Graphical Algebras – A New Approach to Congruence Lattices

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Congruence Lattices and Partition Lattices – History

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- 1970 H. Werner studies the question of which sublattices of a partition lattice on a set X are the sublattices of all congruences for some algebra on X. In his solution, he introduces a new operation on partition lattices, called graphical composition.

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- 1985 M. Haiman uses the notion of graphical composition in the case of series-parallel graphs to study lattices with type-1 embeddings (a notion introduced by B. Jónsson in 1953).

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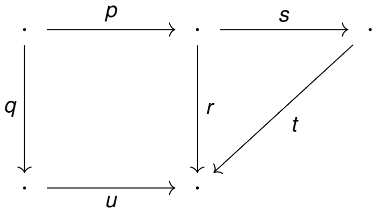
- 2008 V. Repnitskii and J. Tůma show that any algebraic lattice with at most countably many generators is an interval in the subgroup lattice of a locally finite countable group.
- The question of whether every finite lattice is the congruence lattice of a finite algebra is still open.

Graphical Composition of Relations

Given a graph with the edges labelled by relations on a set X:



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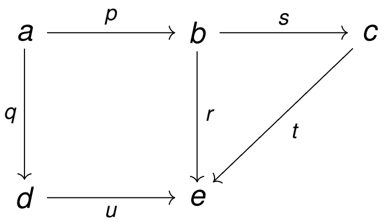

We call a labelling of the vertices by elements of X compatible if the two endpoints of any edge are related by the label of that edge.

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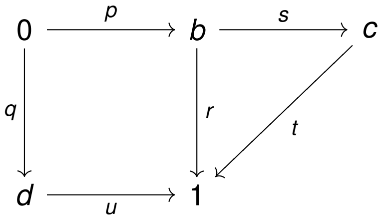

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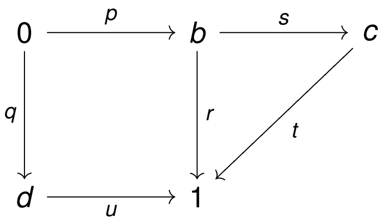

Now, if we have 2 distinguished vertices 0 and 1 of our graph:

We can define a new relation by taking the set of pairs (x, y) such that there is a compatible vertex-labelling which labels 0 with x and 1 with y. So in the above diagram, the relation would relate a and e. (And possibly other elements as well.)

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When the relations used to name the edges are equivalence relations:

- we can ignore loops, since they are all reflexive
- and we can take undirected graphs, since they are all symmetric.
- If we want the result to also be an equivalence relation, then we need to take the symmetric transitive closure of the relation we obtain.

Werner's Results

- Werner showed that a sublattice of a partition lattice is a congruence lattice if and only if it is closed under graphical composition and arbitrary joins.

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- Werner showed that a sublattice of a partition lattice is a congruence lattice if and only if it is closed under graphical composition and arbitrary joins.
- In fact, he showed that it is sufficient to check closure under the graph with vertices elements of X and the edge xy labelled by the smallest equivalence relation in L that relates x and y.

Abstract Graphical Composition

We will consider this notion of graphical composition as extra structure on a lattice, namely a collection of additional operations on the lattice. The operation c going from L-labelled graphs to L.

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We will then ask what additional equations these new operations must satisfy to be the congruences of an algebra.

Necessary Conditions I

- c is order-preserving in the edge-wise partial order on L-labelled graphs.
- If $G_1 \bullet G_2$ is the series composition of G_1 and G_2, then $c(G_1 \bullet G_2) = c(G_1) \vee c(G_2)$.
- If $G_1 /\!\!/ G_2$ is the parallel composition of G_1 and G_2, then $c(G_1 /\!\!/ G_2) \leqslant c(G_1) \wedge c(G_2)$.
- if $/\!\!/_{i \in I} G_i$ is the parallel composition of a collection of graphs, $\{G_i | i \in I\}$, then $c(/\!\!/_{i \in I} G_i) \leqslant \bigwedge_{i \in I} c(G_i)$.
- If $c(G) \leqslant x$, and $G \overset{f}{\rightarrowtail} \hat{G}$ is an embedding of graphs, not necessarily preserving 0 and 1, then $c(\hat{G}) = c(\hat{G} /\!\!/_G x)$, where $\hat{G} /\!\!/_G x$ is the graph formed from $\hat{G}$ by adding an edge between $f(0)$ and $f(1)$, labelled by x.

Necessary Conditions II

- If e is an edge of G labelled by $\bot$, then $c(G) = c(G/e)$, where G/e is the contraction of G along e.
- If e is an edge of G labelled by $\top$, then $c(G) = c(G \backslash e)$, where $G \backslash e$ is the deletion of e from G.
- If G has a set of parallel edges labelled x_i for $i \in I$, then if G' is the graph obtained from G by replacing these edges by a single edge with the same endpoints, labelled $\bigwedge_{i \in I} x_i$, then $c(G) = c(G')$.
- If G^{op} is the same graph as G but with 0 and 1 switched, then $c(G^{\mathrm{op}}) = c(G)$.
- For any graphs G_1, G_2,

$$c(G_1) \wedge c(G_2) = \bigvee_{n \in \mathbb{N}} c((G_1 \bullet G_1^{\mathrm{op}})^{\bullet n} /\!\!/ (G_2 \bullet G_2^{\mathrm{op}})^{\bullet n})$$

Flexible Graphs

Let G be a complete graph with a labelling function $l : E(G) \longrightarrow L$. For any pair of vertices i, j of G, we will let $G_{i,j}$ be the same graph as G, but with chosen vertices $0 = i$, $1 = j$.

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Let G be a complete graph with a labelling function $l : E(G) \longrightarrow L$. For any pair of vertices i, j of G, we will let $G_{i,j}$ be the same graph as G, but with chosen vertices $0 = i$, $1 = j$.

We will say that G is flexible if for any two vertices i, j of G, we have that $l(ij) = c(G_{i,j})$.

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- Now we can show that $c(G) = l_R(01)$, where l_R is the labelling function from the edges of R to L, and 0 and 1 are the vertices of R corresponding to the distinguished vertices 0 and 1 of G.

Flexible Graph Operations

The collection of flexible graphs is closed under:

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These are sufficient to also give closure under:

- Products
- Induced subgraphs

The Importance of Flexible Graphs

Theorem

The notion of graphical composition obtained from a collection of flexible graphs satisfying the triangle inequality, and closed under uncontraction, join and $\perp$-contraction satisfies all the properties that we listed for c, except for the last one.

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Graphical Algebras

Definition

A graphical algebra is a complete lattice L with an operation c from the collection of all graphs with edges labelled by elements of L, to L, such that the collection of flexible graphs for c is closed under uncontraction, join and $\perp$-contraction, all flexible graphs satisfy the triangle inequality, and the final condition

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is satisfied.

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This means that a graphical algebra can only be the congruences of a finite algebra if it has a finite bound.

Distributive 2-Bounded Graphical Algebra

Proposition

For a lattice L, the assignment

$$c(G) = \bigvee_{\text{cuts}\chi} \left(\bigwedge_{x \in \chi} l(x) \right)$$

is a graphical algebra if and only if L is distributive.

Proposition

Any completely distributive lattice admits a unique graphical algebra structure.

Canonical Graphical Algebra on Any Lattice

Theorem

Every lattice admits a graphical algebra, given by choosing the flexible graphs to be complete graphs with the property that the shortest path between any two vertices is always the edge between them.

Bounded Graphical Algebras from General Graphical Algebras

Theorem

If L is a graphical algebra, and α is a cardinal satisfying $\alpha = \alpha + \alpha$, then there is a graphical algebra L_α whose flexible graphs are generated by the flexible graphs of L with at most α vertices, under uncontraction and join.

A New Proof of Gratzer and Schmidt's Result

Theorem

For an appropriately chosen α the canonical α-bounded graphical algebra on any algebraic lattice can be represented as the graphical algebra of congruences on an algebra.

Questions

- Which graphical algebras are representable as congruence graphical algebras? Can we simplify the graphical algebra conditions we have?
- Is there an efficient way to determine whether a given assignment of graphical composites is actually a graphical algebra?
- Which finite, finitely-bounded graphical algebras (call these *finitary* graphical algebras) are congruences of a finite algebra?
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- Which graphical algebras can be represented by special types of algebras – e.g. G-sets?
- What sorts of algebras correspond to special types of graphical algebras, such as ones with small bounds?
- What is the logic of graphical algebras?
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Now, the vertex-labellings of a graph G, used to define graphical composition by Werner, correspond exactly to homomorphisms from G to G_X.

Relative Flexibility

Definition

For graphs G and H, say G is H-flexible if for any edges e of G and e' of H with $l(e') \leqslant l(e)$, there is a sequence of homomorphisms $(G \xrightarrow{f_i} H)_{i=1,\ldots,n}$ such that the images $f_i(e)$ form a path between the endpoints of e'.

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Definition

We say that a collection $\mathcal{G}$ of graphs is *H-flexible* if for an edge e' of H and a of edges $\{e_i \in G_i \in \mathcal{G} | i \in I\}$ for an indexing set I, such that $l(e') \leqslant \bigvee_{i \in I} l(e_i)$, there is a sequence of functions $G_{i_j} \xrightarrow{f_j} H$ such that $f_j(e_{i_j})$ forms a path between the endpoints of e.

The Canonical Graph Characterises Flexibility

Theorem

If G is flexible for the congruence graphical algebra of X, then it is G_X-flexible.

Relative Flexibility Characterises the Canonical Graph

Theorem

Given a flexible graph H such that any flexible G is H-flexible, the graphical algebra L is the congruence graphical algebra of the algebra with underlying set vertices of H, and unary operations endomorphisms of vertices of H that are labelled graph homomorphisms (i.e. decreasing on the labels of edges).

A Simpler Condition to Check

Theorem

Let $\mathcal{G}$ be a collection of graphs such that every flexible graph is a join of uncontractions of graphs in $\mathcal{G}$. If H is a flexible graph such that $\mathcal{G}$ is H-flexible, then L is the congruence graphical algebra of the algebra corresponding to H.

Constructing this graph

Start with any flexible graph H_0.
For any compactly-labelled edge e of H_0, and any set of edges $\{e_i \in G_i | i \in I\}$, such that $l(e) \leqslant \bigvee l(e_i)$, we find a finite subset $J \subseteq I$ such that $l(e) \leqslant_{i \in J} l(e_i)$.

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Now if we let K be the series composition of $\{G_j | j \in J\}$ along the edges $\{e_j | j \in J\}$, we know that there is some n such that $K^{\bullet n} /\!\!/ H_0$ along e can be extended to a flexible graph.

We will now take the directed union of all these extensions, to get a graph H_1, such that for any edge e of H_0, and any set of edges $\{e_i \in G_i | i \in I\}$, such that $l(e) \leqslant \bigvee l(e_i)$, we can find a path in H_1 between the endpoints of e, such that each edge of this path is the image of an edge e_i under a homomorphism $G_i \longrightarrow H_1$.

We repeat this construction, starting from H_1, to get a sequence $H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots$. We let H be the directed union of this sequence.

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Given any edge e of H, and any set of edges $\{e_i \in G_i | i \in I\}$, such that $l(e) \leqslant \bigvee l(e_i)$, we can find a path in H between the endpoints of e, consisting entirely of images of e_i.

Problem

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In fact, our graphical algebra is the graphical algebra of congruences of H for the partial algebra of all partial operations that preserve all the equivalence relations induced by its elements.

Indeed we can restrict to just the partial operations whose domain has at least n elements, for any $n \in \mathbb{N}$.