ON A LATTICE STRUCTURE OF OPERATOR SPACES
IN COMPLETE BORNOLOGICAL LOCALLY CONVEX SPACES

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Abstract. For $X, Y$ complete bornological locally convex spaces, we consider a lattice structure of the space $L(X, Y)$ of all continuous linear operators $L: X \to Y$.

Introduction

The description of theory of complete bornological locally convex spaces (C.B.L.C.S.) we can find in [4], [6], and [3].

In [1], [2] we have developed a technique for an operator valued measure $m : \Delta \to L(X, Y)$, where $\Delta$ is a $\delta$-ring of sets, $L(X, Y)$ the space of all continuous operators $L: X \to Y$, where $X, Y$ are both C.B.L.C.S. In [1] we gave a more detailed explanation of basic $L(X, Y)$-measure set structures (H. Weber, cf. [7], considered these structures particularly from topological aspects.). In connection with it, a Bartle type integral was investigated. In [2], convergences in measure, almost everywhere, almost uniform (and relations between them) were studied.

In the present paper we consider the lattice structure of the range space of such measure $m$, the space $L(X, Y)$.

1. Preliminaries

Let $X, Y$ be two C.B.L.C.S. over the field of real or complex numbers equipped with the bornologies $\mathfrak{B}_X, \mathfrak{B}_Y$. The basis $\mathcal{U}$ of the bornology $\mathfrak{B}_X$ has a marked element $u_0 \in \mathcal{U}$, if $u_0 \subseteq u$ for every $u \in \mathcal{U}$. Let the bases $\mathcal{U}, \mathcal{W}$ be chosen to consist of all $\mathfrak{B}_X-,\mathfrak{B}_Y$- bounded Banach disks in $X, Y$, with marked elements $u_0 \in \mathcal{U}, u_0 \neq \{0\}$, and $w_0 \in \mathcal{W}, w_0 \neq \{0\}$, respectively. Remind that a Banach disk in $X$ is a set which is closed, absolutely convex and the linear span of which is a Banach space. The space $X$ is an inductive limit of Banach spaces $X_u, u \in \mathcal{U}$,

$$X = \lim \text{ind}_{u \in \mathcal{U}} X_u,$$

cf. [4], where $X_u$ is a linear span of $u \in \mathcal{U}$ and $\mathcal{U}$ is directed by inclusion (analogously for $Y$ and $\mathcal{W}$).

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On $\mathcal{U}$ the lattice operations are defined as follows. For $u_1, u_2 \in \mathcal{U}$ we have: $u_1 \wedge u_2 = u_1 \cap u_2$, $u_1 \vee u_2 = u_1 \cup u_2$, where $\text{acs}(u_1 \cup u_2)$ denotes the topological closure of the absolutely convex span of the set. Analogously for $\mathcal{W}$. For $(u_1, w_1), (u_2, w_2) \in \mathcal{U} \times \mathcal{W}$, we write $(u_1, w_1) \ll (u_2, w_2)$ if and only if $u_1 \subset u_2$ and $w_1 \supset w_2$.

2. Lattice structure of $L(X, Y)$

If $p_u$ is Minkowski functional of the set $w \in \mathcal{W}$, then for $u \in \mathcal{U}$, $L \in L(X, Y)$, we put $p_{u,w}(L) = \sup_{x \in u} p_u(L(x))$ (If $w$ does not absorb $L(x), x \in u$, we put $p_{u,w}(L) = \infty$).

Denote by $\mathcal{L}_{u,w} = \{L \in L(X, Y); p_{u,w}(L) < \infty\}, (u, w) \in \mathcal{U} \times \mathcal{W}$, and $\mathcal{L}_{\mathcal{U}, \mathcal{W}} = \bigcup \mathcal{L}_{u,w}; (u, w) \in \mathcal{U} \times \mathcal{W}$. For $(u, w) \in \mathcal{U} \times \mathcal{W}$, a sequence $L_n \in L(X, Y), n = 1, 2, \ldots$, is said to be convergent to $L \in L(X, Y)$ in $\mathcal{L}_{u,w}$ whenever $\lim_{n \to \infty} p_{u,w}(L_n - L) = 0$.

On $\mathcal{L}_{\mathcal{U}, \mathcal{W}}$ define the operations $\wedge, \vee$ and an order $\ll$. For $(u_1, w_1), (u_2, w_2) \in \mathcal{U} \times \mathcal{W},$

$$\mathcal{L}_{u_1,w_1} \vee \mathcal{L}_{u_2,w_2} = \mathcal{L}_{u_1 \cup u_2, w_1 \vee w_2}, \quad \mathcal{L}_{u_1,w_1} \wedge \mathcal{L}_{u_2,w_2} = \mathcal{L}_{u_1 \cap u_2, w_1 \wedge w_2}$$

$$\mathcal{L}_{u_2,w_2} \ll \mathcal{L}_{u_1,w_1} \quad \text{if and only if} \quad (u_1, w_1) \ll (u_2, w_2).$$

It is easy to see that $\wedge, \vee$ are lattice operations.

Theorem 1. The family $\mathcal{L}_{\mathcal{U}, \mathcal{W}}$ of operator spaces is a distributive lattice.

Proof. For $(u_1, w_1), (u_2, w_2), (u_3, w_3) \in \mathcal{U} \times \mathcal{W}$, we have:

$$\mathcal{L}_{u_1,w_1} \vee (\mathcal{L}_{u_2,w_2} \wedge \mathcal{L}_{u_3,w_3}) = \mathcal{L}_{u_1 \cup (u_2 \cap u_3), w_1 \vee (w_2 \wedge w_3)}$$

$$= \mathcal{L}_{u_1 \cup (u_2 \cap u_3), w_1 \vee (w_2 \wedge w_3)}$$

$$= \mathcal{L}_{u_1 \cup (u_2 \cap u_3), (w_1 \vee w_2) \wedge (w_1 \wedge w_2)}$$

$$= \mathcal{L}_{u_1 \cup (u_2 \cap u_3), w_1 \wedge w_2}$$

$$= \mathcal{L}_{u_1 \cup (u_2 \cap u_3), w_1 \wedge w_2} \vee \mathcal{L}_{u_1 \cup (u_2 \cap u_3), w_1 \wedge w_2}$$

$$= (\mathcal{L}_{u_1, w_1} \vee \mathcal{L}_{u_2, w_2}) \wedge (\mathcal{L}_{u_1, w_1} \vee \mathcal{L}_{u_3, w_3}).$$

By [5], Th.2.2, $\mathcal{L}_{\mathcal{U}, \mathcal{W}}$ is a distributive lattice.

The lattice $\mathcal{L}_{\mathcal{U}, \mathcal{W}}$ introduces a topology of an inductive limit on $L(X, Y)$, i.e. there holds the following theorem.

Theorem 2. $L(X, Y)$ is the inductive limit of the subalgebras $L(u, w) \subseteq L(X, Y)$

Proof. For $u \in \mathcal{U}, w \in \mathcal{W}$, it is easy to verify that $L(u, w)$ is vector subspace of $L(X, Y)$ equipped with the topology given by the seminorm $p_{u,w}$.

Show that $\bigcup_{(u, w) \in \mathcal{U} \times \mathcal{W}} L(u, w) = L(X, Y)$. The inclusion $\bigcup_{(u, w) \in \mathcal{U} \times \mathcal{W}} L(u, w) \subseteq L(X, Y)$ is trivial. Show $\bigcup_{(u, w) \in \mathcal{U} \times \mathcal{W}} L(u, w) \supseteq L(X, Y)$. Let $L \in L(X, Y)$, then for each $u \in \mathcal{U}$ there exists $w_{u,L} \in \mathcal{W}$ such that $L \subset w_{u,L}$, i.e. $p_{u,w_{u,L}}(L) \leq 1 < \infty$. Thus, $L \in L(u,w_{u,L}) \subseteq \bigcup_{(u, w) \in \mathcal{U} \times \mathcal{W}} L(u, w)$.

Let $(u_1, w_1), (u_2, w_2) \in \mathcal{U} \times \mathcal{W}$. Consider sequence $L_n \in L(X, Y), n = 1, 2, \ldots$, of operators converges to $L \in L(X, Y)$ in $L(u_2, w_2)$, then it converges to $L$ also in $L(u_1, w_1)$. Indeed, by definition, $(u_1, w_1) \ll (u_2, w_2) \Rightarrow u_1 \subset u_2 \wedge w_1 \supset w_2$. The relation $u_1 \subset u_2$ implies $p_{u_1, w}(L) \leq p_{u_2, w}(L)$ for every $w \in \mathcal{W}$. The inclusion $w_2 \subset w_1$ implies $p_{w_1}(L(x)) \leq p_{w_2}(L(x))$.
for every $x \in X$. From this we have $p_{u,w_1}(L) \leq p_{u,w_2}(L)$ for every $u \in U$. Thus, $p_{u_1,w_1}(L) \leq p_{u_2,w_1}(L) \leq p_{u_2,w_2}(L)$. So, if $(u_1, w_1) \ll (u_2, w_2)$ and $L \in L(X, Y)$, then $p_{u_1,w_1}(L) \leq p_{u_2,w_2}(L)$. This completes the proof.

Note that in the terminology of [6], $L(X, Y)$ (as an inductive limit of seminormed spaces) is a bornological convex vector space, cf. [6], chap. 4, §2, Th. 1.

**Theorem 3.** For every $(u_1, w_1) \in U \times W$, the set

$$\mathcal{J}_{u_1,w_1} = \{L_{u,w} \in \mathcal{L}_{U,W}; L_{u,w} \ll L_{u_1,w_1}, (u, w) \in U \times W\}$$

is an ideal in $\mathcal{L}_{U,W}$.

**Proof.** Let $(p, q), (u, w) \in U \times W$. and $(u_1, w_1) \ll (u, w), (u_1, w_1) \ll (p, q)$. Since $u \wedge p = u \cap p \subseteq u_1, w \vee q = \text{acs}(w \cup q) \subseteq w_1$, then $L_{u_1,w_1} \cap L_{u_1,w_1}$.

Let $(p, q), (u, w) \in U \times W$, and $(u_1, w_1) \ll (p, q)$. Then $L_{u_1,w_1} \cap L_{u_1,w_1} \ll L_{u_1,w_1}$.

Dually to Theorem 3, we obtain the following corollary.

**Corollary 4.** For every $(u_2, w_2) \in U \times W$, the set

$$\mathcal{F}_{u_2,w_2} = \{L_{u,w} \in \mathcal{L}_{U,W}; L_{u,w} \ll L_{u_2,w_2}, (u, w) \in U \times W\},$$

is a filter in $\mathcal{L}_{U,W}$.

**Theorem 5.** Let $(u_1, w_1), (u_2, w_2) \in U \times W$. If $(u_1, w_1) \ll (u_2, w_2)$, then the order interval $[L_{u_2,w_2}, L_{u_1,w_1}] = \mathcal{J}_{u_1,w_1} \cap \mathcal{F}_{u_2,w_2}$ in $\mathcal{L}_{U,W}$ is a Boolean algebra with $L_{u_2,w_2}$ as null and $L_{u_1,w_1}$ as unit.

**Proof.** Let $(u, w) \in U \times W, (u_1, w_1) \ll (u, w) \ll (u_2, w_2)$. Put

$$L_{u_1,w_1} = L_{(u_2,w_2) \cup u_1 \setminus (w_1 \cup w_2)} \in [L_{u_2,w_2}, L_{u_1,w_1}]$$

and show that $L_{u_1,w_1}$ is a complement of $L_{u_1,w_1}$ in $[L_{u_2,w_2}, L_{u_1,w_1}]$. We have:

$$L_{u_1,w_1} \vee L_{u_1,w_1} = L_{u_1,w_1} \vee L_{(u_2,w_2) \cup u_1 \setminus (w_1 \cup w_2)}$$

$$= L_{u \cup (u_2 \setminus u_1 \setminus (w_1 \cup w_2))}$$

$$= L_{u \cup (u_2 \setminus u_1 \setminus (w_1 \cup w_2))}$$

$$= L_{u_1,w_1}.$$ 

Analogously, $L_{u_1,w_1} \wedge L_{u_1,w_1} = L_{u_2,w_2}$. So, $L_{u_2,w_2}$ is the null and $L_{u_1,w_1}$ is the unit of the Boolean algebra $[L_{u_2,w_2}, L_{u_1,w_1}]$. 
References


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