both intervals $x, y \in \mathbb{Q}$ in Definition 1.

3. Pythagorean Tuning. In this section we describe the structure of Pythagorean Tuning.

Theorem 1 *The unique two rational ratio intervals in Definition 1 for 12-ton 2-interval (2/1, 3/2)-scales are*

$$x = 2^8/3^5, y = 3^7/2^{11}.$$

Proof. According to the symmetry,

$$A_2 = \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix}$$

is the unique solution of the Diophantine equation

$$\det \begin{pmatrix} m_{12} & n_{12} \\ m_7 & n_7 \end{pmatrix} = 1, \begin{cases} 0 \leq m_7 \leq m_{12} \\ 0 \leq n_7 \leq n_{12} \end{cases}, \begin{cases} m_{12} + n_{12} = 12, \\ m_7 + n_7 = 7, \end{cases}$$

where $m_{12}, n_{12}, m_7, n_7 \in \mathbb{Z}$.

Indeed, by assumption, $\det A_2 = m_{12}n_7 - m_7 n_{12} = 1, m_{12} = 12 - n_{12}, m_7 = 7 - n_7$. So, $12n_7 - n_{12}n_7 - 7n_{12} + n_{12}n_7 = 1$.

From the possibilities $n_{12} \in \{0,1,2,3,4,5,6,7,8,9,10,11,12\}$ only $n_{12}=5$ satisfies the equality $12n_7 - 7n_{12} = 1$ in $\mathbb{Z}$. Consequently, $n_7 = 3, m_{12} = 7, m_7 = 4$.

It is easy to see that x, y are of the form $2^\alpha 3^\beta, \alpha, \beta \in \mathbb{Z}$. The concrete values of x, y we can obtain from Definition 1, cf. [2]. $\square$

For $x = 2^8/3^5, y = 3^7/2^{11}$, denote by $c = 1, c_\sharp = x, d_\flat = y, d = xy, d_\sharp = x^2y, e_\flat = xy^2, e = x^2y^2, f = x^3y^2, f_\sharp = x^4y^2, g_\flat = x^3y^3, g = x^4y^3, g_\sharp = x^5y^3, a_\flat = x^4y^4, a = x^5y^4, a_\sharp = x^6y^5, b_\flat = x^5y^5, b = x^6y^5 c' = x^7y^5$.

The proof of the following theorem is easy and omitted.

Theorem 2 *Let*

$$P_{r,s,t,u,v} = \{c, r, d, s, e, f, t, g, u, a, v, b, c'\},$$

where $r = c_\sharp, d_\flat; s = d_\sharp, e_\flat; t = f_\sharp, g_\flat; u = g_\sharp, a_\flat; v = a_\sharp, b_\flat$. *Then*

(1) $P_{r,s,t,u,v}$ *are 12-ton 2-interval (2/1, 3/2)-scales,*

(2)

$$P = \bigcup_{r,s,t,u,v} P_{r,s,t,u,v}.$$

Theorem 3 *For* $x = 2^8/3^5, y = 3^7/2^{11}$,

$$P = \{(x^4y^3)^k (x^4y^2) (\mathrm{mod} \quad x^7y^5); k = 0, 1, 2, \ldots, 16\}.$$

Proof. $(x^4y^2) \cdot (x^4y^3)^0 = f_\sharp, (x^4y^2)\frac{x^4y^3}{x^7y^5} = x = c_\sharp, x \cdot (x^4y^3) = x^5y^3 = g_\sharp, (x^5y^3) \cdot x^4y^3x^7y^5 = x^2y = d_\sharp, (x^2y) \cdot (x^4y^3) = x^6y^4 = a_\sharp, (x^6y^4) \cdot \frac{x^4y^3}{x^7y^5} = x^3y^2 = f, (x^3y^2) \cdot \frac{x^4y^3}{x^7y^5} = 1 = c, 1 \cdot (x^4y^3) = g, (x^4y^3) \cdot \frac{x^4y^3}{x^7y^5} = xy = d, (xy) \cdot (x^4y^3) = x^5y^4 = a, (x^5y^4) \cdot \frac{x^4y^3}{x^7y^5} = x^2y^2 = e, (x^2y^2) \cdot (x^4y^3) = x^6y^5 = b, (x^6y^5) \cdot \frac{x^4y^3}{x^7y^5} = x^3y^3 = g_\flat, (x^3y^3) \cdot x^4y^3x^7y^5 = y = d_\flat, y \cdot (x^4y^3) = x^4y^4 = a_\flat, (x^4y^4) \cdot \frac{x^4y^3}{x^7y^5} = xy^2 = e_\flat, (xy^2) \cdot (x^4y^3) = x^5y^5 = b_\flat$. $\square$

Corollary 1 *Pythagorean Tuning is a geometric progression of 17 rationals with the the quotient 3/2 modulo 2.*

4. Images of Equal Temperament and Pythagorean Tuning in the plane. Let $x = 2^8/3^5, y = 3^7/2^{11}$. Denote by $\mathbb{Q}_{x,y} = \{x^\alpha y^\beta; \alpha, \beta \in \mathbb{Z}\}, \mathbb{R}_{x,y} = \{x^\alpha y^\beta; \alpha, \beta \in \mathbb{R}\}$.

Consider the following map $\theta : \mathbb{Q}_{x,y} \to \mathbb{Z}^2, \theta : x^\alpha y^\beta \to (\alpha, \beta)$, and

$$(x^\alpha y^\beta)(x^\gamma y^\delta) \to (\alpha + \gamma, \beta + \delta), (x^\alpha y^\beta)^\gamma \to (\gamma\alpha, \gamma\beta), \ \alpha, \beta, \gamma, \delta \in \mathbb{Z}.$$

Lemma 1 *The morphism θ is an isomorphism.*

Proof. It is enough to show that the map $\theta : \mathbb{Q}_{x,y} \to \mathbb{Z}^2$ is an injection (the other properties of isomorphism are trivial). Suppose that there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Z}$, such that $\alpha_1 \neq \alpha_2, \beta_1 \neq \beta_2$ implies $x^{\alpha_1}y^{\beta_1} = x^{\alpha_2}y^{\beta_2}$. Then $x^{\alpha_1-\alpha_2}y^{\beta_1-\beta_2} = 1$ and $(\alpha_1 - \alpha_2) + (\beta_1 - \beta_2)\log_x y = 0$. This is possible only in the case when $\alpha_1 = \alpha_2, \beta_1 = \beta_2$. A contradiction. Consequently, $\alpha_1 = \alpha_2, \beta_1 = \beta_2$. $\square$

Embed $\mathbb{Z}^2$ identically into the complex plane $\mathbb{C}$ (for the process of the standard enlargements $\mathbb{Z} \to \mathbb{Q} \to \mathbb{R} \to \mathbb{C}$, cf. e.g. [3]). Denote this injection by η. Denote by θ_*^{-1} the extended morphism $\theta_*^{-1} : \mathbb{C} \to \mathbb{R}_{x,y}$. So we have the following commutative diagram, cf. Fig. 1, which defines the embedding $\varphi : \mathbb{Q}_{x,y} \to \mathbb{R}_{x,y}$:

$$\begin{tikzcd} \mathbb{Q}_{x,y} \arrow[r, "\theta"] \arrow[d, hook, "\varphi"'] & \mathbb{Z}^2 \arrow[d, hook, "\eta"] \\ \mathbb{R}_{x,y} & \mathbb{C} \arrow[l, "\theta_*^{-1}"'] \end{tikzcd}$$

Fig.1. The embedding φ

In Fig. 2. we see the image of Pythagorean Tuning in the isomorphism θ.

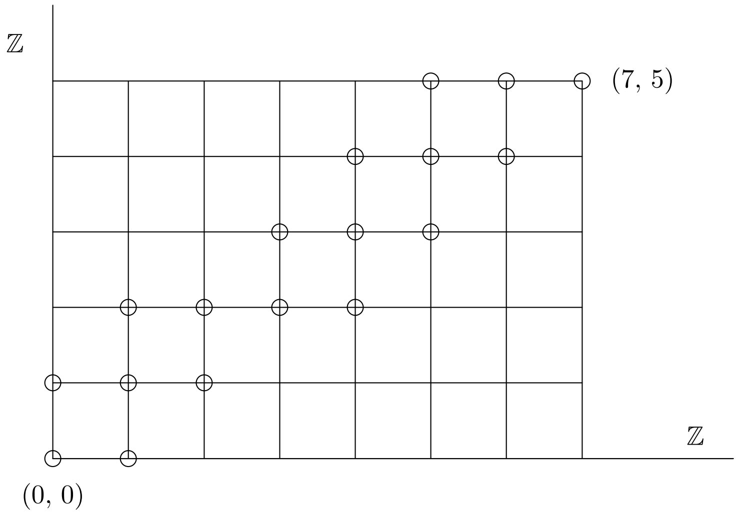

Fig.2. Image of Pythagorean Tuning in the isomorphism θ

Theorem 4 *Let p, q be two lines in the complex plane $\mathbb{C}$, such that $(0,0), (7,5) \in p$; $(12,0), (7,5) \in q$, respectively. Let $\pi : \mathbb{C} \to p$ be the projection of the plane $\mathbb{C}$ into the line p along the line q. Then*

$$W = \theta_*^{-1}(\pi(\eta(\theta(P)))).$$

Proof. See Fig. 3.

Denote by $w_0 = (0,0), w_1 = (\frac{7}{12}, \frac{5}{12}), w_2 = 2 \cdot (\frac{7}{12}, \frac{5}{12}), \ldots, w_{12} = 12 \cdot (\frac{7}{12}, \frac{5}{12})$.

We have:

$$\begin{array}{ccccc} w_0 & = & \pi(\eta(\theta(c))), & \theta_*^{-1}(w_0) = x^0 y^0 & = & 1, \\ w_1 & = & \pi(\eta(\theta(c_\sharp))) = \pi(\eta(\theta(d_\flat))), & \theta_*^{-1}(w_1) = x^{\frac{7}{12}} y^{\frac{5}{12}} = (x^7 y^5)^{\frac{1}{12}} & = & \sqrt[12]{2}, \\ w_2 & = & \pi(\eta(\theta(d))), & \theta_*^{-1}(w_2) = x^{2\cdot\frac{7}{12}} y^{2\cdot\frac{5}{12}} & = & (\sqrt[12]{2})^2, \\ & & \cdots & & & \\ w_{12} & = & \pi(\eta(\theta(c'))), & \theta_*^{-1}(w_{12}) = x^{12\cdot\frac{7}{12}} y^{12\cdot\frac{5}{12}} & = & (\sqrt[12]{2})^{12}. \end{array}$$

$\square$

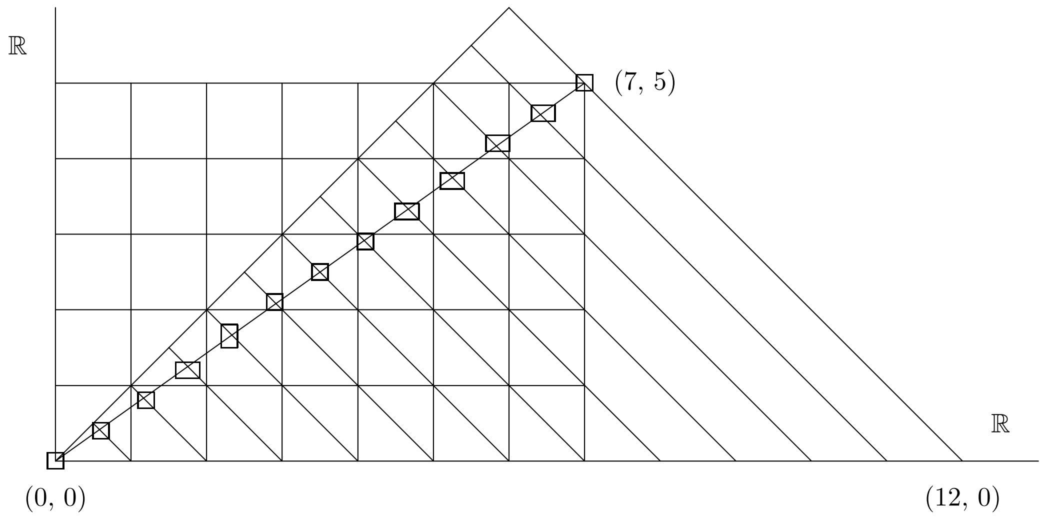

Fig. 3. Image of Equal Temperament in the plane

Corollary 2 *The image of Equal Temperament in Theorem 4 can be obtained also with the projection of the images of the whole-tone scales (the frequency ratio 9/8):* $(c, d, e, g_\flat, a_\flat, b_\flat)$ *and* $(c_\sharp, d_\sharp, f, g, a, b)$ *into the line* p *along the line* q.

References

[1] Lefebvre V.A., *A rational equation for attractive proportions*, Math. Psychology 36 (1992), 100 – 128.

[2] Thirring W., *Über vollkommene Tonsysteme*, Annalen der Physik 47 (1990), 245 – 250.

[3] Feferman S., *The number systems*, Addison – Wesley Publ. Co., Mass. – Palo Alto – London, 1963.