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Diatonic scales summary

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Abstract

We divide the set of all diatonic scales into three classes P, G, R, the intersection of which contains the major diatonic scale. The class G contains Gypsy scales, the class P – Pythagorean heptatonic, and the class R – Redfield Scale. The paper could be of interest for the music theorists, mathematicians, as well as producers of modern key music instruments, interpreters, composers, and electro-acoustical studios.
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1 Semitones

By a diatonic scale we mean usually a 7-degree music scale within the octave in which intervals are not smaller than a semitone and not greater than three semitones (= the *hiat*).

There are many different semitones in music. Each of them has its own good reason for existence (depending on the temperature of the scale). Some examples of semitones:
Pythagorean minor semitone (256/243),
Pythagorean major semitone (2187/2048),
Diatonic semitone (16/15),
Chromatic semitone (25/24),
Praetorius minor semitone ($\sqrt[4]{78125}/16$),
Praetorius major semitone ($8/\sqrt[4]{3125}$),
Co-chromatic semitone (27/25),
Co-diatonic semitone (135/128),
Equal-tempered semitone ($\sqrt[12]{2}$).

The appearance of other intervals between neighbouring tones (the whole tones and/or hiats) in diatonic scales depends on the used semitones.

Various scales (diatonic or nondiatonic) use various numbers of different semitones. Equal temperament uses one semitone. Pythagorean Tuning is constructed by two semitones, analogously Praetorius Tuning. What about diatonic scales in general? It is known that the typical diatonic scale, the major diatonic scale, is constructed e.g. by the semitones: 16/15, 25/24, and 27/25. But it is not the only possibility: the triple $(16/15, 25/24, 135/128)$ of semitones can also serve for constructing of the major diatonic scale. Thus, not greater that three semitones are needed for scale constructing of diatonic scales.

From the dimensional point of view, Equal Temperament can be imaged in the line [the reper: the octave], Pythagorean Tuning (Praetorius Tuning) in the plane [the repers: the octave and perfect fifth (the octave and major third)], and the diatonic major scale [the repers: the octave, perfect fifth, major thirds] in the Euler musical space. Note also that Just Intonation, moreover, needs the fourth dimension, the natural seventh.

2 Definition of Diatonic scale

Although every musician understands what is a diatonic scale, we have find no mathematical definition in the literature. Certainly, the reason is the "great fuzziness" of this notion in music. In this paper we bring a definition of the notion of diatonic scale from the mathematical point of view.

We fix the structure of the 7-valued major diatonic scale (intervals between tones: 9/8, 10/9, 16/15, 9/8, 10/9, 9/8, 16/15) and enlarge it to 12-degree scales such that the resulting sequence will be (a generalization of the geometrical progression) of the form:

$$\Gamma_n = X^{x_n} Y^{y_n} Z^{z_n}, \tag{1}$$

where $X, Y, Z \in (2^{1/24}; 2^{3/24})$,

$$\Gamma_0 = 1, \Gamma_{12} = 2, \tag{2}$$

and n, x_n, y_n, z_n are nonegative integers such that

$$x_n + y_n + x_n = n \tag{3}$$

and

$$0 \le x_1 \le x_2 \cdots \le x_n \cdots, \tag{4}$$

analogously for y_n, z_n.

We suppose the *octave equivalency*, i.e.

$$(\Gamma_{12i+0}, \Gamma_{12i+1}, \ldots, \Gamma_{12i+11}) = 2^i (\Gamma_0, \Gamma_1, \ldots, \Gamma_{11}) \tag{5}$$

where i is a natural number.

According to octave equivalency, consider a variation

$$\mathcal{D} = (\Gamma_{n_1}, \Gamma_{n_2}, \Gamma_{n_3}, \Gamma_{n_4}, \Gamma_{n_5}, \Gamma_{n_6}, \Gamma_{n_7}, \Gamma_{n_8}) \tag{6}$$

from the set

$$\{\Gamma_n; n = 0, 1, \ldots, 11\} \tag{7}$$

such that

$$1 \leq n_{i+1} - n_i \leq 3,\ i = 1, 2, 3, 4, 5, 6, 7. \tag{8}$$

Let $S_n = X^{x_n} Y^{y_n} Z^{z_n}, Q_n = U^{u_n} V^{v_n} T^{t_n}$ be two generalized geometrical progressions such that $S_0 = 1, S_{12} = 2, Q_0 = 1, Q_{12} = 2$, and $X, Y, Z, U, V, T \in (2^{1/24}; 2^{3/24})$. A map

$$\theta : (S_n) \to (Q_n) \tag{9}$$

is a *homomorphism of generalized geometrical progressions* if for every n non-negative integer number, $S_n = X^{x_n} Y^{y_n} Z^{z_n} \Rightarrow Q_n = U^{x_n} V^{y_n} T^{z_n}$.

By a *diatonic scale* we understand the variation $\mathcal{D}$ according to homomorphisms of generalized geometrical progressions.

The idea to explain the structure of various tone systems via the generalization of the notion of geometrical progression was successfully applied to Pythagorean Tuning (and Equal Temperament) in [1], [6], [10], and to Just Intonation in [2]. An other approach to music scales uses continued fractions, [5].

3 Three classes

For the proof of the following theorem, see [3].

Theorem 1 *According to the symmetry, all rational triples (X, Y, Z) generating generalized geometrical progressions such that (1) (2) (3) (4) with the diatonic major scale are the following:*

$$(25/24, 135/128, 16/15), \tag{10}$$

$$(256/243, 135/128, 16/15), \tag{11}$$

$$(25/24, 16/15, 27/25). \tag{12}$$

All superparticular ratios for numbers 2,3, and 5, are exactly: 2/1, 3/2, 4/3, 5/4, 6/5, 9/8, 10/9, 16/15, 25/24, and 81/80. It is easy to verify Table 1.

The following three Tables 2, 3, and 4 shows the result of enlargement of the diatonic major scale $(C, D, E, F, G, A, B, C')$ to 12-degree scales. There are 96 12-degree scales such that they are generalized geometrical progressions. From these 12-degree scales we choose diatonic scales (not considering homomorphism).

Table 1
Superparticular ratios
$X = 25/24, Y = 16/15, Z = 27/25, V = 135/128, W = 256/243$

2/1	$X^5Y^4Z^3$	$X^2V^3Y^7$	$W^2V^5Y^5$
3/2	$X^3Y^2Z^2$	XV^2Y^4	WV^3Y^3
4/3	X^2Y^2Z	XVY^3	WV^2Y^2
5/4	X^2YZ	XVY^2	WV^2Y
6/5	XYZ	VY^2	VY^2
9/8	XZ	VY	VY
10/9	XY	XY	WV
16/15	Y	Y	Y
25/24	X	X	WVY^{-1}
81/80	$Y^{-1}Z$	$X^{-1}V$	$W^{-1}Y$

Table 2
Class R
$X = 25/24, Y = 16/15, Z = 27/25$

$X^0Y^0Z^0$	1	C
$X^1Y^0Z^0$	25/24	$C_\sharp$
$X^0Y^0Z^1$	27/25	$D_\flat$
$X^1Y^0Z^1$	9/8	D
$X^2Y^0Z^1$	75/64	$D_\sharp$
$X^1Y^1Z^1$	6/5	$E_\flat$
$X^2Y^1Z^1$	5/4	E
$X^2Y^2Z^1$	4/3	F
$X^3Y^2Z^1$	25/18	$F_\sharp$
$X^2Y^2Z^2$	36/25	$G_\flat$
$X^3Y^2Z^2$	3/2	G
$X^4Y^2Z^2$	25/16	$G_\sharp$
$X^3Y^3Z^2$	8/5	$A_\flat$
$X^4Y^3Z^2$	5/3	A
$X^5Y^3Z^2$	125/72	$A_\sharp$
$X^4Y^3Z^3$	9/5	$B_\flat$
$X^5Y^3Z^3$	15/8	B
$X^5Y^4Z^3$	2	C'

Table 3
Class G
$X = 25/24, Y = 16/15, V = 135/128$

$X^0Y^0V^0$	1	C
$X^0Y^0V^1$	135/128	$C_\sharp$
$X^0Y^1V^0$	16/15	$D_\flat$
$X^0Y^1V^1$	9/8	D
$X^1Y^1V^1$	75/64	$D_\sharp$
$X^0Y^2V^1$	6/5	$E_\flat$
$X^1Y^2V^1$	5/4	E
$X^1Y^3V^1$	4/3	F
$X^1Y^3V^2$	45/32	$F_\sharp$
$X^1Y^4V^1$	64/45	$G_\flat$
$X^1Y^4V^2$	3/2	G
$X^2Y^4V^2$	25/16	$G_\sharp$
$X^1Y^5V^2$	8/5	$A_\flat$
$X^2Y^5V^2$	5/3	A
$X^2Y^5V^3$	225/128	$A_\sharp$
$X^2Y^6V^2$	16/9	$B_\flat$
$X^2Y^6V^3$	15/8	B
$X^2Y^7V^3$	2/1	C'

Table 4
Class P
$W = 256/243, Y = 16/15, V = 135/128$

$W^0Y^0V^0$	1	C
$W^0Y^0V^1$	135/128	$C_\sharp$
$W^0Y^1V^0$	16/15	$D_\flat$
$W^0Y^1V^1$	9/8	D
$W^1Y^1V^1$	32/27	$D_\sharp$
$W^0Y^1V^2$	1215/1024	$E_\flat$
$W^1Y^1V^2$	5/4	E
$W^1Y^2V^2$	4/3	F
$W^1Y^2V^3$	45/32	$F_\sharp$
$W^1Y^3V^2$	64/45	$G_\flat$
$W^1Y^3V^3$	3/2	G
$W^2Y^3V^3$	128/81	$G_\sharp$
$W^1Y^3V^4$	405/256	$A_\flat$
$W^2Y^3V^4$	5/3	A
$W^2Y^3V^5$	225/128	$A_\sharp$
$W^2Y^4V^4$	16/9	$B_\flat$
$W^2Y^4V^5$	15/8	B
$W^2Y^5V^5$	2	C'

4 Structure of the all diatonic scales set

Denote the classes od diatonic scales given by Table 2, Table 3, and Table 4, as R, G, P corresponding triplets (X, Y, Z), (X, Y, V), (W, Y, V), respectively.

Theorem 2 *The class R contains the Redfield diatonic scale.*

Proof. The Redfield diatonic scale, [8], [9], is defined by the sequence of intervals between the neighbour tones: (10/9, 9/8, 16/15, 9/8, 10/9, 9/8, 16/15). We see (Table 2) that the sequence

$$(E_\flat, F, G, A_\flat, B_\flat, C', D', E'_\flat) \tag{13}$$

satisfies the requirement, where $D' = 2D, E_\flat = 2E'_\flat$. □

Theorem 3 *The class G contains the Gypsy major and minor scales.*

Proof.

(a) The Gypsy major scale, [8], is defined by the sequence of intervals between the neighbour tones: (16/15, 9/8 · 25/24, 16/15, 9/8, 16/15, 9/8 · 25/24, 16/15). We see (Table 3) that the sequence

$$(C, D_\flat, E, F, G, A_\flat, B, C') \tag{14}$$

satisfies the requirement.

(b) The Gypsy minor scale, [8], is defined by the sequence of intervals between the neighbour tones: (9/8, 16/15, 9/8 · 25/24, 16/15, 16/15, 9/8 · 25/24, 16/15). We see (Table 3) that the sequence

$$(A, B, C', D'_\sharp, E', F', G'_\sharp, A') \tag{15}$$

satisfies the requirement, where $D'_\sharp = D_\sharp, E' = 2E, F' = 2F$. □

Theorem 4 *The class P contains the Pythagorean heptatonic.*

Proof. The Pythagorean heptatonic, [8], is defined by the sequence of intervals between the neighbour tones: (9/8, 9/8, 256/243, 9/8, 9/8, 9/8, 256/243). We see (Table 4) that the sequence

$$(D_\sharp, F, G, G_\sharp, B_\flat, C', D', D'_\sharp) \tag{16}$$

satisfies the requirement, where $D' = 2D, D'_\sharp = 2D_\sharp$. □

The following three theorems can be verified directly.

Theorem 5 *The class G contains no Redfield scale and no Pythagorean heptatonic.*

Theorem 6 *The class R contains no Gypsy scale and no Pythagorean heptatonic.*

Theorem 7 *The class P contains no Redfield scale and no Gypsy scale.*

The other diatonic scales we obtain from classes G, P, R via homomorphism (and specially, isomorphism). We do not describe them in this paper and note now only some important special cases.

If $X = Y = Z = V = W = \sqrt[12]{2}$, then Table 2, Table 3, and Table 4 define Equal Temperament. Another interesting simplification of the general case via homomorphism we obtain in the following theorem which can be verified directly.

Theorem 8 *If*
(a) $Y = Z = a, X = b$ (see Table 2), or
(b) $X = Y = a, V = b$ (see Table 3), or
(c) $W = Y = a, V = b$ (see Table 4),
and $a = 256/243, b = 2187/2048$ (or $a = 8/\sqrt[4]{3125}, b = \sqrt[4]{78125}/16$), then Table 2 or Table 3 or Table 4 contain Pythagorean (or Praetorius) Tuning.

Corollary 1 *Pythagorean Tuning (reduced to a 12-valued one) and Praetorius Tuning are isomorphic.*

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