Quantifiers on algebras of pseudo-basic logic

Jiří Rachůnek

Palacký University in Olomouc

Czech Republic

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$\mathcal{PBL}$ ... propositional calculus of the pseudo-basic (fuzzy) logic (Hájek)

connectives: $\&, \rightarrow, \rightsquigarrow, \wedge, \vee,$

truth constant: $\bar{0}$

deduction rules (two modus ponens and implications):

(MP$\rightarrow$) $$\frac{\varphi,\ \varphi \rightarrow \psi}{\psi}$$

(MP$\rightsquigarrow$) $$\frac{\varphi,\ \varphi \rightsquigarrow \psi}{\psi}$$

(Imp$\rightarrow$) $$\frac{\varphi \rightarrow \psi}{\varphi \rightsquigarrow \psi}$$

(Imp$\rightsquigarrow$) $$\frac{\varphi \rightsquigarrow \psi}{\varphi \rightarrow \psi}$$

axioms:

(A1) $(\psi \to \chi) \to ((\varphi \to \psi) \to (\varphi \to \chi))$,
$(\psi \rightsquigarrow \chi) \rightsquigarrow ((\varphi \rightsquigarrow \psi) \rightsquigarrow (\varphi \rightsquigarrow \chi))$;

(A2) $(\varphi \,\&\, \psi) \to \varphi$;

(A3) $(\varphi \,\&\, \psi) \to \psi$;

(A4) $(\varphi \wedge \psi) \leftrightarrow ((\varphi \to \psi) \,\&\, \varphi) \leftrightarrow ((\psi \to \varphi) \,\&\, \psi)$,
$(\varphi \wedge \psi) \leftrightsquigarrow (\varphi \,\&\, (\varphi \rightsquigarrow \psi)) \leftrightsquigarrow (\psi \,\&\, (\psi \rightsquigarrow \varphi))$;

(A5) $(\varphi \to (\psi \to \chi)) \leftrightarrow ((\varphi \,\&\, \psi) \to \chi)$,
$(\varphi \rightsquigarrow (\psi \rightsquigarrow \chi)) \leftrightarrow ((\psi \,\&\, \varphi) \rightsquigarrow \chi)$;

(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$,
$((\varphi \rightsquigarrow \psi) \rightsquigarrow \chi) \rightsquigarrow (((\psi \rightsquigarrow \varphi) \rightsquigarrow \chi) \rightsquigarrow \chi)$;

(A7) $\bar{0} \to \varphi$;

(A8) $(\varphi \vee \psi) \leftrightarrow (((\varphi \rightsquigarrow \psi) \to \psi) \wedge ((\psi \rightsquigarrow \varphi) \to \varphi))$,
$(\varphi \vee \psi) \leftrightarrow (((\varphi \to \psi) \rightsquigarrow \psi) \wedge ((\psi \to \varphi) \rightsquigarrow \varphi))$.

$\mathcal{MPBL}$ ... monadic pseudo-basic propositional logic

contains $\mathcal{PBL}$ with axioms:

(M1) $\varphi \to \exists\varphi, \ \ \varphi \rightsquigarrow \exists\varphi$;

(M2) $\forall\varphi \to \varphi, \ \ \forall\varphi \rightsquigarrow \varphi$;

(M3) $\forall(\varphi \to \exists\psi) \equiv \exists\varphi \to \exists\psi, \ \ \forall(\varphi \rightsquigarrow \exists\psi) \equiv \exists\varphi \rightsquigarrow \exists\psi$;

(M4) $\forall(\exists\varphi \to \psi) \equiv \exists\varphi \to \forall\psi, \ \ \forall(\exists\varphi \rightsquigarrow \psi) \equiv \exists\varphi \rightsquigarrow \forall\psi$;

(M5) $\forall(\varphi \vee \exists\psi) \equiv \forall\varphi \vee \exists\psi$;

(M6) $\exists\bar{0} \equiv \bar{0}$;

(M7) $\exists\forall\varphi \equiv \forall\varphi$;

(M8) $\forall\forall\varphi \equiv \varphi$;

(M9) $\exists(\exists\varphi \& \exists\psi) \equiv \exists\varphi \& \exists\psi$.

deduction rules: (MP$\to$), (MP$\rightsquigarrow$), (Imp$\to$), (Imp$\rightsquigarrow$) and necessitation

(Nec) $\dfrac{\varphi}{\forall\varphi}$.

First-order language L based on $\{\&, \to, \rightsquigarrow, \wedge, \vee, \bar{0}, \exists, \forall\}$

Monadic propositional logic L_m based on $\{\&, \to, \rightsquigarrow, \wedge, \vee, \bar{0}, \exists, \forall\}$.

x ... a fixed variable in L.
For any propositional variable p in L_m choose a monadic predicate $F_p(x)$ in L. Then it is possible to identify formulas of L_m and monadic formulas of L containing x. The mapping $\Delta : \ Form(L_m) \longrightarrow Form(L)$:

(1) $\Delta(p) = F_p(x)$, p ... any propositional variable,

(2) $\Delta(\varphi \circ \psi) = \Delta(\varphi) \circ \Delta(\psi)$, for any $\circ \in \{\&, \to, \rightsquigarrow, \vee, \wedge\}$,

(3) $\Delta(\exists\varphi) = \exists x \Delta(\varphi)$,

(4) $\Delta(\forall\varphi) = \forall x \Delta(\varphi)$.

monadic Boolean algebras (Halmos)

monadic Heyting algebras (Monteiro, Varsavsky, Bezhanisvili, Harding, ...)

monadic *MV*-algebras (Rutledge, Grigolia, Di Nola, Belluce, Lettieri, Georgescu, Iorgulescu, Leustean)

monadic *GMV*-algebras (Rachůnek, Šalounová)

monadic *BL*-algebras (Grigolia)

Pseudo-BL-algebra (psBL-algebra) $A = (A;\ \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$, type $\langle 2, 2, 2, 2, 2, 0, 0, \rangle$ (Di Nola, Georgescu, Iorgulescu)

Axioms:

(i) $(A;\ \odot, 1)$ is a monoid (need not be commutative).

(ii) $(A;\ \vee, \wedge, 0, 1)$ is a bounded lattice.

(iii) $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$ for any x, y, $z \in A$.

(iv) $(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$.

(v) $(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$.

Monadic pseudo-BL–algebra (MpsBL-algebra)

$A = (A;\ \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1, \exists, \forall) = (A;\ \exists, \forall)$,

type $\langle 2, 2, 2, 2, 2, 0, 0, 1, 1 \rangle$

$A = (A;\ \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ is a pseudo-BL–algebra and for each $x, y \in A$:

(vi) $x \to \exists x = 1, \quad x \rightsquigarrow \exists x = 1;$

(vii) $\forall x \to x = 1, \quad \forall x \rightsquigarrow x = 1;$

(viii) $\forall(x \to \exists y) = \exists x \to \exists y, \quad \forall(x \rightsquigarrow \exists y) = \exists x \rightsquigarrow \exists y;$

(ix) $\forall(\exists x \to y) = \exists x \to \forall y, \quad \forall(\exists x \rightsquigarrow y) = \exists x \rightsquigarrow \forall y;$

(x) $\forall(x \vee \exists y) = \forall x \vee \exists y;$

(xi) $\exists \forall x = \forall x;$

(xii) $\forall \forall x = \forall x;$

(xiii) $\exists(\exists x \odot \exists y) = \exists x \odot \exists y;$

(xiv) $\exists(x \odot x) = \exists x \odot \exists x.$

Theorem

If $A = (A;\ \exists, \forall)$ is an MpsBL-algebra, then $(x^- := x \to 0, x^\sim := 0)$:

(1) $(\exists x)^- = \forall(x^-)$, $(\exists x)^\sim = \forall(x^\sim)$;

(2) $(\exists x)^{-\sim} = (\forall(x^-))^\sim$, $(\exists x)^{\sim-} = (\forall(x^\sim))^-$;

(3) $(\exists(x^-))^\sim = \forall(x^{-\sim})$, $(\exists(x^\sim))^- = \forall(x^{\sim-})$;

(4) $(\forall(x^{-\sim}))^{-\sim} = \forall(x^{-\sim})$, $\forall(x^{\sim-}))^{\sim-} = \forall(x^{\sim-})$;

(5) $(\exists(x^{-\sim}))^{-\sim} = (\exists x)^{-\sim} \geq \exists(x^{-\sim})$, $(\exists(x^{\sim-}) = (\exists x)^{\sim-} \geq \exists(x^{\sim-})$.

psBL-algebra A ... good if $x^{-\sim} = x^{\sim-}$, for every $x \in A$.

Theorem

If $(A\,; \exists, \forall)$ is an MpsBL-algebra such that the psBL-algebra A is good, then for every $x \in A$:

(6) $(\forall(x^-))^\sim = (\forall x^\sim)^-$;

(7) $(\exists(x^-))^\sim = (\exists(x^\sim))^-$.

$(A\,;\exists,\forall)$ ... MpsBL-algebra

$A_{\exists\forall} := \{x \in A \ : x = \exists x\} = \{x \in A \ : x = \forall x\}.$

Theorem

If $(A\,;\exists,\forall)$ is a MpsBL-algebra, then $A_{\exists\forall}$ is a subalgebra of A.

$A = (A;\ \oplus, \odot, ^-, ^\sim, 0, 1)$ ... GMV-algebra

$x \to y := x^- \oplus y = (x \odot y^\sim)^-$

$x \rightsquigarrow y := y \oplus x^\sim = (y^- \odot x)^\sim$

$x \vee y := x \oplus (y \odot x^-),\ \ x \wedge y := (x^- \oplus y) \odot x$

$(A;\ \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ is a good psBL-algebra.

$A = (A;\ \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ ... a good psBL-algebra

$x \oplus y := (x^- \odot y^-)^\sim = (x^\sim \odot y^\sim)^-$

$x^- := x \to 0,\ x^\sim := x \rightsquigarrow 0$

A ... GMV-algebra, $\exists : A \longrightarrow A$

$(A; \exists)$ ... monadic GMV-algebra if

(E1) $x \leq \exists x$;

(E2) $\exists(x \vee y) = \exists x \vee \exists y$;

(E3) $\exists((\exists x)^{-}) = (\exists x)^{-}$, $\exists((\exists x)^{\sim}) = (\exists x)^{\sim}$;

(E4) $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$;

(E5) $\exists(x \odot x) = \exists x \odot \exists x$;

(E6) $\exists(x \oplus x) = \exists x \oplus \exists x$.

$\forall x := (\exists x^{-})^{\sim} = (\exists x^{\sim})^{-}$

Theorem

Let $(A; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1, \exists, \forall)$ be a good MpsBL-algebra such that $(A; \oplus, \odot, ^{-}, ^{\sim}, 0, 1)$ is a GMV-algebra. Then $(A; \oplus, \odot, ^{-}, ^{\sim}, 0, 1, \exists) = (A; \exists)$ is a monadic GMV-algebra.

A ... Heyting algebra, $\exists : A \longrightarrow A$, $\forall : A \longrightarrow A$

$(A; \exists, \forall)$ ... monadic Heyting algebra if

(H1) $\forall x \leq x$;

(H2) $x \leq \exists x$

(H3) $\forall(x \wedge y) = \forall x \wedge \forall y$;

(H4) $\exists(x \vee y) = \exists x \vee \exists y$;

(H5) $\forall 1 = 1$;

(H6) $\exists 0 = 0$;

(H7) $\forall \exists x = \forall x$;

(H8) $\exists \forall x = \forall x$;

(H9) $\exists(\exists x \wedge y) = \exists x \wedge \exists y$.

Heyting algebra + BL-algebra = Gödel algebra

Theorem

Let $(A; \exists, \forall)$ be a MpsBL-algebra such that A is a Gödel algebra. Then $(A; \exists, \forall)$ is a monadic Heyting (Gödel) algebra.

M ... psBL-algebra, $X \neq \emptyset$ ... a set
M^X ... direct power of M, psBL-algebra
M^X contains a subalgebra isomorphic to M.

$p \in M^X$, $R(p) := \{p(x) : x \in X\}$

A ... a subalgebra of M^X

A ... a functional MpsBL-algebra if

(i) for every $p \in A$ there exist

$$\sup_M R(p) = \bigvee R(p), \ \inf_M R(p) = \bigwedge R(p);$$

(ii) for every $p \in A$, the constant functions $\exists p$ and $\forall p$ defined by

$$\exists p(x) := \bigvee R(p), \ \forall p(x) := \bigwedge R(p),$$

for any $x \in X$, belong to A.

Theorem

If M is a psBL-algebra, $X \neq \emptyset$ and $A \subseteq M^X$ is a functional MpsBL-algebra, then $(A; \exists, \forall)$ is a MpsBL-algebra.

A ... psBL-algebra

$\emptyset \neq F \subseteq A$ ... filter of A if

(i) $x, y \in F \Longrightarrow x \odot y \in F$;

(ii) $x \in F, y \in M, x \leq y \Longrightarrow y \in F$.

$D \subseteq A$ ... deductive system of A if

(iii) $1 \in D$;

(iv) $x \in D, x \rightarrow y \in D \Longrightarrow y \in D$.

filters = deductive systems

$\mathcal{F}(A)$ ... complete lattice of all filters of A

$(A; \exists, \forall)$ ... MpsBL-algebra, $F \in \mathcal{F}(A)$
F ... monadic filter (m-filter) of $(A; \exists, \forall)$ if $x \in F \Longrightarrow \forall x \in F$.

$\mathcal{F}(A; \exists, \forall)$ ... complete lattice of all m-filters of $(A; \exists, \forall)$

Theorem

If $(A; \exists, \forall)$ is a MpsBL-algebra, then the lattice $\mathcal{F}(A; \exists, \forall)$ is isomorphic to the lattice $\mathcal{F}(A_{\exists\forall})$ of all filters of the psBL-algebra $A_{\exists\forall}$.

F ... filter of a psBL-algebra A

F ... normal if $x \to y \in F$ iff $x \rightsquigarrow y \in F$, for any $x, y \in A$

$(A; \exists, \forall)$ ... MpsBL-algebra, θ ... congruence on A

θ ... *m*-congruence on $(A; \exists, \forall)$ if $(x, y) \in \theta \Longrightarrow (\forall x, \forall y) \in \theta$ for any $x, y \in A$.

Theorem

For any MpsBL-algebra there is a one-to-one correspondence between its *m*-congruences and normal *m*-filters.

A ... psBL-algebra, F ... filter of A

F ... prime if $x \vee y = 1$ implies $x \in F$ or $y \in F$

F ... normal filter, F ... prime iff A/F is a psBL-chain

A psBL-algebra A ... representable if A is a subdirect product of psBL-chains, iff there is a system $\mathcal{S}$ of normal prime filters such that $\bigcap \mathcal{S} = \{1\}$.

Theorem

Let $(A; \exists, \forall)$ be a MpsBL-algebra satisfying the identity $\forall(x \vee y) = \forall x \vee \forall y$. Then $(A; \exists, \forall)$ is a subdirect product of linearly ordered MpsBL-algebras if and only if A is a reprentable psBL-algebra.

A ... psBL-algebra, B ... subalgebra of A

B ... relatively complete if for each $a \in A$, the set $\{b \in B : a \leq b\}$ has a least element $\bigwedge_{a \leq b \in B} b$, and the set $\{b \in B : b \leq a\}$ has a greatest element $\bigvee_{a \geq b \in B} b$.

B ... relatively complete subalgebra of A
B ... m-relatively complete if for every $a \in A$ and $x \in B$ such that $x \geq a \odot a$ there is $v \in B$ such that $v \geq a$ and $v \odot v \leq x$.

Theorem

If $(A; \exists, \forall)$ is a MpsBL-algebra, then $A_{\exists\forall}$ is an m-relatively complete subalgebra of A.

Theorem

There is a 1-1 correspondence between MpsBL-algebras and pairs (A, B), where B is an m-relatively complete subalgebra of A.

A ... psBL-algebra, B ... subalgebra of A, $h : B \longrightarrow A$

$\exists_h : A \longrightarrow B$ ... a left adjoint mapping to h if

$$\exists_h(a) \leq x \Longleftrightarrow a \leq h(x),$$

for every $a \in A$, $x \in B$.

$\exists_h(a \odot a) = \exists_h(a) \odot \exists_h(a)$, for every $a \in A$, then $\exists_h$ ... a left m-adjoint mapping to h.

A right adjoint mapping $\forall_h$ to h ... dual to left adjoint mapping

$\forall_h(a \odot a) = \forall_h(a) \odot \forall_h(a)$ for every $a \in A$ ... a right m-adjoint mapping to h.

Theorem

There is a 1-1 correspondence between pairs (A, B), where B is an m-relatively complete subalgebra of a psBL-algebra A, and pairs (A, B), where B is a subalgebra of a psBL-algebra A such that the canonical embedding $h : B \hookrightarrow A$ has a left and a right m-adjoint mapping.

Theorem

There are 1-1 correspondences among

1. MpsBL-algebras;
2. pairs (A, B), where B is an m-relatively complete subalgebra of a psBL-algebra A;
3. pairs (A, B), where B is a subalgebra of a psBL-algebra A such that the canonical embedding $h : B \hookrightarrow A$ has a left and a right m-adjoint mapping.