Relative pseudocomplementation on directoids

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Outline

Directoids

Definition

By a directoid is meant a groupoid $\mathcal{D} = (D; \sqcap)$ satisfying identities

(D1) $x \sqcap x = x$,

(D2) $x \sqcap y = y \sqcap x$,

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Directoids are partially ordered sets

Every directoid $\mathcal{D} = (D; \sqcap)$ can be converted into an ordered set $(D; \leq)$ via

$$x \leq y \iff x \sqcap y = x$$

Conversely, every ordered set $(D; \leq)$ can be organized into a directoid taking

$$\begin{aligned} x \sqcap y = y \sqcap x \in L(x, y) &\iff x \parallel y, \\ x \sqcap y = y \sqcap x = x &\iff x \leq y. \end{aligned}$$

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Relative pseudocomplementation on semilattices

Let $(S; \wedge)$ be a $\wedge$-semilattice, $a, b \in S$. By a relative pseudocomplement of a with respect to b in S we mean the greatest element among $x \in S$ satisfying

$$a \wedge x \leq b.$$

Or equivalently (But only if it exists!), $a * b$ is the greatest element among $x \in S$ satisfying

$$a \wedge x = a \wedge b.$$

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There is a greatest $x \in N_5$ with $c \wedge x = c \wedge a$.

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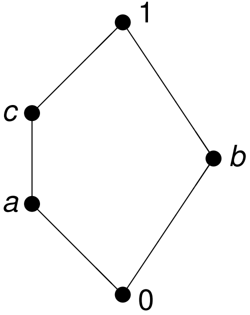

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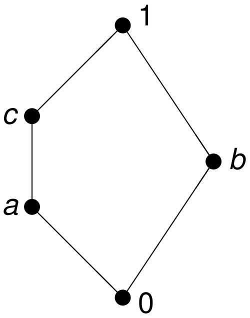

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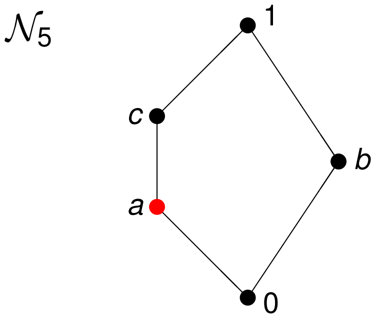

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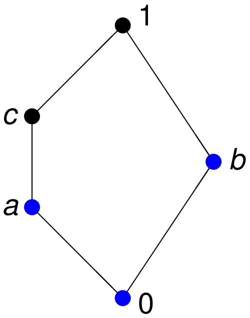

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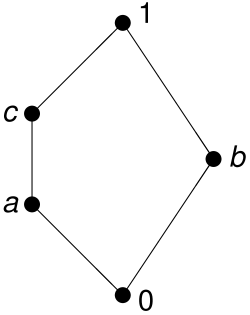

There is a greatest $x \in N_5$ with $c \wedge x = c \wedge a$.
$c * a$ does not exist.

Analogy for directoids?

In general,

$$x \leq y \quad \not\Rightarrow \quad x \sqcap z \leq y \sqcap z.$$

$\mathcal{D}$:

There is the greatest $x \in D$ such that $a \sqcap x \leq b$.

But c is not the greatest $x \in D$ such that $a \sqcap x = a \sqcap b$.

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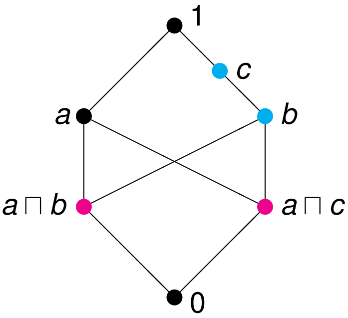

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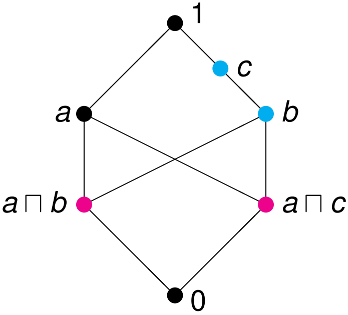

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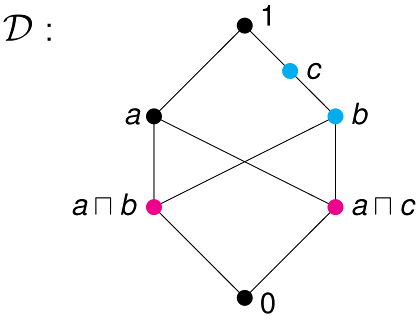

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Definition (by I. Chajda)

Let $\mathcal{D} = (D; \sqcap)$ be a directoid and $a, b \in D$. An element x is called a relative pseudocomplement of a with respect to b if it is a greatest element of D such that

$$a \sqcap x = a \sqcap b.$$

It is denoted by $a * b$.

A directoid $\mathcal{D}$ is relatively pseudocomplemented if there exists $a * b$ for every $a, b \in D$.

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Identities characterizing relative pseudocomplementation on directoids

Theorem

Let $(D; \sqcap)$ be a directoid and $*$ a binary operation on D. Then $\mathcal{D} = (D; \sqcap, *)$ is a relatively pseudocomplemented directoid if and only if it satisfies the following identities

(S1) $x \sqcap (x * y) = x \sqcap y$

(S2) $(x * y) \sqcap y = y$

(S3) $x * y = x * (x \sqcap y)$

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Axiom system of RPCD

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Let $\mathcal{D} = (D; \sqcap, *)$ be an algebra with two binary operations. Then $\mathcal{D}$ is a relatively pseudocomplemented directoid if and only if it satisfies the identities (D2), (D3), (S1), (S2) and (S3), i.e.

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Adjointness property for RPCD

In relatively pseudocomplemented $\wedge$-semilattices,

$$a \wedge x \leq b \iff x \leq a * b. \qquad (APS)$$

In relatively pseudocomplemented directoids,

$$a \sqcap x = a \sqcap b \implies x \leq a * b, \qquad (I)$$

but not conversely in general.

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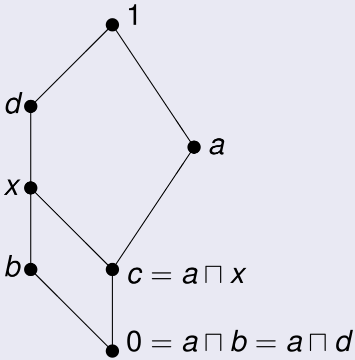

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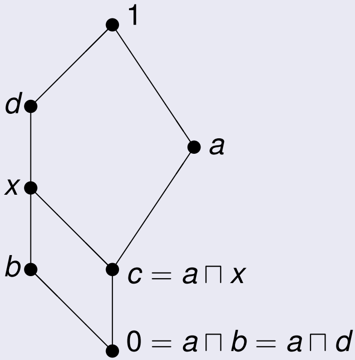

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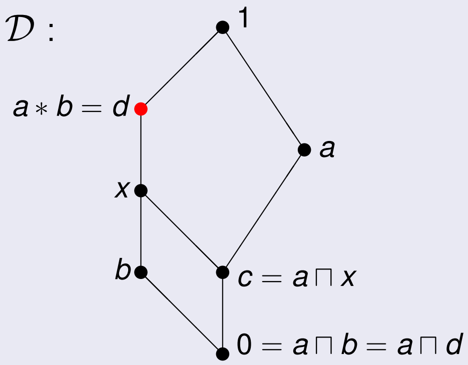

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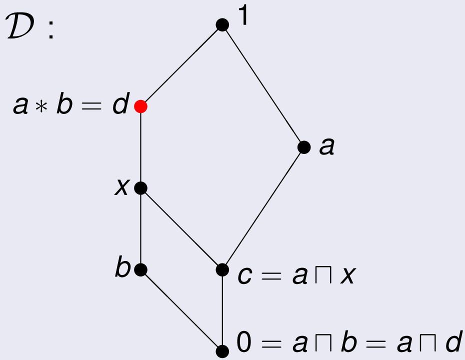

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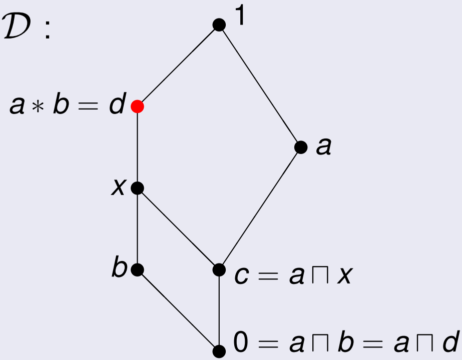

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So the right hand side of (I) must be completed to obtain a condition in the form of an equivalence.

Theorem

Let $(D; \sqcap)$ be a directoid and $*$ be a binary operation on D. Then $\mathcal{D} = (D; \sqcap, *)$ is a relatively pseudocomplemented directoid if and only if the following adjointness property holds

$$a \sqcap x = a \sqcap b \quad \text{iff} \quad x \leq a * b \ \text{and} \ a \sqcap (a * b) = a \sqcap x. \quad (APD)$$

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Variety of RPCD is CD

Theorem

The terms

$$\begin{aligned} t_0(x,y,z) &= x, \\ t_1(x,y,z) &= x \sqcap [((z*y) \sqcap (x*z)) * (x*y)], \\ t_2(x,y,z) &= x \sqcap (y*z), \\ t_3(x,y,z) &= z \sqcap [((z*x)*(x*y)) * (z*y)], \\ t_4(x,y,z) &= z \end{aligned}$$

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The question of CP for the variety $\mathcal{V}$ of RPCD

- The problem is open.
- We are able to find proper subvarieties of $\mathcal{V}$ which are CP.
- The variety $\mathcal{R}$ of relatively pseudocomplemented semilattices.
- $\mathcal{W}$ . . . relatively pseudocomplemented directoids satisfying the identity

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$\mathcal{R} \subset \mathcal{W}$

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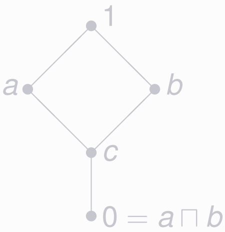

$*$	0	c	a	b	1
0	1	1	1	1	1
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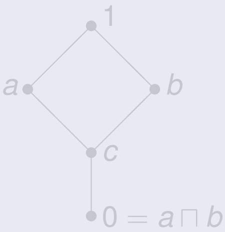

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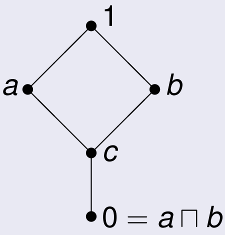

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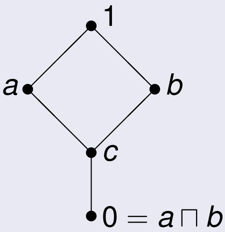

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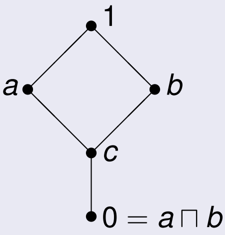

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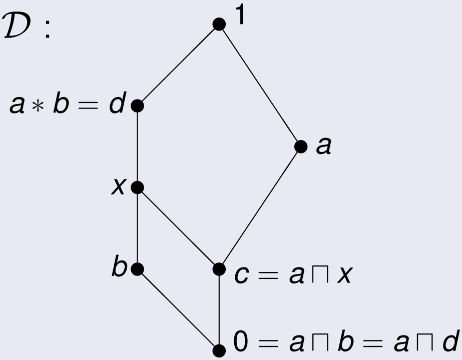

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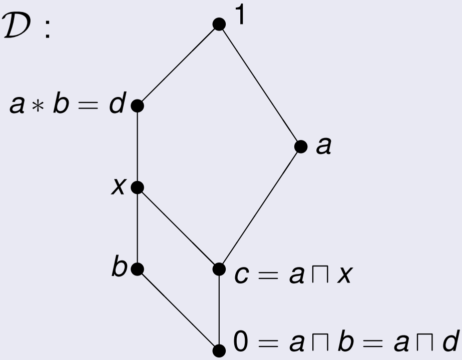

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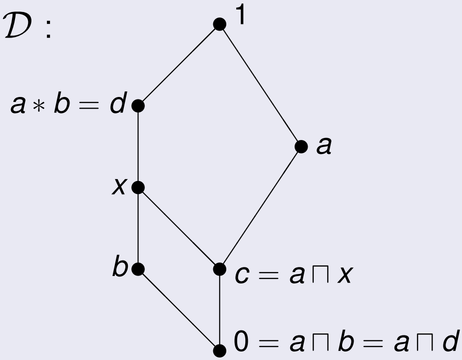

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- $\mathcal{D} \oplus \mathbf{1}$ ... the directoid constructed from a directoid $\mathcal{D}$ with a greatest element q by adding a new greatest element 1
- If $\mathcal{D}$ is a RPCD then also $\mathcal{D} \oplus \mathbf{1}$ is.

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For any relatively pseudocomplemented directoid $\mathcal{D}$, the directoid $\mathcal{D} \oplus \mathbf{1}$ is a subdirectly irreducible member of $\mathcal{V}$.

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Are there any other SI members in $\mathcal{V}$?

$\mathcal{N}_5$:

Two non-trivial (congruence) partitions:

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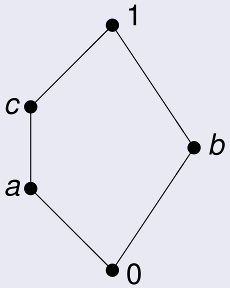

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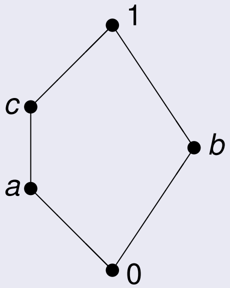

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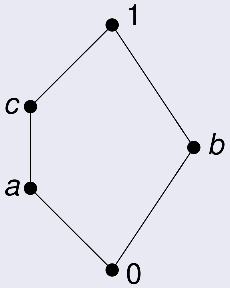

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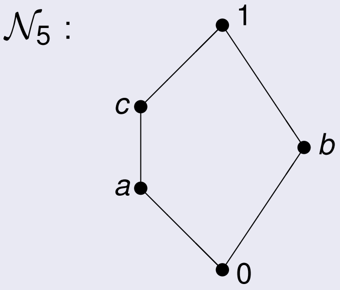

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Outline

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Pseudocomplemented directoids.
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Relatively pseudocomplemented directoids.
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