Complementary quasiorder lattices of monounary algebras

D. Jakubíková–Studenovská and M. Petrejčíková
danica.studenovska@upjs.sk, maria.petrejcikova@upjs.sk

Stará Lesná, Slovakia
5.9. – 11.9.2008

Introduction

$\mathcal{A} = (A, f)$
a **monounary algebra**

$\mathcal{A} = (A, f)$

a **monounary algebra**

α

quasiorder of $\mathcal{A}$

a binary relation on $\mathcal{A}$, which is reflexive, transitive and compatible with f

Introduction

$\mathcal{A} = (A, f)$
a **monounary algebra**

α
quasiorder of $\mathcal{A}$
a binary relation on $\mathcal{A}$, which is reflexive, transitive and compatible with f

$\operatorname{Quord} \mathcal{A}$
all quasiorders of (A, f)

Introduction

$\mathcal{A} = (A, f)$

a **monounary algebra**

α

quasiorder of $\mathcal{A}$

a binary relation on $\mathcal{A}$, which is reflexive, transitive and compatible with f

$\mathrm{Quord}\,\mathcal{A}$

all quasiorders of (A, f)

$(\mathrm{Quord}\ \mathcal{A}, \subseteq)$

lattice of all quasiorders of (A, f)

Introduction

$\mathcal{A} = (A, f)$
a **monounary algebra**

α
quasiorder of $\mathcal{A}$
a binary relation on $\mathcal{A}$, which is reflexive, transitive and compatible with f

$\mathrm{Quord}\,\mathcal{A}$
all quasiorders of (A, f)

$(\mathrm{Quord}\ \mathcal{A}, \subseteq)$
lattice of all quasiorders of (A, f)

$\Delta = \{(a, a) : a \in A\}$ ... the smallest quasiorder
A^2 ... the greatest quasiorder

Introduction

$\mathcal{A} = (A, f)$
a **monounary algebra**

Introduction

$\mathcal{A} = (A, f)$

a **monounary algebra**

- $c \in A$ is **cyclic** if $f^k(c) = c$ for some $k \in N$,

$\mathcal{A} = (A, f)$
a **monounary algebra**

- $c \in A$ is **cyclic** if $f^k(c) = c$ for some $k \in N$,
- the set of all cyclic elements of some connected component of (A, f) is a **cycle** of (A, f).

Motivation

AIM

$(\text{Quord } \mathcal{A}, \subseteq)$ complementary lattice $\iff$ some conditions for $\mathcal{A} = (A, f)$

Motivation

$$\xrightarrow{necessary}$$

(Quord $\mathcal{A}, \subseteq$)
complementary lattice

conditions for
$\mathcal{A} = (A, f)$

Motivation

$\xrightarrow{necessary}$

$(\text{Quord } \mathcal{A}, \subseteq)$
complementary lattice

conditions for
$\mathcal{A} = (A, f)$

1. each connected component of $\mathcal{A}$ contains a cycle,
2. there is $n \in N$ such that each cycle of $\mathcal{A}$ has n elements,
3. n is square-free,
4. for each $a \in A$, the element $f(a)$ is cyclic.

(picture)

Motivation

$$\xrightarrow{necessary}$$

$(\text{Quord } \mathcal{A}, \subseteq)$
complementary lattice

conditions for
$\mathcal{A} = (A, f)$

1. each connected component of $\mathcal{A}$ contains a cycle,
2. there is $n \in N$ such that each cycle of $\mathcal{A}$ has n elements,
3. n is square-free,
4. for each $a \in A$, the element $f(a)$ is cyclic.

(picture)

$$\xleftarrow{sufficient}$$

Motivation

$\xrightarrow{necessary}$

$(\text{Quord } \mathcal{A}, \subseteq)$

complementary lattice

conditions for

$\mathcal{A} = (A, f)$

1. each connected component of $\mathcal{A}$ contains a cycle,
2. there is $n \in N$ such that each cycle of $\mathcal{A}$ has n elements,
3. n is square-free,
4. for each $a \in A$, the element $f(a)$ is cyclic.

(picture)

$\xleftarrow{sufficient}$

$|A/r| = 1;$

r is an equivalence on A

Motivation

$\xrightarrow{necessary}$

$(\mathrm{Quord}\ \mathcal{A}, \subseteq)$
complementary lattice

conditions for
$\mathcal{A} = (A, f)$

1. each connected component of $\mathcal{A}$ contains a cycle,
2. there is $n \in N$ such that each cycle of $\mathcal{A}$ has n elements,
3. n is square-free,
4. for each $a \in A$, the element $f(a)$ is cyclic.

(picture)

$\xleftarrow{sufficient}$

Motivation

$\xrightarrow{necessary}$

$(\text{Quord } \mathcal{A}, \subseteq)$
complementary lattice

conditions for
$\mathcal{A} = (A, f)$

1. each connected component of $\mathcal{A}$ contains a cycle,
2. there is $n \in N$ such that each cycle of $\mathcal{A}$ has n elements,
3. n is square-free,
4. for each $a \in A$, the element $f(a)$ is cyclic.

(picture)

$\xleftarrow{sufficient}$

$|A/r| = 1;$

r is an equivalence on A

HYPOTHESIS

$(\text{Quord}\ \mathcal{A}, \subseteq)$

complementary lattice

$\xrightarrow{necessary}$

$\xleftarrow{sufficient}$

conditions for

$\mathcal{A} = (A, f)$

1. each connected component of $\mathcal{A}$ contains a cycle,
2. there is $n \in N$ such that each cycle of $\mathcal{A}$ has n elements,
3. n is square-free,
4. for each $a \in A$, the element $f(a)$ is cyclic.

Necessary condition

Assume: $(A, f) \ldots$ a monounary algebra ⥀

$\mathrm{Quord}\,(A, f) \ldots$ complementary lattice

Necessary condition

Assume: (A, f) ... a monounary algebra
$\text{Quord}\,(A, f)$ ... complementary lattice

Lemma

Let (B, f) be a subalgebra of the algebra (A, f). Then the lattice $\text{Quord}\,(B, f)$ is complementary.

Necessary condition

Assume: $(A, f) \ldots$ a monounary algebra
$\mathrm{Quord}\,(A, f) \ldots$ complementary lattice

↺

Lemma

Let (B, f) be a subalgebra of the algebra (A, f). Then the lattice $\mathrm{Quord}\,(B, f)$ is complementary.

Lemma

If $x \in A$, then there is $m \in \mathbb{N}$ such that $f^{m+1}(x) = f(x)$.

Necessary condition

Assume: (A, f) ... a monounary algebra
$\text{Quord}\,(A, f)$ ... complementary lattice ⥀

Lemma

Let (B, f) be a subalgebra of the algebra (A, f). Then the lattice $\text{Quord}\,(B, f)$ is complementary.

Lemma

If $x \in A$, then there is $m \in \mathbb{N}$ such that $f^{m+1}(x) = f(x)$.

Lemma

All cycles of (A, f) have the same number of elements.

Necessary condition

Assume: (A, f) ... a monounary algebra
$\mathrm{Quord}\,(A, f)$ ... complementary lattice ↻

Lemma

Let (B, f) be a subalgebra of the algebra (A, f). Then the lattice $\mathrm{Quord}\,(B, f)$ is complementary.

Lemma

If $x \in A$, then there is $m \in \mathbb{N}$ such that $f^{m+1}(x) = f(x)$.

Lemma

All cycles of (A, f) have the same number of elements.

Lemma

If C is a cycle of (A, f) with n elements, then $n = 1$ or n is a product of mutually distinct primes (square-free).

Sufficient condition

Assume: $(A, f) \ldots$ a monounary algebra such that;

- each connected component of $\mathcal{A} = (A, f)$ contains a cycle,
- there is $n \in N$ such that each cycle of $\mathcal{A}$ has n elements,
- n is square-free,
- for each $a \in A$, the element $f(a)$ is cyclic.

Sufficient condition - equivalence relations r

Sufficient condition - equivalence relations r

- For $\alpha \in \mathrm{Quord}\,(A, f)$, define α^{-1}:

$$(b, a) \in \alpha^{-1} \iff (a, b) \in \alpha.$$

Sufficient condition - equivalence relations r

- For $\alpha \in \mathrm{Quord}\,(A, f)$, define α^{-1}:

$$(b, a) \in \alpha^{-1} \Longleftrightarrow (a, b) \in \alpha.$$

- For $a \in A$ denote by $C(a)$ the cycle, containing $f(a)$.

Sufficient condition - equivalence relations r

- For $\alpha \in \mathrm{Quord}\,(A, f)$, define α^{-1}:

$$(b, a) \in \alpha^{-1} \Longleftrightarrow (a, b) \in \alpha.$$

- For $a \in A$ denote by $C(a)$ the cycle, containing $f(a)$.
- Relation R: If B, D are cycles of (A, f), then $B\ R\ D$, if there are $k \in N$, cycles $B = C_0, C_1, \dots, C_k = D$, elements $c_0 \in C_0, c_1 \in C_1, \dots, c_k \in C_k$ such that for each $i \in \{0, 1, \dots, k-1\}$, $(c_i, c_{i+1}) \in \alpha \cup \alpha^{-1}$.

Sufficient condition - equivalence relations r

- For $\alpha \in \mathrm{Quord}\,(A, f)$, define α^{-1}:

$$(b, a) \in \alpha^{-1} \Longleftrightarrow (a, b) \in \alpha.$$

- For $a \in A$ denote by $C(a)$ the cycle, containing $f(a)$.
- Relation R: If B, D are cycles of (A, f), then $B\ R\ D$, if there are $k \in N$, cycles $B = C_0, C_1, \ldots, C_k = D$, elements $c_0 \in C_0, c_1 \in C_1, \ldots, c_k \in C_k$ such that for each $i \in \{0, 1, \ldots, k-1\}$, $(c_i, c_{i+1}) \in \alpha \cup \alpha^{-1}$.
- For $a, b \in A$, set

$$a\ r\ b \Longleftrightarrow C(a)\ R\ C(b).$$

Sufficient condition - equivalence relations r

- For $\alpha \in \mathrm{Quord}\,(A, f)$, define α^{-1}:

$$(b, a) \in \alpha^{-1} \Longleftrightarrow (a, b) \in \alpha.$$

- For $a \in A$ denote by $C(a)$ the cycle, containing $f(a)$.
- Relation R: If B, D are cycles of (A, f), then $B\ R\ D$, if there are $k \in N$, cycles $B = C_0, C_1, \ldots, C_k = D$, elements $c_0 \in C_0, c_1 \in C_1, \ldots, c_k \in C_k$ such that for each $i \in \{0, 1, \ldots, k-1\}$, $(c_i, c_{i+1}) \in \alpha \cup \alpha^{-1}$.
- For $a, b \in A$, set

$$a\ r\ b \Longleftrightarrow C(a)\ R\ C(b).$$

The relation r is an equivalence on A.

Lemma:

If $a, b \in A$ belong to the same connected component, then $a\ r\ b$.

(example)

Sufficient condition

$A/r = \{A_j : j \in J\}$

Theorem

Let $\alpha \in \mathrm{Quord}\,(A, f)$, $j \in J$. *Then there exists a complement* β_j *of* $\alpha_j = \alpha \restriction A_j$ *in the lattice* $\mathrm{Quord}\,(A_j, f)$.

Sufficient condition

$A/r = \{A_j : j \in J\}$

Theorem

Let $\alpha \in \mathrm{Quord}\,(A, f)$, $j \in J$. *Then there exists a complement* β_j *of* $\alpha_j = \alpha \restriction A_j$ *in the lattice* $\mathrm{Quord}\,(A_j, f)$.

Theorem

If $\alpha \in \mathrm{Quord}\,(A, f)$ *and* $|A/r| = 1$, *then the conditions*

- *each connected component of* (A, f) *contains a cycle,*
- *there is* $n \in N$ *such that each cycle of* (A, f) *has* n *elements,*
- n *is square-free,*
- *for each* $a \in A$, *the element* $f(a)$ *is cyclic*

are necessary and sufficient for the existence of a complement of α *in the lattice* $\mathrm{Quord}\,(A, f)$.

Sufficient condition - auxiliary results

$A/r = \{A_j : j \in J\}, |J| = 1$

Let $\alpha \in \mathrm{Quord}\,(A, f)$.

- A': all noncyclic elements x of A such that $(x, f^n(x)) \notin \alpha$ and $(f^n(x), x) \notin \alpha$.

Sufficient condition - auxiliary results

$A/r = \{A_j : j \in J\}, |J| = 1$

Let $\alpha \in \mathrm{Quord}\,(A, f)$.

- A': all noncyclic elements x of A such that $(x, f^n(x)) \notin \alpha$ and $(f^n(x), x) \notin \alpha$.
- ρ on A': $(a, b) \in \rho$ if $a, b \in A'$, $f(a) = f(b)$ and there are $k \in N$ and $a = u_0, u_1, \ldots, u_k = b$ elements of A' such that $(\forall i \in \{0, \ldots, k-1\})(f(a) = f(u_i), (u_i, u_{i+1}) \in \alpha \cup \alpha^{-1})$.

Sufficient condition - auxiliary results

$A/r = \{A_j : j \in J\}, |J| = 1$

Let $\alpha \in \mathrm{Quord}\,(A, f)$.

- A': all noncyclic elements x of A such that $(x, f^n(x)) \notin \alpha$ and $(f^n(x), x) \notin \alpha$.
- ρ on A': $(a, b) \in \rho$ if $a, b \in A'$, $f(a) = f(b)$ and there are $k \in N$ and $a = u_0, u_1, \ldots, u_k = b$ elements of A' such that $(\forall i \in \{0, \ldots, k-1\})(f(a) = f(u_i), (u_i, u_{i+1}) \in \alpha \cup \alpha^{-1})$.

Sufficient condition - auxiliary results

$A/r = \{A_j : j \in J\}, |J| = 1$

Let $\alpha \in \mathrm{Quord}\,(A, f)$.

- A': all noncyclic elements x of A such that $(x, f^n(x)) \notin \alpha$ and $(f^n(x), x) \notin \alpha$.
- ρ on A': $(a, b) \in \rho$ if $a, b \in A'$, $f(a) = f(b)$ and there are $k \in N$ and $a = u_0, u_1, \ldots, u_k = b$ elements of A' such that $(\forall i \in \{0, \ldots, k-1\})(f(a) = f(u_i), (u_i, u_{i+1}) \in \alpha \cup \alpha^{-1})$.

The relation ρ is an equivalence on A'.

Lemma:

For each $D \in A'/\rho$ there are $P(D) \subseteq D$ and $p(D) \in P(D)$ such that

1. $(\forall x \in D \setminus P(D))(\exists y \in P(D))((x, y) \in \alpha, (y, x) \in \alpha)$;
2. $(\forall x, y \in P(D))((x, y) \in \alpha \Rightarrow (y, x) \notin \alpha)$.

Sufficient condition - construction

$\beta = \beta_j, \; \beta_j \in \mathrm{Quord}\,(A_j, f)$

- *Step (a).* Let x, y belong to the same cycle C, $y = f^k(x)$, $\alpha \restriction C = \theta_d$, d/n and let $e = \frac{n}{d}$. We set $(x, y) \in \beta$ if and only if e/k.

(example)

Sufficient condition - construction

$\beta = \beta_j, \ \beta_j \in \mathrm{Quord}\,(A_j, f)$

- *Step (a).* Let x, y belong to the same cycle C, $y = f^k(x)$, $\alpha \restriction C = \theta_d$, d/n and let $e = \frac{n}{d}$. We set $(x, y) \in \beta$ if and only if e/k.
- *Step (b).* Let $x \in C_1$, $y \in C_2$, where C_1 and C_2 are distinct cycles. We put $(x, y) \in \beta$ if and only if there are $a \in C_1$ and $b \in C_2$ with $(b, a) \in \alpha, (a, b) \notin \alpha$.

(example)

Sufficient condition - construction

$\beta = \beta_j,\ \beta_j \in \mathrm{Quord}\,(A_j, f)$

- *Step (a).* Let x, y belong to the same cycle C, $y = f^k(x)$, $\alpha \restriction C = \theta_d$, d/n and let $e = \frac{n}{d}$. We set $(x, y) \in \beta$ if and only if e/k.
- *Step (b).* Let $x \in C_1$, $y \in C_2$, where C_1 and C_2 are distinct cycles. We put $(x, y) \in \beta$ if and only if there are $a \in C_1$ and $b \in C_2$ with $(b, a) \in \alpha, (a, b) \notin \alpha$.
- *Step (c).* Suppose that $x, y \in P(D)$ for some $D \in A'/\rho$. Then $(x, y) \in \beta$ if and only if and $(y, x) \in \alpha$.

(example)

Sufficient condition - construction

$\beta = \beta_j,\ \beta_j \in \mathrm{Quord}\,(A_j, f)$

- *Step (a).* Let x, y belong to the same cycle C, $y = f^k(x)$, $\alpha \restriction C = \theta_d$, d/n and let $e = \frac{n}{d}$. We set $(x, y) \in \beta$ if and only if e/k.
- *Step (b).* Let $x \in C_1$, $y \in C_2$, where C_1 and C_2 are distinct cycles. We put $(x, y) \in \beta$ if and only if there are $a \in C_1$ and $b \in C_2$ with $(b, a) \in \alpha, (a, b) \notin \alpha$.
- *Step (c).* Suppose that $x, y \in P(D)$ for some $D \in A'/\rho$. Then $(x, y) \in \beta$ if and only if and $(y, x) \in \alpha$.
- *Step (d1).* Suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $y \notin A'$, then $(x, y) \in \beta$ if and only if $(f^n(y), y) \notin \alpha, (y, f^n(y)) \in \alpha, x = f^k(y), e/k$.

(example)

Sufficient condition - construction

$\beta = \beta_j,\ \beta_j \in \mathrm{Quord}\,(A_j, f)$

- *Step (d'1).* Suppose that y belongs to a cycle C, x is noncyclic, $C(x) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $x \notin A'$, then $(x, y) \in \beta$ if and only if $(f^n(x), x) \in \alpha, (x, f^n(x)) \notin \alpha, y = f^k(x), e/k.$
- *Step (d2).* Suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $y \in A'$, then $(x, y) \in \beta$ if and only if there is $D \in A'/\rho$ such that $y \in P(D), x = f^k(y), e/k$ and $(y, p(D)) \in \alpha$.
- *Step (d'2).* Suppose that y belongs to a cycle C, x is noncyclic, $C(x) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $x \in A'$, then $(x, y) \in \beta$ if and only if there is $D \in A'/\rho$ such that $x \in P(D), y = f^k(x), e/k$ and $(x, p(D)) \in \alpha$.
- *Step (e).* Suppose that x, y satisfy none of the assumptions of the previous steps. Then $(x, y) \in \beta$ if and only if $(x, f^n(x)) \in \beta$, $(f^n(x), f^n(y)) \in \beta$, $(f^n(y), y) \in \beta$.

Assumption and denotation

Assumption and denotation

- (A, f) is a monounary algebra;
  - each connected component of (A, f) contains a cycle,
  - there is $n \in N$ such that each cycle of (A, f) has n elements,
  - n is square-free,
  - for each $a \in A$, the element $f(a)$ is cyclic,

Sufficient condition - general case

Assumption and denotation

- (A, f) is a monounary algebra;
  - each connected component of (A, f) contains a cycle,
  - there is $n \in N$ such that each cycle of (A, f) has n elements,
  - n is square-free,
  - for each $a \in A$, the element $f(a)$ is cyclic,
- $\alpha \in \mathrm{Quord}\,(A, f)$,

Sufficient condition - general case

Assumption and denotation

- (A, f) is a monounary algebra;
  - each connected component of (A, f) contains a cycle,
  - there is $n \in N$ such that each cycle of (A, f) has n elements,
  - n is square-free,
  - for each $a \in A$, the element $f(a)$ is cyclic,
- $\alpha \in \mathrm{Quord}\,(A, f)$,
- r_α an equivalence r on A (as above) depends on α,

Assumption and denotation

- (A, f) is a monounary algebra;
  - each connected component of (A, f) contains a cycle,
  - there is $n \in N$ such that each cycle of (A, f) has n elements,
  - n is square-free,
  - for each $a \in A$, the element $f(a)$ is cyclic,
- $\alpha \in \mathrm{Quord}\,(A, f)$,
- r_α an equivalence r on A (as above) depends on α,
- $|J| > 1$; $A/r_\alpha = \{A_j : j \in J\}$ and for each $i \in J$:

Assumption and denotation

- (A, f) is a monounary algebra;
  - each connected component of (A, f) contains a cycle,
  - there is $n \in N$ such that each cycle of (A, f) has n elements,
  - n is square-free,
  - for each $a \in A$, the element $f(a)$ is cyclic,
- $\alpha \in \mathrm{Quord}\,(A, f)$,
- r_α an equivalence r on A (as above) depends on α,
- $|J| > 1$; $A/r_\alpha = \{A_j : j \in J\}$ and for each $i \in J$:
  - c_i a fixed cyclic element of chosen cycle C_i in A_i;

Sufficient condition - general case

Assumption and denotation

- (A, f) is a monounary algebra;
  - each connected component of (A, f) contains a cycle,
  - there is $n \in N$ such that each cycle of (A, f) has n elements,
  - n is square-free,
  - for each $a \in A$, the element $f(a)$ is cyclic,
- $\alpha \in \mathrm{Quord}\,(A, f)$,
- r_α an equivalence r on A (as above) depends on α,
- $|J| > 1$; $A/r_\alpha = \{A_j : j \in J\}$ and for each $i \in J$:
  - c_i a fixed cyclic element of chosen cycle C_i in A_i;
  - $\alpha \restriction C_i \quad \ldots$ a congruence of the cycle C_i,

Sufficient condition - general case

Assumption and denotation

- (A, f) is a monounary algebra;
  - each connected component of (A, f) contains a cycle,
  - there is $n \in N$ such that each cycle of (A, f) has n elements,
  - n is square-free,
  - for each $a \in A$, the element $f(a)$ is cyclic,
- $\alpha \in \mathrm{Quord}\,(A, f)$,
- r_α an equivalence r on A (as above) depends on α,
- $|J| > 1$; $A/r_\alpha = \{A_j : j \in J\}$ and for each $i \in J$:
  - c_i a fixed cyclic element of chosen cycle C_i in A_i;
  - $\alpha \restriction C_i$ ... a congruence of the cycle C_i,
  - $d_i \in \mathbb{N}$; $\alpha \restriction C_i$ is the smallest congruence containing the pair $(c_i, f^{d_i}(c_i))$,

Berman: if $n \in \mathbb{N}$, then θ_d is a congruence of an n-element cycle $(C, f) \Leftrightarrow$ if there is $d \in \mathbb{N}$ such that d/n. For each $x \in C$, θ_d is the smallest congruence containing the pair $(x, f^d(x))$.

Sufficient condition - general case

Notice:

$$(x, f^j(x)) \in \alpha \restriction C_i, \text{ for each } x \in C_i, j \in \mathbb{N} \Leftrightarrow d_i/j.$$

Sufficient condition - general case

Notice:

$$(x, f^j(x)) \in \alpha \restriction C_i, \text{ for each } x \in C_i, j \in \mathbb{N} \Leftrightarrow d_i / j.$$

- the set of all d_i is finite $\{d_1, d_2, \ldots, d_s\}$ and let $\{1, 2, \ldots, s\} \subseteq J$,
- d the greatest common divisor of $d_1, d_2, \ldots, d_s$,

Sufficient condition - general case

Notice:

$$(x, f^j(x)) \in \alpha \restriction C_i, \text{ for each } x \in C_i, j \in \mathbb{N} \Leftrightarrow d_i/j.$$

- the set of all d_i is finite $\{d_1, d_2, \ldots, d_s\}$ and let $\{1, 2, \ldots, s\} \subseteq J$,
- d the greatest common divisor of $d_1, d_2, \ldots, d_s$,

Notice:

$$(f^l(c_i), f^k(c_i)) \in \theta(c_i, f^d(c_i)), \text{ for } d, l, k \in \mathbb{N} \Leftrightarrow d/l - k.$$

Sufficient condition - general case

Notice:

$$(x, f^j(x)) \in \alpha \upharpoonright C_i, \text{ for each } x \in C_i, j \in \mathbb{N} \Leftrightarrow d_i/j.$$

- the set of all d_i is finite $\{d_1, d_2, \ldots, d_s\}$ and let $\{1, 2, \ldots, s\} \subseteq J$,
- d the greatest common divisor of $d_1, d_2, \ldots, d_s$,

Notice:

$$(f^l(c_i), f^k(c_i)) \in \theta(c_i, f^d(c_i)), \text{ for } d, l, k \in \mathbb{N} \Leftrightarrow d/l - k.$$

Lemma

There exist positive integers $q_1, q_2, \ldots, q_s$ and q such that

$$1 + qn = q_1 \frac{d_1}{d} + q_2 \frac{d_2}{d} + \cdots + q_s \frac{d_s}{d}.$$

Denotation of complement β

Denotation of complement β

- $\gamma = \{(f^k(c_i), f^k(c_j)) : i, j \in J, k \in \mathbb{N}\}, \qquad \gamma \in \mathrm{Quord}\,(A, f),$

Sufficient condition - general case

Denotation of complement β

- $\gamma = \{(f^k(c_i), f^k(c_j)) : i, j \in J, k \in \mathbb{N}\}$, $\quad \gamma \in \mathrm{Quord}\,(A, f)$,
- $\alpha'_i = \theta(c_i, f^d(c_i)) \vee \alpha_i$, $i \in J$

$\alpha' = \bigcup_{j \in J} \alpha'_i$, $\alpha' \in \mathrm{Quord}\,(A, f)$ and $r_{\alpha'} = r_\alpha$,
(example)

Denotation of complement β

- $\gamma = \{(f^k(c_i), f^k(c_j)) : i, j \in J, k \in \mathbb{N}\}, \qquad \gamma \in \mathrm{Quord}\,(A, f),$
- $\alpha'_i = \theta(c_i, f^d(c_i)) \vee \alpha_i,\ i \in J$

  $\alpha' = \bigcup_{j \in J} \alpha'_i$, $\alpha' \in \mathrm{Quord}\,(A, f)$ and $r_{\alpha'} = r_\alpha$,
  (example)

By the previous results there exists

- β'_i ... a complement of α'_i in $\mathrm{Quord}\,(A_i, f)$,

Denotation of complement β

- $\gamma = \{(f^k(c_i), f^k(c_j)) : i, j \in J, k \in \mathbb{N}\}$, $\gamma \in \mathrm{Quord}\,(A, f)$,
- $\alpha_i' = \theta(c_i, f^d(c_i)) \vee \alpha_i$, $i \in J$

$\alpha' = \bigcup_{j \in J} \alpha_i'$, $\alpha' \in \mathrm{Quord}\,(A, f)$ and $r_{\alpha'} = r_\alpha$,
(example)

By the previous results there exists

- β_i' ... a complement of α_i' in $\mathrm{Quord}\,(A_i, f)$,
- $\beta_i' \upharpoonright C_i = \theta(c_i, f^{\frac{n}{d}}(c_i))$, from construction.

(example)

Lemma

Let $i \in J$, $l, k \in \mathbb{N}$. Then $(f^l(c_i), f^k(c_i)) \in \alpha_i \vee \beta_i'$ if and only if $\frac{d_i}{d} / l - k$.

Denotation of complement β

Sufficient condition - general case

Denotation of complement β

$$\beta = \gamma \ \vee \ \bigvee_{j \in J} \beta'_j,$$

$\beta \in \mathrm{Quord}\,(A, f).$

*(*example)

Denotation of complement β

$$\beta = \gamma \ \vee \ \bigvee_{j \in J} \beta'_j,$$

$\beta \in \mathrm{Quord}\,(A, f).$

(example)

??β is a complement of α in $\mathrm{Quord}\,(A, f)$??

Sufficient condition - general case

Meet

Lemma

If $(x, y) \in \alpha \wedge \beta$*, then* $x = y$*.*

Sufficient condition - general case

Meet

Lemma

If $(x, y) \in \alpha \wedge \beta$, *then* $x = y$.

Proof:

Let $(x, y) \in \alpha \wedge \beta$, $x \neq y$;

$\rightarrow$ $(x, y) \in \alpha$, there is $i \in J$ such that $(x, y) \in \alpha_i'$

$(x, y \in A_i,\ (x, y) \in \alpha_i)$,

Meet

Lemma

If $(x,y) \in \alpha \wedge \beta$, *then* $x = y$.

Proof:

Let $(x,y) \in \alpha \wedge \beta$, $x \neq y$;

$\rightarrow (x,y) \in \alpha$, there is $i \in J$ such that $(x,y) \in \alpha_i'$

$(x, y \in A_i,\ (x,y) \in \alpha_i)$,

$\rightarrow (x,y) \in \beta$, $\beta = \gamma \ \vee \ \bigvee_{j \in J} \beta_j'$

Sufficient condition - general case

Meet

Lemma

If $(x,y) \in \alpha \wedge \beta$, *then* $x = y$.

Proof:

Let $(x,y) \in \alpha \wedge \beta$, $x \neq y$;

$\rightarrow (x,y) \in \alpha$, there is $i \in J$ such that $(x,y) \in \alpha'_i$

$(x, y \in A_i,\ (x,y) \in \alpha_i)$,

$\rightarrow (x,y) \in \beta$, $\beta = \gamma \ \vee \ \bigvee_{j \in J} \beta'_j$

$\searrow (x,y) \notin \beta'_i$, β'_i is a complement to α'_i

($\alpha_i \cap \beta'_i = \alpha'_i \cap \beta'_i$, + assumption $x \neq y$),

Sufficient condition - general case

Meet

Lemma

If $(x, y) \in \alpha \wedge \beta$, *then* $x = y$.

Proof:

Let $(x, y) \in \alpha \wedge \beta$, $x \neq y$;

$\rightarrow (x, y) \in \alpha$, there is $i \in J$ such that $(x, y) \in \alpha'_i$

$(x, y \in A_i, (x, y) \in \alpha_i)$,

$\rightarrow (x, y) \in \beta$, $\beta = \gamma \ \vee \ \bigvee_{j \in J} \beta'_j$

$\searrow (x, y) \notin \beta'_i$, β'_i is a complement to α'_i

($\alpha_i \cap \beta'_i = \alpha'_i \cap \beta'_i$, + assumption $x \neq y$),

$\searrow$ there is the shortest chain $x = u_0, u_1, \ldots, u_m = y$, $m > 1$;

either for any k, $(u_k, u_{k+1}) \in \gamma$ or $(u_k, u_{k+1}) \in \bigvee_{j \in J} \beta'_j$.

Meet

Lemma

If $(x, y) \in \alpha \wedge \beta$, then $x = y$.

Proof:

Let $(x, y) \in \alpha \wedge \beta$, $x \neq y$;

$\rightarrow (x, y) \in \alpha$, there is $i \in J$ such that $(x, y) \in \alpha'_i$

$(x, y \in A_i,\ (x, y) \in \alpha_i)$,

$\rightarrow (x, y) \in \beta$, $\beta = \gamma \vee \bigvee_{j \in J} \beta'_j$

$\searrow (x, y) \notin \beta'_i$, β'_i is a complement to α'_i

$(\alpha_i \cap \beta'_i = \alpha'_i \cap \beta'_i$, + assumption $x \neq y)$,

$\searrow$ there is the shortest chain $x = u_0, u_1, \ldots, u_m = y$, $m > 1$;

either for any k, $(u_k, u_{k+1}) \in \gamma$ or $(u_k, u_{k+1}) \in \bigvee_{j \in J} \beta'_j$.

Notice: $u_0, u_1, \ldots, u_m$ are distinct and if $(u_k, u_{k+1}) \in \gamma$,

then $(u_{k+1}, u_{k+2}) \in \bigvee_{j \in J} \beta'_j$ (similarly for the second possibility).

Sufficient condition - general case (meet)

$$\boxed{\beta = \gamma \ \vee \ \bigvee_{j \in J} \beta'_j}$$

For each k there is $i_k \in J$ with $u_k \in A_{i_k}$. From the definition of β we get:

$$(u_k, u_{k+1}) \in \gamma \Rightarrow u_k = f^{t_k}(c_{i_k}), u_{k+1} = f^{t_{k+1}}(c_{i_{k+1}}),$$

$$i_k \neq i_{k+1}, t_k = t_{k+1},$$

$$(u_k, u_{k+1}) \in \beta'_j \implies i_k = i_{k+1},$$

$$u_k = f^{t_k}(c_{i_k}), u_{k+1} = f^{t_{k+1}}(c_{i_{k+1}}), (u_k, u_{k+1}) \in \beta'_j \Rightarrow i_k = j,$$

$$\frac{n}{d} / t_k - t_{k+1}.$$

We have either

$x = u_0 \ \gamma \ u_1 \ \beta'_j \ u_2 \ \gamma \ u_3 \ \ldots$ *or* $x = u_0 \ \beta'_j \ u_1 \ \gamma \ u_2 \ \beta'_j \ u_3 \ \ldots .$

Sufficient condition - general case (meet)

Assume $x = u_0 \; \beta'_j \; u_1 \; \gamma \; u_2 \; \beta'_j \; u_3 \; \ldots$ and that

Sufficient condition - general case (meet)

Assume $x = u_0 \; \beta'_j \; u_1 \; \gamma \; u_2 \; \beta'_j \; u_3 \; \ldots$ and that $u_{m-1} \in A_i$, m is odd.
There exists a positive integer t_k;

$$u_k = f^{t_k}(c_{i_k}), \text{ for each } 0 < k \leqq m$$

(definition of γ)

In view of above,

$$t_1 = t_2, \; \tfrac{n}{d}/t_2 - t_3, \; t_3 = t_4, \; \tfrac{n}{d}/t_4 - t_5, \; \ldots, \; t_{m-2} = t_{m-1}.$$

Then

$$\frac{n}{d}/(t_1 - t_2) + (t_2 - t_3) + (t_4 - t_5) + \cdots + (t_{m-3} - t_{m-2}) + (t_{m-2} - t_{m-1}) =$$

$$= t_1 - t_{m-1},$$

hence $(u_1, u_{m-1}) \in \beta'_{i_0}$ and $(u_0, u_1) \in \beta'_{i_0}$,$(u_{m-1}, u_m) \in \beta'_{i_0}$

$(x, y) = (u_0, u_m) \in \beta'_{i_0}$, a contradiction.

$\diamond$

Sufficient condition - general case

Join

Lemma

$\alpha \vee \beta = A \times A.$

Sufficient condition - general case

Join

Lemma

$\alpha \vee \beta = A \times A.$

Proof:

?? $(x, y) \in \alpha \vee \beta$ for every $x, y \in A$?? i.e.

?? there are $m \in \mathbb{N} \cup \{0\}$ and a chain of elements $x = u_0, u_1, u_2, \dots,$ $u_m = y \in A$ such that either $(u_k, u_{k+1}) \in \gamma$ or $(u_k, u_{k+1}) \in \alpha_j \vee \beta'_j$ for some $j \in J$ is valid for each $0 \leq k < m$??
Assume that $x \neq y$. We will investigate:

1. $x \in C_1$, $y = f(x)$,
2. $i \in J$, $x, y \in C_i$,
3. $i \in J$, $x \in A_i$, $y \in C_i$, (and symmetric case)
4. $i, j \in J$, $x \in A_i$, $y \in A_j$.

and we will use the previous cases for the proof of a new one.

◇

Complementarity - main result

~~HYPOTHESIS~~
THEOREM

$$(\text{Quord } \mathcal{A}, \subseteq) \quad \xrightarrow{necessary} \quad \text{conditions for}$$

complementary lattice $\xleftarrow{sufficient}$ conditions for $\mathcal{A} = (A, f)$

1. each connected component of $\mathcal{A}$ contains a cycle,
2. there is $n \in N$ such that each cycle of $\mathcal{A}$ has n elements,
3. n is square-free,
4. for each $a \in A$, the element $f(a)$ is cyclic.

Complementarity - main result

Theorem

Let (A, f) *be a monounary algebra. Then the conditions*

- *each connected component of* (A, f) *contains a cycle,*
- *there is* $n \in \mathbb{N}$ *such that each cycle of* (A, f) *has* n *elements,*
- n *is square-free,*
- *for each* $a \in A$*, the element* $f(a)$ *is cyclic*

are **necessary** *and* **sufficient** *for the lattice* $\mathrm{Quord}\,(A, f)$ *to be complementary.*

Complementarity - main result

Theorem

Let (A, f) be a monounary algebra. Then the conditions

- *each connected component of (A, f) contains a cycle,*
- *there is $n \in \mathbb{N}$ such that each cycle of (A, f) has n elements,*
- *n is square-free,*
- *for each $a \in A$, the element $f(a)$ is cyclic*

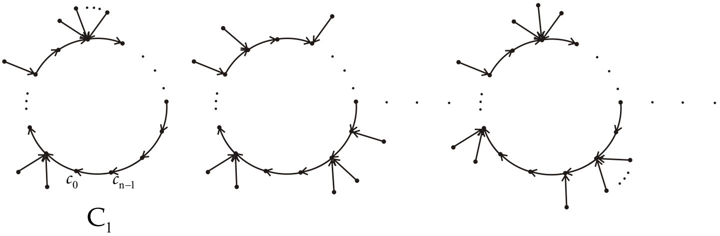
are **necessary** *and* **sufficient** *for the lattice $\mathrm{Quord}\,(A, f)$ to be complementary.*

Theorem

Let (A, f) be a monounary algebra. The lattice $\mathrm{Quord}\,(A, f)$ is Boolean if and only if either $|A| \leq 2$ or (A, f) is connected with a cycle C of (A, f) such that $|A| \leq |C| + 1$ and $|C|$ is square-free.

The picture of a monounary algebra $\mathcal{A} = (A, f)$

$\mathcal{A} = (A, f)$

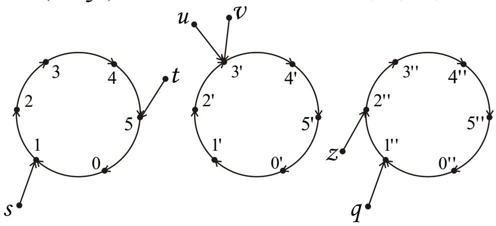


which satisfies conditions:

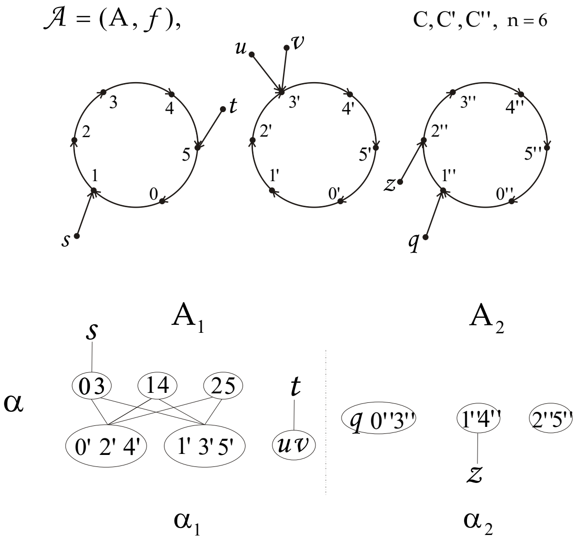
1. each connected component of $\mathcal{A}$ contains a cycle,
2. there is $n \in N$ such that each cycle of $\mathcal{A}$ has n elements,
3. n is square-free,
4. for each $a \in A$, the element $f(a)$ is cyclic.

Example

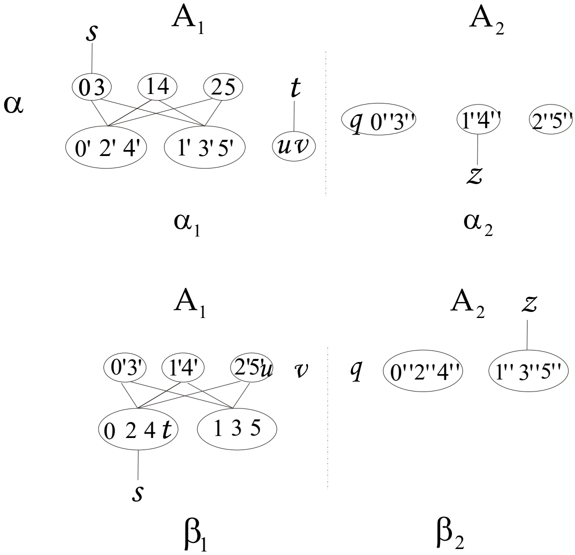
$\mathcal{A} = (A, f)$, C, C', C'', $n = 6$



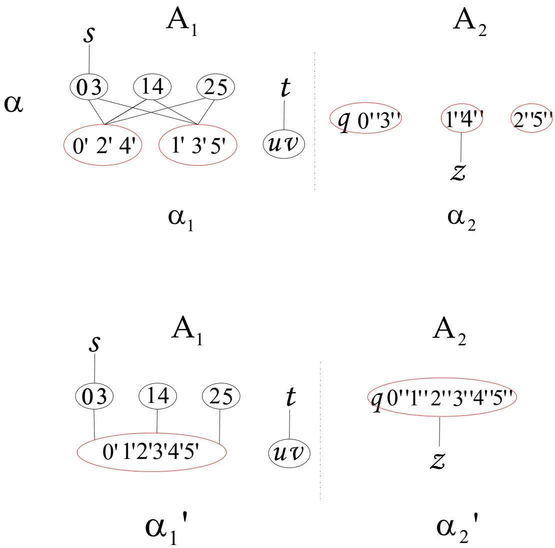
Example



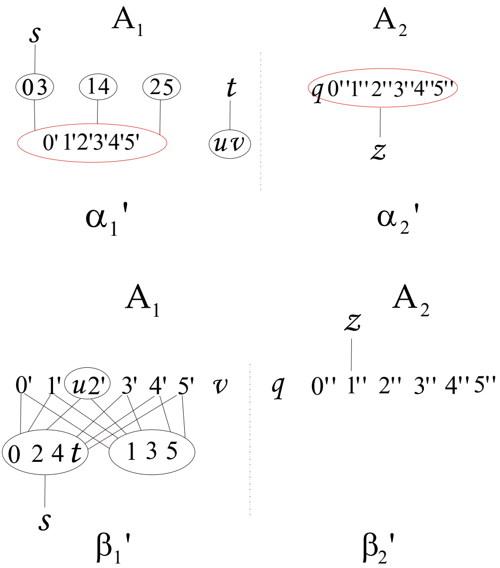
$\mathcal{A} = (A, f)$,
C, C', C'', n = 6
u
v
t
s
z
q
3
4
2
5
1
0
3'
4'
2'
5'
1'
0'
3''
4''
2''
5''
1''
0''
A_1
A_2
α
s
03
14
25
t
0' 2' 4'
1' 3' 5'
uv
q 0''3''
1''4''
2''5''
z
α_1
α_2


A1
A2
s
03
14
25
t
α
0' 2' 4'
1' 3' 5'
uv
q 0''3''
1''4''
2''5''
z
α1
α2
A1
A2
z
0'3'
1'4'
2'5'u
v
q
0''2''4''
1'' 3''5''
0 2 4 t
1 3 5
s
β1
β2

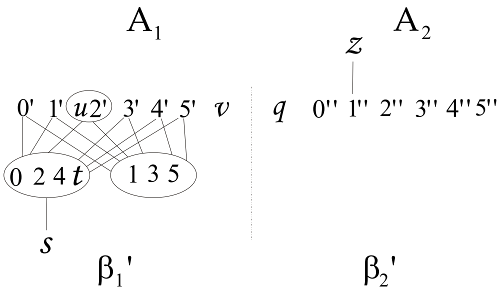
Example: $\alpha_i' = \theta(c_i, f^d(c_i)) \vee \alpha_i$, $i \in J$



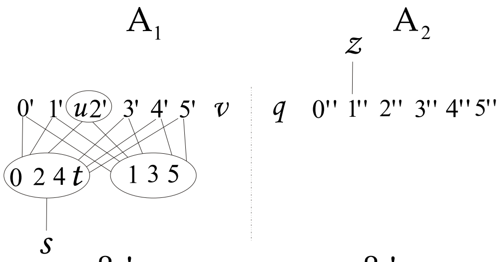
Example: α_i' and β_i'



Example: β_i' and β

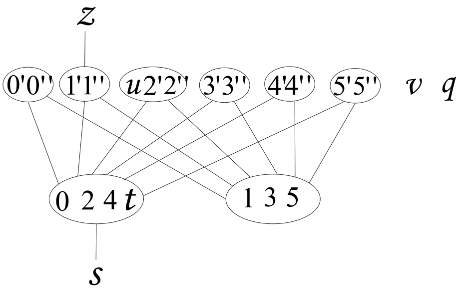


Example: β_i' and β



β_1' β_2'

β



Example: β complement to the α