Quasiorders on monounary algebras

Danica Jakubíková-Studenovská, Košice
Reinhard Pöschel, Dresden
Sándor Radeleczki, Miskolc

SSAOS, Stará Lesná
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Introduction

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- an algebra $\mathcal{A}$

the lattice $\operatorname{Quord}\mathcal{A}$ **of all quasiorders of an algebra** $\mathcal{A}$

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- **quasiorder of** $\mathcal{A}$ $=$ a binary relation on $\mathcal{A}$, which is
  - reflexive
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  - compatible with all fundamental operations of $\mathcal{A}$
- common generalization of congruences and compatible partial orders of an algebra

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- $u \in L$ is $\wedge$-**irreducible** if

$$u = v_1 \ \wedge \ v_2 \implies (u = v_1 \ \vee \ u = v_2)$$

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  - Find some **necessary conditions** under which a quasiorder on (A, f) is meet-irreducible in the lattice $\mathrm{Quord}\,(A, f)$
  - Characterize **quasilinear** quasiorders on (A, f) which are meet-irreducible in the lattice $\mathrm{Quord}\,(A, f)$

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- $[c]_{\ker f}$ consists of two disjoint sets C_1, C_2 (one of them can be empty):

  for $a \in C_i$, $b \in C_j$, $i, j \in \{1, 2\}$

$$a \lesssim_q b \iff i \leq j$$

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Lemma

Assume $(\vartriangle)$

- $a \gtrsim_q f(a) \implies a \sim_q f(a) \sim_q 0$

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Lemma

Assume $(\vartriangle)$

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- $a \lesssim_q f(a) \implies a \prec_q f(a)$ *or* $a \sim_q f(a) \sim_q 0$

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Definition

$\lambda \in \mathrm{Lord}\,(A, f)$ is called an f-**chain** if either λ or λ^{-1} is equal to the transitive hull of $f^{\bullet} \cup \Delta$.

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Theorem

$\lambda \in \mathrm{Lord}\,(A, f)$ is meet-irreducible in the lattice $\mathrm{Quord}\,(A, f)$ if and only if λ is an f-chain.

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$\Rightarrow \qquad n \prec_\lambda n-1 \prec_\lambda \cdots \prec_\lambda 2 \prec_\lambda 1 \prec_\lambda 0$

x in λ precedes the only element, it is $f(x)$

$f(m) = m - 1$ for $m > 0$

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$\Leftarrow$ $\quad \lambda$ linear, an f-chain $\Rightarrow A = \{0, \ldots, n\}$,
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$i - i = f^i(i) \lesssim_q f^i(j) = j - i, \; j - i \lesssim_q 1$

$0 \lesssim_q 1 \quad \square$