Sectional switching mappings in semilattices

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Outline

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Introduction

It was shown in [Chajda I., Emanovský P.] that a certain algebra can be derived from a bounded lattice having an antitone involution on every section. More generally, consider a $\vee$-semilattice $\mathscr{S} = (S, \vee)$ with a greatest element 1. An interval $[a, 1]$ for $a \in S$ is called a section. A mapping f of $[a, 1]$ into itself is called a switching mapping if $f(a) = 1$, $f(1) = a$ and for $x \in [a, 1]$, $a \neq x \neq 1$ we have $a \neq f(x) \neq 1$. We study $\vee$-semilattices with switching mappings on all the sections. If for $p, q \in S$, $p \leq q$, the mapping on the section $[q, 1]$ is determined by that of $[p, 1]$, we say that the compatibility condition is satisfied. We will get conditions for antitony of switching mappings and a connection with complementation in sections will be shown.

Outline

Basic concepts

A mapping $f : A \to A$ is an **involution** whenever $f(f(x)) = x$ for all $x \in A$.

A mapping f of an ordered set $(A, \leq)$ into itself is **antitone** provided $x \leq y$ implies $f(y) \leq f(x)$.

Let $\mathscr{L} = (L, \vee, \wedge, 1)$ be a from above bounded lattice. For each $a \in L$ we call the interval $[a, 1]$ a **section** of $\mathscr{L}$.

The lattice $\mathscr{L}$ is called **sectionally involuted lattice** if for each $a \in L$ there exists an antitone involution $x \mapsto x^a$ on the section $[a, 1]$.

Basic concepts

Remark.

(a) For each $a \in L$ the mapping $x \mapsto x^a$ is a dual isomorphism of the interval $[a, 1]$; thus, for all $x, y \in [a, 1]$, De Morgan laws

$$(x \vee y)^a = x^a \wedge y^a \quad \text{and} \quad (x \wedge y)^a = x^a \vee y^a$$

are satisfied,

(b) For each $a \in L$ we have $a^a = 1$, $1^a = 1$,

(c) If $\mathscr{L}$ is bounded, i.e. $L = [0, 1]$, there exists an antitone involution $x \mapsto x^0$ on the whole $\mathscr{L}$ and $0^0 = 1$ and $1^0 = 0$.

Outline

Bounded lattices with sectional antitone involutions

Proposition

Let $\mathscr{L} = (L, \vee, \wedge, 0, 1)$ be a bounded sectionally involuted lattice and let $x \cdot y$ be defined by the rule $x \cdot y := (x \vee y)^y$. Then the following identities are satisfied for all $x, y, z \in L$:

(1) $1 \cdot x = x, \quad x \cdot 1 = 1, \quad 0 \cdot x = 1,$

(2) $(x \cdot y) \cdot y = (y \cdot x) \cdot x,$

(3) $(((x \cdot y) \cdot y) \cdot z) \cdot (x \cdot z) = 1.$

Outline

Semilattice with sectional mappings

Let $\mathscr{S} = (S, \vee, 1)$ be a join-semilattice with the greatest element 1 where for each $a \in S$ there is a mapping on the section $[a, 1]$; such a structure will be called a **semilattice with sectional mappings.**

Let $\mathscr{S} = (S, \vee, 1)$ be a semilattice with sectional mappings. Define the so-called **induced operation** on S by the rule

$$x \cdot y = (x \vee y)^y$$

Evidently, "$\cdot$" is everywhere defined binary operation on S since $x \vee y \in [y, 1]$ for any $x, y \in S$. Also conversely, if "$\cdot$" is induced on S then for each $a \in S$ and $x \in [a, 1]$ we have

$$x \cdot a = (x \vee a)^a = x^a$$

Semilattice with sectional mappings

Lemma

Let $\mathscr{S} = (S, \vee, \cdot, 1)$ *be a semilattice with sectional involutions. The following conditions are equivalent for* $a \in S$*:*

(a) $x \mapsto x^a$ *is antitone,*

(b) *the section* $[a, 1]$ *is a lattice where*

$x \wedge_a y = (x^a \vee y^a)^a$ (*De Morgan law*).

Switching mappings

We say that a mapping $x \mapsto x^a$ on the section $[a, 1]$ is **weakly switching** if $a^a = 1$ and $1^a = a$. In other words, a weakly switching mapping "switch" the bound elements of the section.

Lemma

Let $\mathscr{S} = (S, \vee, \cdot, 1)$ be a semilattice with sectional mappings.
(a) *If for each $a \in S$ the sectional mapping $x \mapsto x^a$ is an involution then the induced operation satisfies the identity*
(I1) $(x \cdot y) \cdot y = (y \cdot x) \cdot x = x \vee y$
(b) *If for each $a \in S$ the sectional mapping $x \mapsto x^a$ is weakly switching and the induced operation satisfies* (I1) *then every sectional mapping is an involution.*

Switching mappings

A weakly switching mapping $x \mapsto x^a$ will be called a **switching mapping** if $a \neq x^a \neq 1$ for each $x \in [a,1]$ with $a \neq x \neq 1$.

Observation. Every join-semilattice $\mathscr{S} = (S, \vee, 1)$ with a greatest element can be considered as a semilattice with sectional switching mappings. One can take for each $a \in S$ and every $x \in [a,1]$ $a^a = 1$, $1^a = a$ and $x^a = x$ for $a \neq x \neq 1$. Hence, our concept is really universal and very natural for semilattices.

Lemma

Let $\mathscr{S} = (S, \vee, \cdot, 1)$ be a semilattice with sectional switching mappings, let $\leq$ be its induced order. Then

$$x \leq y \quad \text{if and only if} \quad x \cdot y = 1.$$

Switching mappings

Lemma

Let $\mathscr{S} = (S, \vee, \cdot, 1)$ *be a semilattice with sectional weakly switching mappings. Then* $\mathscr{S}$ *satisfies the identities*

(I2) $\quad x \cdot x = 1, \qquad 1 \cdot x = x, \qquad x \cdot 1 = 1.$

Theorem

Let $\mathscr{S} = (S, \vee, \cdot, 1)$ *be a semilattice with sectional switching mappings.*

(a) *If* $\mathscr{S}$ *satisfies the identity*

(I3) $(((x \cdot y) \cdot y) \cdot z) \cdot (x \cdot z) = 1$

then every switching mapping on $\mathscr{S}$ *is antitone.*

(b) *If every sectional switching mapping on* $\mathscr{S}$ *is an involution then it is antitone if and only if* $\mathscr{S}$ *satisfies* (I3).

Outline

The compatibility condition

We will consider a semilattice with sectional mappings where the mapping in a smaller section is determined by that of a bigger one. More precisely, we say that $\mathscr{S} = (S, \vee, \cdot, 1)$ satisfies the **compatibility condition** if

$p \leq q \leq x$ implies $x^q = x^p \vee q$. (CC)

It is easy to verify that (CC) can be equivalently expressed as the following identity

$(y \vee z) \cdot (x \vee y) = ((y \vee z) \cdot x) \vee (x \vee y)$. (CCI)

since $x \leq x \vee y \leq x \vee y \vee z$ and

$(y \vee z) \cdot (x \vee y) = (x \vee y \vee z)^{(x \vee y)}$

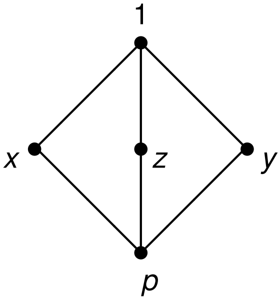
$(y \vee z) \cdot x = (x \vee y \vee z)^x$.

The compatibility condition

Let us also note that the compatibility condition is satisfied for complementation in any boolean semilattice [Abbott J. C.] and in any orthomodular semilattice [Abbott J. C.] , its modification holds also for semilattices with sectionally antitony involutions which are implication algebras for MV-algebras, see [Chajda I., Halaš R., Kühr J.]
We are going to show that (CC) does not imply neither antitony nor involutiveness of switching mappings.

The compatibility condition

Example: Let $\mathscr{S} = (\{p, x, y, z, 1\}, \vee, \cdot, 1)$ be a semilattice where the sectional mappings are given as follows:

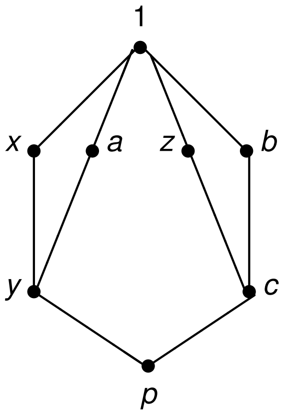


in $[p, 1]$: $x^p = y^p = z$, $z^p = y$, $p^p = 1$, $1^p = p$,
in $[x, 1]$: $x^x = 1$, $1^x = x$,
in $[y, 1]$: $y^y = 1$, $1^y = y$,
in $[z, 1]$: $z^z = 1$, $1^z = z$,
in $[1, 1]$: $1^1 = 1$.

One can easily verify that (CC) is satisfied by $\mathscr{S}$ and all the sectional mappings are antitone switching mappings. However, $a \mapsto a^p$ is not a bijection since $x \neq y$ but $x^p = y^p$.

The compatibility condition

Example: Let $\mathscr{S} = (\{p, a, b, c, x, y, z, 1\}, \vee, \cdot, 1)$ be a semilattice where



$x^p = a$, $y^p = c$, $z^p = b$, $a^p = x$,
$c^p = y$, $b^p = z$, $p^p = 1$, $1^p = p$,
$x^y = a$, $a^y = x$, $y^y = 1$, $1^y = y$,
$z^c = b$, $b^c = z$, $c^c = 1$, $1^c = c$,
$x^x = 1$, $1^x = x$,
$a^a = 1$, $1^a = a$,
$z^z = 1$, $1^z = z$,
$b^b = 1$, $1^b = b$, $1^1 = 1$.

The switching mappings satisfy (CC) but $v \mapsto v^p$ is not antitone since $y \leq x$ but $x^p = a$, $y^p = c$ are incomparable.

The compatibility condition

Lemma

Let $\mathscr{S} = (S, \vee, \cdot, 1)$ be a semilattice with sectional mappings satisfying the compatibility condition. Then
(a) $x \vee x^p = 1$ *for each* $p \in S$ *and each* $x \in [p, 1]$*;*
(b) *if* $z \mapsto z^p$ *is a switching mapping for* $p \neq 1$ *then* $x^p \neq x$ *and if* $x < y$ *then* $x^p \neq y^p$ *for each* $x, y \in [p, 1]$*;*
(c) *if all the sectional mappings are switching then no section of* $\mathscr{S}$ *can be a chain with more than two elements.*

The compatibility condition

Let us recall that a join semilattice $\mathscr{S} = (S, \vee, 1)$ with 1 where for $p \in S$ the section $[p, 1]$ is a lattice $([p, 1], \vee, \wedge_p)$ is called a **nearlattice** (the concept was introduced by [M. Sholander] in 1950-ies).

Theorem

Let $\mathscr{S} = (S, \vee, \cdot, 1)$ be a nearlattice with sectional switching mappings satisfying the compatibility condition. If $x \mapsto x^p$ is antitone on $[p, 1]$ then x^p is a complement of x for each $x \in [p, 1]$.

The compatibility condition

Theorem

Let $\mathscr{S} = (S, \vee, \cdot, 1)$ be a semilattice with sectionally antitone involutions satisfying the compatibility condition. Then for each $p \in S$ the section $[p, 1]$ is an orthomodular lattice where x^p is an orthocomplement of $x \in [p, 1]$.

Theorem

Let $\mathscr{S} = (S, \vee, \cdot, 1)$ be a semilattice with sectionally antitone involutions. If for $p \in S$ and each $x, y \in [p, 1]$ it holds

$$(x^p \vee y)^p \vee x^p = (y^p \vee x)^p \vee y^p \qquad (\star)$$

then $([p, 1], \vee, \wedge_p)$ is a Boolean algebra.

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Thank you for your attention.