Products of Tree Languages.

K. Denecke and N. Sarasit

Institute of Mathematics, Am Neuen Palais 10,
University of Potsdam, 14469 Potsdam, Germany

SSAOS 2009, September 10

♦ A set of terms L (tree language of type τ) is recognizable iff there exist a finite algebra $\mathcal{A}$, a homomorphism $\varphi : \mathcal{F}_\tau(X) \to \mathcal{A}$, a subset $A' \subseteq A$ such that $\varphi^{-1}(A') = L$.

♦ A set of terms L (tree language of type τ) is recognizable iff there exist a finite algebra $\mathcal{A}$, a homomorphism $\varphi : \mathcal{F}_\tau(X) \to \mathcal{A}$, a subset $A' \subseteq A$ such that $\varphi^{-1}(A') = L$.

♦ $Rec(\tau)$-the set of all recognizable tree languages of type τ,

♦ A set of terms L (tree language of type τ) is recognizable iff there exist a finite algebra $\mathcal{A}$, a homomorphism $\varphi : \mathcal{F}_\tau(X) \to \mathcal{A}$, a subset $A' \subseteq A$ such that $\varphi^{-1}(A') = L$.

♦ $Rec(\tau)$-the set of all recognizable tree languages of type τ,

$Rec(\tau)$ is closed under $\cup$, $\cap$, complement set.

♦ Eilenberg's variety theorem (S. Eilenberg, Automata, languages and Machines,1976) which gives a one-to- one corespondence
pseudovarieties of finite semigroups $\Leftrightarrow$ varieties of recognizable languages

♦ Eilenberg's variety theorem (S. Eilenberg, Automata, languages and Machines,1976) which gives a one-to- one corespondence pseudovarieties of finite semigroups $\Leftrightarrow$ varieties of recognizable languages

♦ M. Steinby (Syntactic algebras and varieties of recognizable sets,1979) extended the variety theorem to recognizable tree languages

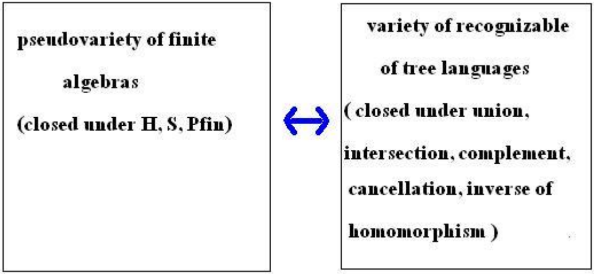
♦ Eilenberg's variety theorem (S. Eilenberg, Automata, languages and Machines,1976) which gives a one-to- one corespondence pseudovarieties of finite semigroups ⇔ varieties of recognizable languages

♦ M. Steinby (Syntactic algebras and varieties of recognizable sets,1979) extended the variety theorem to recognizable tree languages

♦ Almeida's generalization (1990) includes also varieties of filters of congruences

◆ Eilenberg- type correspondence

pseudovarieties of finite algebras $\Leftrightarrow$ varieties of recognizable tree languages



★ Problem: Superposition of tree languages is not included in the operations which define a variety of tree languages, but $Rec(\tau)$ is also closed under superposition,

1.

superposition $\rightsquigarrow$ binary operation, associative $\rightsquigarrow$ semigroup

2. What is pseudovarieties correspondence to the varieties of tree languages which closed under the superposition?

We consider an indexed set of operation symbols f_i where f_i is n_i-ary for every $i \in I$, and a finite alphabet $X_n := \{x_1, ..., x_n\}$. Let $\tau := (n_i)_{i \in I}, n_i \geq 1$ be the sequence of all arities of operation symbols f_i. The sequence τ is called the type of the terms. Then the set $W_\tau(X_n)$ of all n-ary terms of type τ is inductively defined in the following way:

(i) For all $1 \leq j \leq n$ the variables x_j are n -ary terms of type τ.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms and if f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ.

Every subset of $W_\tau(X) := \bigcup_{n \geq 1} W_\tau(X_n)$ is called a tree language of type τ.

The power set $\mathcal{P}(W_\tau(X))$ of the set of all terms of type τ.

The power set $\mathcal{P}(W_\tau(X))$ of the set of all terms of type τ.
$\hat{S}^n_g : \mathcal{P}(W_\tau(X))^{n+1} \to \mathcal{P}(W_\tau(X))$ is inductively defined by the following steps:

The power set $\mathcal{P}(W_\tau(X))$ of the set of all terms of type τ.
$\hat{S}_g^n : \mathcal{P}(W_\tau(X))^{n+1} \to \mathcal{P}(W_\tau(X))$ is inductively defined by the following steps:Let $n \in \mathbb{N}^+$ $(:= \mathbb{N} \setminus \{0\})$ and let $B, B_1, \ldots, B_n \in \mathcal{P}(W_\tau(X))$ such that $B, B_1, \ldots, B_n \neq \emptyset$.

(i) If $B = \{x_i\}$ for $1 \leq i \leq n$, then $\hat{S}_g^n(\{x_i\}, B_1, \ldots, B_n) := B_i$, and if $B = \{x_i\}$ for $n < i$, then $\hat{S}_g^n(\{x_i\}, B_1, \ldots, B_n) := \{x_i\}$.

(ii) If $B = \{f_i(t_1, \ldots, t_{n_i})\}$ and if we assume that $\hat{S}_g^n(\{t_j\}, B_1, \ldots, B_n)$ for $1 \leq j \leq n_i$ are already defined, then $\hat{S}_g^n(\{f_i(t_1, \ldots, t_{n_i})\}, B_1, \ldots, B_n) := \{f_i(r_1, \ldots, r_{n_i}) \mid r_j \in \hat{S}_g^n(\{t_j\}, B_1, \ldots, B_n), 1 \leq j \leq n_i\}$.

(iii) If B is an arbitrary non-empty subset of $W_\tau(X)$, then we define $\hat{S}_g^n(B, B_1, \ldots, B_n) := \bigcup_{b \in B} \hat{S}_g^n(\{b\}, B_1, \ldots, B_n)$.

If one of the sets $B, B_1, \ldots, B_n$ is empty, we define $\hat{S}_g^n(B, B_1, \ldots, B_n) = \emptyset$.

Using the operation $\hat{S}^n_g$ for every $n \geq 1$ and $i \leq n$ we define a binary operation $\cdot_{x_i}$ in the following way:

$$B_1 \cdot_{x_i} B_2 := \hat{S}^n_g(B_1, \{x_1\}, \ldots, \{x_{i-1}\}, B_2, \{x_{i+1}\}, \ldots, \{x_n\})$$

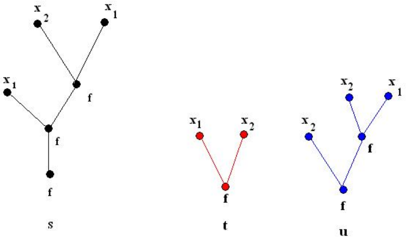
for all $B_1, B_2 \subseteq W_\tau(X)$.

$\cdot_{x_i}$ is associative $\rightsquigarrow$ $(\mathcal{P}(W_\tau(X); \cdot_{x_i})$ is a semigroup.

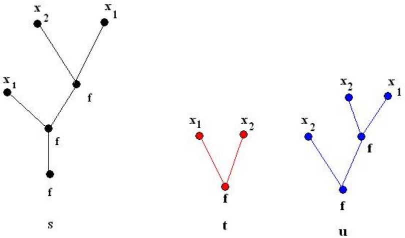
Let $\tau = (2)$, f a binary operation and $X_2 = \{x_1, x_2\}$.
Then $W_\tau(X_2) = \{x_1, x_2, f(x_1, x_1), f(x_1, x_2), f(x_2, x_1),$
$f(x_2, x_2), f(x_1, f(x_1, x_1)), f(x_1, f(x_1, x_2)), ...\}$

Let $\tau = (2)$, f a binary operation and $X_2 = \{x_1, x_2\}$.
Then $W_\tau(X_2) = \{x_1, x_2, f(x_1, x_1), f(x_1, x_2), f(x_2, x_1),$
$f(x_2, x_2), f(x_1, f(x_1, x_1)), f(x_1, f(x_1, x_2)), ...\}$
Let $A = \{f(f(x_1, f(x_2, x_1)))\}$ and $B = \{f(x_1, x_2), f(x_2, f(x_2, x_1))\}$.

Let $\tau = (2)$, f a binary operation and $X_2 = \{x_1, x_2\}$.
Then $W_\tau(X_2) = \{x_1, x_2, f(x_1, x_1), f(x_1, x_2), f(x_2, x_1),$
$f(x_2, x_2), f(x_1, f(x_1, x_1)), f(x_1, f(x_1, x_2)), \ldots\}$
Let $A = \{f(f(x_1, f(x_2, x_1)))\}$ and $B = \{f(x_1, x_2), f(x_2, f(x_2, x_1))\}$.

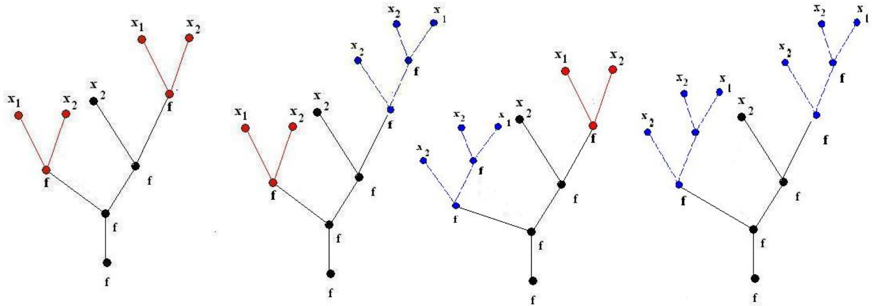


Let $\tau = (2)$, f a binary operation and $X_2 = \{x_1, x_2\}$.
Then $W_\tau(X_2) = \{x_1, x_2, f(x_1, x_1), f(x_1, x_2), f(x_2, x_1),$
$f(x_2, x_2), f(x_1, f(x_1, x_1)), f(x_1, f(x_1, x_2)), \ldots\}$
Let $A = \{f(f(x_1, f(x_2, x_1)))\}$ and $B = \{f(x_1, x_2), f(x_2, f(x_2, x_1))\}$.



$A \cdot_{x_1} B$?

if $n = 3$ then we get $A \cdot_{x_1} B = \hat{S}^3_g(A, B, \{x_2\}, \{x_3\}) =>$



$Var(A)$- set of all variables occurring in all terms of A.

$Var(A)$- set of all variables occurring in all terms of A.

Proposition 1: Let $A, B \in \mathcal{P}(W_\tau(X))$ and let $i \neq j \in \{1, \ldots, n\}$. Then

(i) If $x_j \notin Var(A)$, then there is a set $A' \in \mathcal{P}(W_\tau(X))$ such that $A \cdot_{x_i} B = A' \cdot_{x_j} B$ for all $B \in \mathcal{P}(W_\tau(X))$.

(ii) If $x_i \notin Var(A)$, then $A \cdot_{x_i} B = A$.

(iii) $x_i \notin Var(A \cdot_{x_i} B)$ if and only if $x_i \notin Var(A)$ or $x_i \notin Var(B)$.

(iv) $x_i \notin A \cdot_{x_i} B$ if and only if $x_i \notin A$ or $x_i \notin B$.

(v) If $x_i \in A \cdot_{x_i} B$, then $B \subseteq A \cdot_{x_i} B$.

(vi) If $x_j \in A \cdot_{x_i} B$ and $x_j \notin A$, then $B \subseteq A \cdot_{x_i} B$ and $x_j \in B$.

(vii) If $x_j \in A \cdot_{x_i} B$, then $A \cap X_n \neq \emptyset$.

Lemma 2: Let $A, B \in \mathcal{P}(W_\tau(X)$ and $x_i \in Var(A)$.

If $A = B \cdot_{x_i} A$ or $A = A \cdot_{x_i} B$ then $x_i \in B$.

Lemma 2: Let $A, B \in \mathcal{P}(W_\tau(X))$ and $x_i \in Var(A)$.

If $A = B \cdot_{x_i} A$ or $A = A \cdot_{x_i} B$ then $x_i \in B$.

Idempotent and regular elements.

Theorem 3: Let $A \in \mathcal{P}(W_\tau(X))$.
A is an idempotent element of $(\mathcal{P}(W_\tau(X)); \cdot_{x_i})$ if and only if it is regular if and only if it has finite order.

As a consequence, every element which has finite order has order 1. This means that $(\mathcal{P}(W_\tau(X)); \cdot_{x_i})$ has only idempotent elements or elements with infinite order and we can find examples which show that the collection of all idempotent elements does not form a subsemigroup.

We consider the cases $A\mathcal{L}B$ and $A\mathcal{R}B$ for $x_i \in Var(A)$ and $x_i \notin Var(A)$.

Theorem 4: Let $A, B \in \mathcal{P}(W_\tau(X))$. Then

(i) Let $x_i \notin Var(A)$. Then $A\mathcal{L}B$ if and only if $x_i \notin Var(B)$.

(ii) Let $x_i \in Var(A)$. Then $A\mathcal{L}B$ if and only if $A = B$, i.e. $\mathcal{L}$ is the diagonal $\Delta_{\mathcal{P}(W_\tau(X))}$.

For Green's relation $\mathcal{R}$ we have:

Theorem 5: Let $A, B \in \mathcal{P}(W_\tau(X))$. Then

(i) Let $x_i \notin Var(A)$. Then $A\mathcal{R}B$ if and only if $A = B$.

(ii) Let $x_i \in Var(A)$. If $A\mathcal{R}B$, then $x_i \in Var(B)$ and

$$\{a \mid a \in A \text{ and } x_i \notin Var(\{a\})\} = \{b \mid b \in B \text{ and } x_i \notin Var(\{b\})\}.$$

For any $x_i \in X$ the x_i-iteration A^{*x_i} of a tree language $A \in \mathcal{P}(W_\tau(X))$ is defined as the union $\bigcup_{k\geq 0} A^{k,x_i}$ where

$A^{0,x_i} := \{x_i\}$ and
$A^{j,x_i} := (A^{j-1,x_i} \cdot_{x_i} A) \cup A^{j-1,x_i}$.

For any $x_i \in X$ the x_i-iteration A^{*x_i} of a tree language $A \in \mathcal{P}(W_\tau(X))$ is defined as the union $\bigcup_{k \geq 0} A^{k,x_i}$ where

$A^{0,x_i} := \{x_i\}$ and
$A^{j,x_i} := (A^{j-1,x_i} \cdot_{x_i} A) \cup A^{j-1,x_i}$.

Proposition 6: Let $A \in \mathcal{P}(W_\tau(X))$.

Then $A^{*x_i} = \bigcup_{k \geq 0} A^k$.

Theorem 7: Let $A \in \mathcal{P}(W_\tau(X))$ and $x_i \in Var(A)$. Then the following conditions are equivalent:

(i) $x_i \in A$.

(ii) $(A)^{*x_i} = (A)^{*x_i} \cdot_{x_i} A \cdot_{x_i} (A)^{*x_i}$.

(iii) $(A)^{*x_i}$ is an idempotent in $(\mathcal{P}(W_\tau(X)); \cdot_{x_i})$.

THANK YOU FOR YOUR ATTENTION