Atomicity in Archimedean lattice effect algebras

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Introduction

Lattice effect algebras may be carriers of states or probabilities when events are unsharp, fuzzy or imprecise. They are a common generalization of orthomodular lattices and MV-algebras.

Every compactly generated lattice effect algebra is atomic.

Every modular Archimedean lattice effect algebra with compact top element is atomic.

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Basic definition – effect algebras

Definition (D. Foulis and M.K. Bennett, 1994)

A partial algebra $(E;\oplus,0,1)$ is called an **effect algebra** if 0, 1 are two distinct elements and $\oplus$ is a partially defined binary operation on E which satisfy the following conditions for any $x,y,z\in E$:

(Ei) $x\oplus y=y\oplus x$ if $x\oplus y$ is defined,

(Eii) $(x\oplus y)\oplus z=x\oplus(y\oplus z)$ if one side is defined,

(Eiii) for every $x\in E$ there exists a unique $y\in E$ such that $x\oplus y=1$ (we put $x'=y$),

(Eiv) if $1\oplus x$ is defined then $x=0$.

Example

Let $E=[0,1]\subseteq\mathbb{R}$. We put $x\oplus y=x+y$ iff $x+y\le 1$. Hence $\frac{3}{4}\oplus\frac{4}{5}$ does not exist in E.

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On every effect algebra E the partial order $\leq$ and a partial binary operation $\ominus$ can be introduced as follows:

$$x \leq y \text{ and } y \ominus x = z \text{ iff } x \oplus z \text{ is defined and } x \oplus z = y\,.$$

If E with the defined partial order is a (complete) lattice then $(E;\oplus,0,1)$ is called a *(complete) lattice effect algebra*.

If, moreover, E is a modular or distributive lattice then E is called *modular* or *distributive* effect algebra.

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Basic definitions – Archimedean atomic effect algebras

An effect algebra E is *Archimedean* if for all $x \in E$, $x \neq 0$ there exists positive integer $n_x = max\{n \in \mathbb{N} \mid nx = \underbrace{x \oplus x \oplus \cdots \oplus x}_{n\text{-times}} \text{ exists}\}$.

A minimal nonzero element of an effect algebra E is called an *atom* and E is called *atomic* if under every nonzero element of E there is an atom. An element $u \in E$ is called *finite* if either $u = 0$ or there is a finite sequence $\{a_1, a_2, \ldots, a_n\}$ of not necessarily different atoms of E such that $u = a_1 \oplus a_2 \oplus \cdots \oplus a_n$.

Examples

- Every finite effect algebra is atomic and Archimedean.
- Every complete lattice effect algebra is Archimedean (see Z. Riečanová, Demonstratio Mathematica 33 (2000), 443–452).

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Definition

(1) An element a of a lattice L is called *compact* iff, for any $D \subseteq L$, $a \leq \bigvee D$ implies $a \leq \bigvee F$ for some finite $F \subseteq D$.

(2) A lattice L is called *compactly generated* iff every element of L is a join of compact elements.

Theorem

(1) Every compactly generated lattice effect algebra E is atomic.
(2) If E is an Archimedean lattice effect algebra then every compact element is a finite join of finite elements.
(3) The condition that E is Archimedean in (2) cannot be omitted (e.g., the Chang MV-effect algebra $E = \{0, a, 2a, 3a, \ldots, (3a)', (2a)', a', 1\}$ is not Archimedean, every $x \in E$ is compact and 1 can not be presented as a finite join of finite elements $0, a, 2a, 3a, \ldots, ka, \ldots$.

Compact elements of lattice effect algebras

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Atomicity of modular lattice effect algebras

Theorem

Let E be a modular lattice effect algebra and let $F = \{x \in E \mid x \text{ is finite}\}$ such that $\bigvee F = 1$. Then E is atomic.

Corollary

Let E be a modular lattice effect algebra. Let at least one block M of E be Archimedean and atomic. Then E is atomic.

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Compact elements and atomicity of modular lattice effect algebras

Qiang Lei, Junde Wu and Ronglu Li in 2009 have shown the following

Lemma

Let E be a complete atomic distributive lattice effect algebra and G be the set of all finite elements. Then G is an ideal of E.

Examples

Let B be an infinite complete atomic Boolean algebra, C a finite chain MV-algebra. Then

1. The set G of finite elements of the horizontal sum of B and C is not closed under order (namely, the top element is finite but the coatoms from B are not finite).
2. The set G of finite elements of the horizontal sum of two copies of B is not closed under join (namely, the join of two atoms in different copies of B is the top element which is not finite).

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Theorem

Let E be a modular lattice effect algebra and let $x, y \in E$ be finite. Then $[0, x]$ is a complete lattice of finite height and $x \vee y$ is finite. Moreover, the set G of all finite elements of E is an ideal of E.

Proposition

Let E be a modular Archimedean lattice effect algebra, $u \in E$ a compact element. Then u is finite.

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Let E be a modular Archimedean lattice effect algebra and let $C = \{x \in E \mid x \text{ is compact}\}$ such that $\bigvee C = 1$. Then E is atomic.

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Let E be an effect algebra. A map $\omega : E \to [0,1]$ is called a *state* on E if $\omega(0) = 0$, $\omega(1) = 1$ and $\omega(x \oplus y) = \omega(x) + \omega(y)$ whenever $x \oplus y$ exists in E.

It is easy to check that the notion of a state ω on an orthomodular lattice L coincides with the notion of a state on its derived effect algebra L.

It is because $x \leq y'$ iff $x \oplus y$ exists in L, hence $\omega(x \vee y) = \omega(x \oplus y) = \omega(x) + \omega(y)$ whenever $x \leq y'$.

A state ω is called (o)*-continuous* (*order-continuous*) if, for every net $(x_\alpha)_{\alpha \in \mathscr{E}}$ of elements of E,

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Applications: Existence of states on non-orthomodular Archimedean atomic modular effect algebras

Theorem

Let E be a modular Archimedean atomic lattice effect algebra that is not orthomodular. Then there exists an (o)-continuous state ω on E, which is subadditive (i.e., $\omega(a \vee b) \leq \omega(a) + \omega(b)$).

Sketch of the proof:

1. Since E is non-orthomodular there is an atom a of E, $a \leq a'$ such that $E \cong [0, n_a a] \times [0, (n_a a)']$.
2. Since $n_a a$ is finite we have that the interval $[0, n_a a]$ is a complete modular atomic lattice effect algebra. From Z. Riečanová, 2004 we get a subadditive (o)-continuous state ω_a on $[0, n_a a]$.
3. We define $\omega : E \to [0,1] \subseteq \mathbb{R}$ by setting $\omega(x) = \omega_a(y)$, for every $x = y \oplus z$, $y \in [0, (n_a a)]$, $z \in [0, (n_a a)']$.

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Thank you for your attention.