Characterizations of posets via weak states

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Abstract

Weak states on posets are defined which are in some analogy to states on orthomodular posets used in axiomatic quantum mechanics. It is shown how certain properties of the set of weak states characterize certain properties of the underlying poset.

Orthomodular posets serve as algebraic models for logics in axiomatic quantum mechanics. States on them are considered which reflect the properties of states of the corresponding physical system. A crucial property of such states is monotonicity. In analogy to these states we define so-called weak states on an arbitrary poset. These weak states are also monotonous and play some role in the characterization of certain algebraic models of quantum systems (cf. [Länger–Mączyński, 2005]). We use properties of the set of weak states in order to characterize certain properties of the underlying poset. In this context semilattices play an important role. For the theory of semilattices see the monograph [Chajda–Halaš–Kühr, 2007].

In the following let $\mathscr{P} = (P, \leq)$ be an arbitrary but fixed non-empty poset.

For every positive integer n and $a_1, \ldots, a_n \in P$ put $L(a_1, \ldots, a_n) := \{x \in P \mid x \leq a_1, \ldots, a_n\}$ and $U(a_1, \ldots, a_n) := \{x \in P \mid x \geq a_1, \ldots, a_n\}$. $\mathscr{P}$ is called **connected** if its Hasse diagram is a connected graph. $\mathscr{P}$ is called **upward directed** if $U(a, b) \neq \emptyset$ for any $a, b \in P$.

Now we define weak states on posets.

Definition

Let $f : P \to [0,1]$. We call f a **0-weak state** on $\mathscr{P}$ if both f is monotonous and $(f^{-1}(\{0\}), \leq)$ has a greatest element which we denote by $\alpha(f)$. Let $W_0(\mathscr{P})$ denote the set of all 0-weak states on $\mathscr{P}$.

We call f a **1-weak state** on $\mathscr{P}$ if both f is monotonous and $(f^{-1}(\{1\}), \leq)$ has a least element which we denote by $\beta(f)$. Let $W_1(\mathscr{P})$ denote the set of all 1-weak states on $\mathscr{P}$.

Next we give some examples of weak states.

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Next we give some examples of weak states.

Definition

For $(a,s) \in P \times (0,1]$ let $f_{a,s}$ denote the mapping from P to $[0,1]$ defined by

$$f_{a,s}(x) := \begin{cases} 0 & \text{if } x \leq a \\ s & \text{otherwise.} \end{cases}$$

We call the mappings of the form $f_{a,1}$ – which we will shortly denote by f_a – **canonical 0-weak states** on $\mathscr{P}$ and denote their set by $C_0(\mathscr{P})$.

For $(a,s) \in P \times [0,1)$ let $g_{a,s}$ denote the mapping from P to $[0,1]$ defined by

$$g_{a,s}(x) := \begin{cases} 1 & \text{if } x \geq a \\ s & \text{otherwise.} \end{cases}$$

We call the mappings of the form $g_{a,0}$ – which we will shortly denote by g_a – **canonical 1-weak states** on $\mathscr{P}$ and denote their set by $C_1(\mathscr{P})$.

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We call the mappings of the form $g_{a,0}$ – which we will shortly denote by g_a – **canonical 1-weak states** on $\mathscr{P}$ and denote their set by $C_1(\mathscr{P})$.

(Remark.) $f_{a,s} = f_{b,t}$ if and only if either $(a,s) = (b,t)$ or $a = b$ is the greatest element of $\mathscr{P}$. $g_{a,s} = g_{b,t}$ if and only if either $(a,s) = (b,t)$ or $a = b$ is the least element of $\mathscr{P}$.

Lemma

If $(a,s) \in P \times (0,1]$ then $f_{a,s} \in W_0(\mathscr{P})$ and $\alpha(f_{a,s}) = a$. If $(a,s) \in P \times [0,1)$ then $g_{a,s} \in W_1(\mathscr{P})$ and $\beta(g_{a,s}) = a$.

For weak states we consider pointwise arithmetic mean and pointwise multiplication which not necessarily yields a weak state.

Definition

For all $f, g \in [0,1]^P$ we define $f \oplus g, fg \in [0,1]^P$ by $(f \oplus g)(x) := (f(x) + g(x))/2$ and $(fg)(x) := f(x)g(x)$ for all $x \in P$.

(In the proofs we often use the following formulas:)

Lemma

$$(f \oplus g)^{-1}(\{0\}) = L(\alpha(f), \alpha(g)) \textit{ for all } f, g \in W_0(\mathscr{P})$$
$$(f \oplus g)^{-1}(\{1\}) = U(\beta(f), \beta(g)) \textit{ for all } f, g \in W_1(\mathscr{P})$$
$$(fg)^{-1}(\{0\}) = L(\alpha(f)) \cup L(\alpha(g)) \textit{ for all } f, g \in W_0(\mathscr{P})$$
$$(fg)^{-1}(\{1\}) = U(\beta(f), \beta(g)) \textit{ for all } f, g \in W_1(\mathscr{P})$$

Now we start our characterization results.

Theorem

The following are equivalent:

(i) $\mathscr{P}$ *is connected.*

(ii) *For all* $a, b \in P$ *there exist a positive integer* n *and* $a_0, \ldots, a_n \in P$ *with* $a_0 = a$ *and* $a_n = b$ *such that for all* $i = 1, \ldots, n$ *either* $(f_{a_{i-1}} \oplus f_{a_i})^{-1}(\{0\}) \neq \emptyset$ *or* $(g_{a_{i-1}} g_{a_i})^{-1}(\{1\}) \neq \emptyset$.

Theorem

Let $a, b \in P$. *Then the following are equivalent:*

(i) P *is the disjoint union of* $(a]$ *and* $[b)$.

(ii) $f_a = g_b$

Theorem

The following are equivalent:

(i) $\mathscr{P}$ *is upward directed.*

(ii) $(fg)^{-1}(\{1\}) \neq \emptyset$ *for all* $f, g \in W_1(\mathscr{P})$.

Theorem

The following are equivalent:

(i) $\mathscr{P}$ *is a join-semilattice.*

(ii) $W_1(\mathscr{P})$ *is a subsemigroup of* $([0,1],\cdot)^P$.

Further we see that if $W_1(\mathscr{P})$ is a subsemigroup of $([0,1],\cdot)^P$ then it contains a subsemigroup which is both isomorphic to $(P,\vee)$ and a homomorphic image of $\mathscr{W}_1(\mathscr{P}) := (W_1(\mathscr{P}),\cdot)$.

Theorem

If $\mathscr{P}$ is a join-semilattice $(P,\vee)$ then $C_1(\mathscr{P})$ is the set of all idempotents of $\mathscr{W}_1(\mathscr{P})$ and a subsemilattice of $\mathscr{W}_1(\mathscr{P})$ which is both isomorphic to $(P,\vee)$ and a homomorphic image of $\mathscr{W}_1(\mathscr{P})$.

Theorem

Assume $\mathscr{P}$ to be a join-semilattice. Then the following are equivalent:

(i) *$\mathscr{P}$ has a least element.*

(ii) *$(W_1(\mathscr{P}),\cdot)$ has a unit element.*

Theorem

Assume $\mathscr{P}$ to be a join-semilattice. Then the following are equivalent:

(i) $\mathscr{P}$ *has a greatest element.*

(ii) $\mathscr{W}_1(\mathscr{P})$ *has a zero element.*

Theorem

The following are equivalent:

(i) $\mathscr{P}$ *is a chain.*

(ii) $\mathscr{P}$ *is a meet-semilattice* $(P, \wedge)$ *and every canonical* 0*-weak state on* $\mathscr{P}$ *is a homomorphism from* $(P, \wedge)$ *to* $(\{0, 1\}, \cdot)$*.*

(iii) $W_0(\mathscr{P})$ *is a subsemigroup of* $([0, 1], \cdot)^P$*.*

Theorem

If $\mathscr{P}$ is a chain then $C_0(\mathscr{P})$ is the set of all idempotents of $\mathscr{W}_0(\mathscr{P})$ and a subsemilattice of $\mathscr{W}_0(\mathscr{P})$ which is both isomorphic to $(P,\vee)$ and a homomorphic image of $\mathscr{W}_0(\mathscr{P})$.

For the following theorem we recall some definitions from semigroup theory.

Definition

Let $\mathscr{S} = (S,\cdot)$ be a semigroup. $\mathscr{S}$ is called **regular** if for every $a \in S$ there exists an element b of S with $aba = a$. $\mathscr{S}$ is called **cancellative** if whenever $a,b,c \in S$ and either $ac = bc$ or $ca = cb$ then $a = b$.

Theorem

The following are equivalent:

(i) $\mathscr{P}$ *is trivial.*

(ii) $W_0(\mathscr{P})$ *is a regular subsemigroup of* $([0,1],\cdot)^P$.

(iii) $W_1(\mathscr{P})$ *is a regular subsemigroup of* $([0,1],\cdot)^P$.

(iv) $W_1(\mathscr{P})$ *is a cancellative subsemigroup of* $([0,1],\cdot)^P$.

(v) $W_0(\mathscr{P})$ *is a subgroup of* $([0,1],\cdot)^P$.

(vi) $W_1(\mathscr{P})$ *is a subgroup of* $([0,1],\cdot)^P$.

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