Compatible quasiorders and linear orders of (monounary) algebras

R. Pöschel, S. Radeleczki, D. Jakubíková-Studenovská

Technische Universität Dresden
University Miskolc, University Košice

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Outline

Notions and notations

Linear orders

The structure of the lattice Quord$\langle A, f\rangle$

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compatible quasiorders

$\langle A, F\rangle$ universal algebra

compatible (invariant) relation $q \subseteq A \times A$:
For each $f \in F$ (n-ary) we have $f \triangleright q$ (f preserves q), i.e.

$$(a_1, b_1), \ldots, (a_n, b_n) \in q \implies (f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in q\,.$$

Lord$\langle A, F\rangle$ compatible linear orders
Pord$\langle A, F\rangle$ compatible partial orders (refl., trans., antisymmetric)

Generalization of Pord$\langle A, F\rangle$ and Con$\langle A, F\rangle$:
Quord$\langle A, F\rangle$ compatible *quasiorders* (reflexive, transitive)

Remark

(Quord$\langle A, F\rangle, \subseteq$) is a lattice and it is a complete sublattice of the lattice (Quord$(A), \subseteq$) of all quasiorders on A.

Problem

Describe the quasiorder lattice Quord$\langle A, F\rangle$.

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Reduction to (mono)unary algebras

$H :=$ unary polynomial operations of $\langle A, F\rangle$ (i.e. $H = \langle F \cup C\rangle^{(1)}$).
Then

$$\mathrm{Quord}\langle A, F\rangle = \mathrm{Quord}\langle A, H\rangle$$
$$\mathrm{Quord}\langle A, H\rangle = \bigcap_{f \in H} \mathrm{Quord}\langle A, f\rangle.$$

Note $\langle A, f\rangle$ is a monounary algebra ($f : A \to A$).

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linear extensions of partial orders

Recall:
Every partial order has a linear extension

Theorem (Dushnik & Miller)

Every partial order is the intersection of its linear extensions.

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Is this true also for compatible partial orders and compatible linear orders? (or to what extent is this true)

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compatible linear orders of $\langle A, f\rangle$

$\langle A, f\rangle$ monounary algebra
When exists a compatible linear order?

Theorem (Szigeti)

$$\mathrm{Lord}\langle A, f\rangle \neq \emptyset \iff \langle A, f\rangle \textit{ acyclic.}$$

Definition
$\langle A, f\rangle$ is *acyclic* $:\iff \forall a \in A \forall n \in \mathbb{N}_+ : f^n(a) = a \implies f(a) = a$
(no cycles in the graph of f, except loops)

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Characterization of compatible linear orders

There exists a full description for the compatible linear orders of an acyclic monounary algebra $\langle A, f\rangle$:

For simplicity, let the graph $f^\bullet := \{(x, f(x)) \mid x \in A\}$ be connected and let f has a (single) fixed point 1, formally: $\forall x, y \in A \exists m \in \mathbb{N} : f^m(x) = f^m(y)$. Then:

Theorem

Every $R \in \mathrm{Lord}\langle A, f\rangle$ is uniquely characterized by a family $(R_B)_{B\in A/\ker f}$ of linear orders on each equivalence class $B \in A/\ker f$:

$$R = \Delta \cup \{(a, b) \mid \exists n \in \mathbb{N} \exists B \in A/\ker f : (f^n(a), f^n(b)) \in R_B \setminus \Delta\}.$$

$\Delta := \{(a, a) \mid a \in A$ (equality relation)
$f^{n+1}(x) := f(f^n(x))$, $n \in \mathbb{N}$, f^0 is the identity mapping, $f^0(x) = x$.

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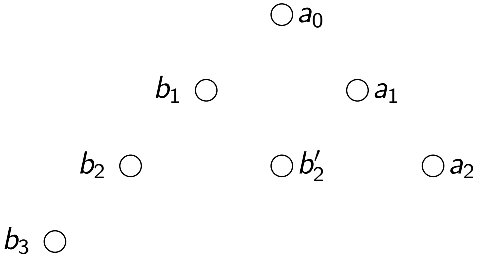
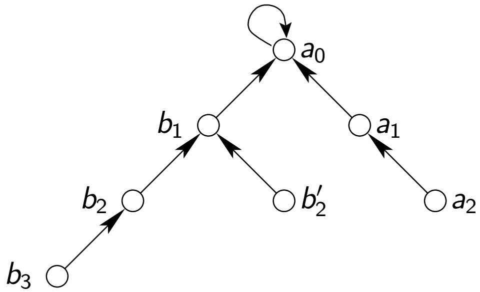


Example

f



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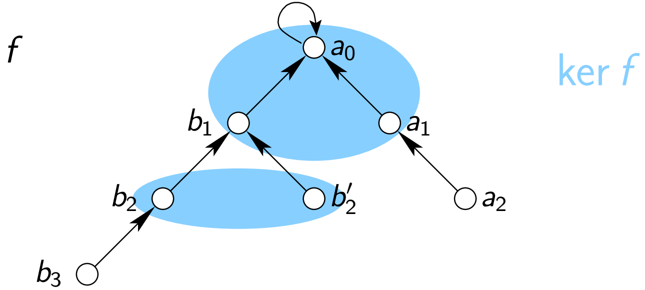

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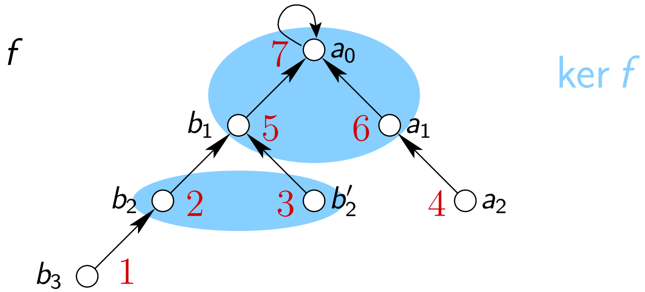

linear order
induced by $b_2 < b_2'$ and $b_1 < a_1 < a_0$ on blocks

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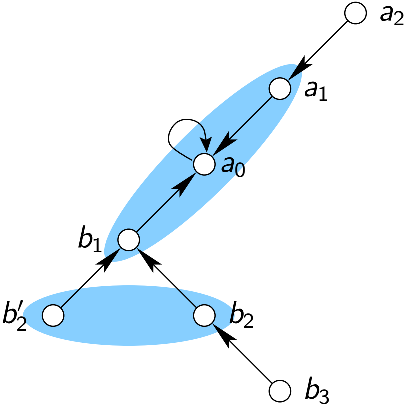
the same f



ker f

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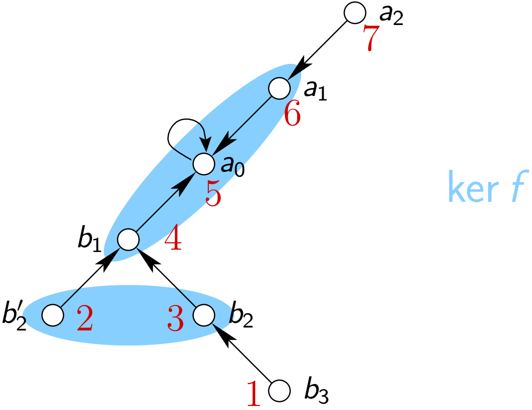

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Types

General assumption from now on:
$\langle A, f\rangle$ (finite) acyclic, connected and with exactly one fixed point 1.

Definition
Set of *types*:

$$\mathcal{T}_f := \{\beta \in \mathrm{Quord}\langle A, f\rangle \mid \beta \subseteq \ker f\} = \{\beta \in \mathrm{Quord}(A) \mid \beta \subseteq \ker f\}$$

For $\beta \in \mathcal{T}_f$ we have $\beta = \bigcup\{\beta_B \mid B \in A/\ker f\}$
where $\beta_B := \beta \cap B^2$ for each block B of $\ker f$.
Therefore $\mathcal{T}_f \cong \prod_{B \in A/\ker f} \mathrm{Quord}(B)$.

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Typ of a quasiorder q: $q^{\min} := q \cap \ker f$.

(Clearly, $q^{\min} \in \mathcal{T}_f$, note that $\ker f \in \mathrm{Con}\langle A, f\rangle \subseteq \mathrm{Quord}\langle A, f\rangle$.)
$\beta := q^{\min}$ is the least quasiorder of type β.

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The lattice Quord$\langle A, f \rangle$

For type $\beta \in \mathcal{T}_f$ let $Q(\beta) := \{q \in \text{Quord}\langle A, f \rangle \mid q^{\min} = \beta\}$.

Theorem

(1) $\text{Quord}\langle A, f \rangle = \bigcup\{Q(\beta) \mid \beta \in \mathcal{T}_f\}$. *Each set* $Q(\beta)$ *is a semi-interval of the form*

$$Q(\beta) = \bigcup_{i \in I} [\beta, q_i]$$

(union of intervals in $\text{Quord}\langle A, f \rangle$ *all with least element* β*).*

(2) $\mathcal{T}_f$ *is a sublattice of* $\text{Quord}\langle A, f \rangle$ *and* $\mathcal{T}_f \cong \prod_{B \in A/\ker f} \text{Quord}(B)$.

(3) ${}^{\min} : \text{Quord}\langle A, f \rangle \to \mathcal{T}_f : q \mapsto q^{\min}$ *is a* $\wedge$*-semilattice homomorphism (in particular an order-homomorphism)*

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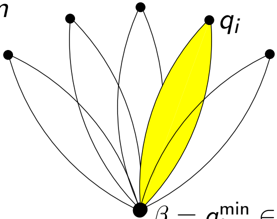

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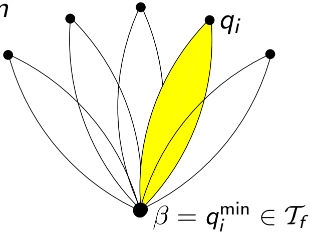

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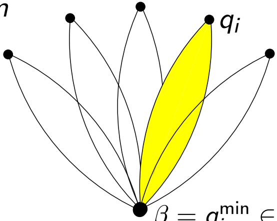
The lattice Quord$\langle A, f \rangle$

For type $\beta \in \mathcal{T}_f$ let $Q(\beta) := \{q \in \text{Quord}\langle A, f \rangle \mid q^{\min} = \beta\}$.

Theorem

(1) $\text{Quord}\langle A, f \rangle = \bigcup\{Q(\beta) \mid \beta \in \mathcal{T}_f\}$. *Each set* $Q(\beta)$ *is a semi-interval of the form*

$$Q(\beta) = \bigcup_{i \in I} [\beta, q_i]$$



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Special case: partial orders

Lemma

Let $\beta \in \mathcal{T}_f$ be a partial order. Then

$$\beta^{\max} := \Delta \cup \{(a, b) \mid \exists n \in \mathbb{N} : (f^n(a), f^n(b)) \in \beta \setminus \Delta\}$$

is a compatible partial order, i.e. $\beta^{\max} \in \mathrm{Pord}\langle A, f \rangle$.

Proposition

Let $\beta \in \mathcal{T}_f$ be a partial order. Then

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Linear extensions of compatible partial orders

The analogon to the Theorem of Dushnik & Miller for compatible orders:

Theorem

Let $q \in \mathrm{Pord}\langle A, f\rangle$ be a compatible partial order of type $\beta = q \cap \ker f$. Then

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Proof

$\beta^{\max} = \bigcap\{R \in \mathrm{Lord}\langle A, f\rangle \mid q \subseteq R\}$
"$\subseteq$": Let $(a, b) \in \beta^{\max}$ and R compatible linear extension of q (thus also of $\beta \subseteq q$). By definition of $\beta^{\max}$ there exists $n \in \mathbb{N}$, such that $(f^n(a), f^n(b)) \in \beta \setminus \Delta \subseteq R \setminus \Delta$. By Lemma below this implies $(a, b) \in R$; thus $\beta^{\max} \subseteq R$.

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Lemma

For $R \in \mathrm{Lord}\langle A, f\rangle$ we have:

$$(f^i(a), f^i(b)) \in R \setminus \Delta \iff (a, b) \in R \setminus \Delta \text{ and } f^i(a) \neq f^i(b).$$

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$$\beta^{\max} = \bigcap\{R \in \mathrm{Lord}\langle A, f\rangle \mid q \subseteq R\}$$

"$\supseteq$": On each block $B \in A/\ker f$ the restriction $\beta_B := \beta \cap B^2 = q \cap B^2$ is the intersection of all linear extensions (on B) (Dushnik & Miller). Linear extensions on each block uniquely define a compatible linear extension of β, thus $\bigcap\{R \in \mathrm{Lord}\langle A, f\rangle \mid q \subseteq R\}$ has type β, i.e. belongs to $Q(\beta) = [\beta, \beta^{\max}]$.

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Generalization: linear $\longrightarrow$ quasilinear

In general, $\beta^{\max} := \Delta \cup \{(a, b) \mid \exists n \in \mathbb{N} : (f^n(a), f^n(b)) \in \beta \setminus \Delta\}$ does not belong to $Q(\beta)$ because it is not transitive, i.e. not a quasiorder.

How to describe the maximal elements q_i of the semi-intervals $Q(\beta)$?

Answer:

They are the intersection of all compatible *quasilinear* extensions.

Moreover, $\beta^{\max} = \bigcup_{i \in I} q_i$.

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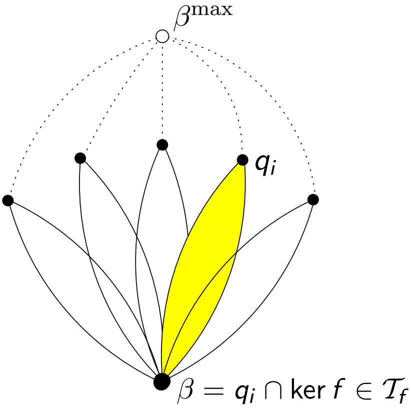

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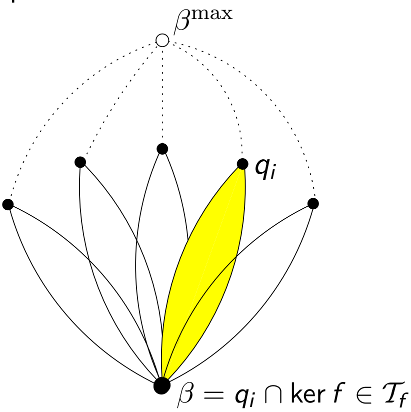

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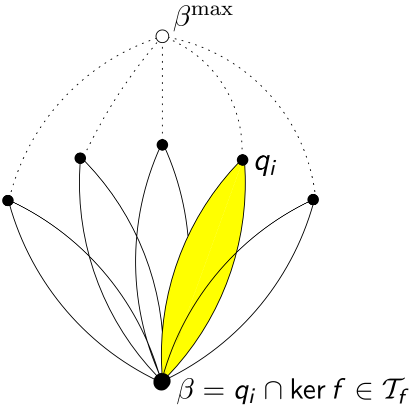

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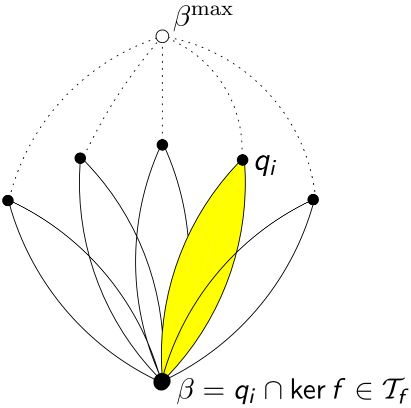

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quasilinear quasiorders

Definition

A quasiorder q is called *quasilinear* if q/q_0 given by

$$([a]_{q_0}, [b]_{q_0}) \in q/q_0 :\iff \exists u \in [a]_{q_0}\, \exists v \in [b]_{q_0} : (u, v) \in q$$

is a linear order on the factor set A/q_0
(where $q_0 = q \cap q^{-1}$, note $q_0 \in \mathrm{Con}\langle A, f\rangle$).

Remark: $q \in \mathrm{QLord}(A, f)$ is uniquely determined by q_0 and q/q_0, i.e. by a congruence $\theta \in \mathrm{Con}(A, f)$ and a linear order $\hat{\lambda} \in \mathrm{Quord}(A/\theta, \hat{f})$ (which in turn is given by linear orders on the blocks of $\ker \hat{f}$)

Theorem

Let β be some type and let q be a maximal element in the semi-interval $Q(\beta)$. Then q is the intersection of all its quasilinear extensions:

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$\vee$- or $\wedge$-irreducible quasiorders

join in Quord(A, f): $q_1 \vee q_2 = (q_1 \cup q_2)^{\text{tra}}$

$\vee$-irreducible quasiorders: 1-generated

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meet in Quord(A, f): $q_1 \wedge q_2 = q_1 \cap q_2$

Problem

Characterize the $\wedge$-irreducible quasiorders

Characterize the $\wedge$-irreducible quasilinear quasiorders

(partial) answer: next talk by Danica Jakubíková-Studenovská

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Thank you for your attention

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