On a general criterion for classification of homomorphism-homogeneous relational structures

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Homogeneous structures

Homogeneity of structures...

...is a classical notion of model theory. There is a well-developed theory about constructions of homogeneous structures due to the work of Roland Fraïssé.

The classification of homogeneous relational structures...

...is a widely studied area. There are many combinatorial results on this subject:

- Homogeneous graphs by Gardiner(1976), Lachlan and Woodrow(1980)
- k-homogeneous graphs by Gol'fand and Klin (1978)
- Homogeneous posets by Schmerl (1979)
- Homogeneous tournaments by Lachlan (1984)
- Homogeneous digraphs by Cherlin (1998)
- Homogeneous permutations by Cameron (2002)

Homomorphism-homogeneous structures

The concept of HH-structures...

...was introduced by Cameron and Nešetřil (2002) as an interesting and natural generalization of the notion of homogeneity. It opened a question of characterization and classification of such structures.

The state of the art:

- HH-graphs by Cameron and Nešetřil (2006)
- HH-strict orders by Cameron and Lockett (2009)
- HH-posets by Mašulović (2007)
- Finite HH-tournaments with loops by Ilić, Mašulović and Rajković (2008)
- So far, this was more a combinatorial problem driven by curiosity.

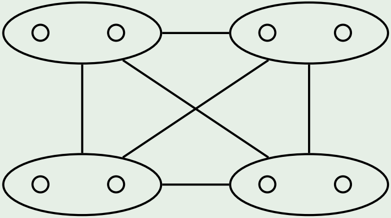
Meet HH-structures

Homomorphism-homogeneity

A structure **A** is **homomorphism-homogeneous** if every local homomorphism of **A** extends to an endomorphism of **A**.

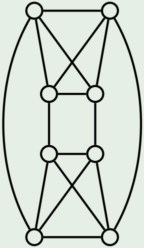
Example

$K_{2,2,2,2}$ with all loops



Example

$K_2 + K_4$ with all loops



A trip to model theory

The one-point extension property:

A relational structure **A** has **the one-point extension property** if for every finite substructure **B** of **A**, every $b \in A \setminus B$ and every homomorphism $f : \mathbf{B} \to \mathbf{A}$, there exists a homomorphism $g : \mathbf{B} \cup \{b\} \to \mathbf{A}$ which extends f.

Weak centers:

Given a relational structure $\mathbf{A} = (A, (\varrho_{\mathbf{A}})_{\varrho \in R})$ and its finite substructure $\mathbf{B} = (B, (\varrho_{\mathbf{B}})_{\varrho \in R})$. Some $c \in A$ is

a weak center of B

if for every $b \in B$ there exists a $\varrho \in R$, $b_3, \ldots, b_{\mathrm{ar}(\varrho)} \in B$ and a $\pi \in \mathrm{Sym}\{1, \ldots, \mathrm{ar}(\varrho)\}$ such that

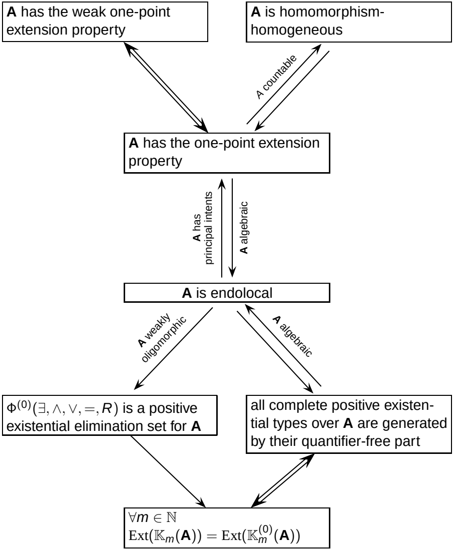
$$(c, b, b_3, \ldots, b_{\mathrm{ar}(\varrho)})^{\pi} \in \varrho_{\mathbf{A}}.$$

The weak one-point extension property:

A relational structure $\mathbf{A} = (A, (\varrho_A)_{\varrho \in R})$ has **the weak one-point-extension property** if for every finite substructure **B** of **A**, every homomorphism $f : \mathbf{B} \to \mathbf{A}$ and every weak center c of **B** there is a homomorphism $g : \mathbf{B} \cup \{c\} \to \mathbf{A}$ such that $g{\restriction}_B = f$.

Main theorem

Let $\mathbf{A} = (A, (\varrho_{\mathbf{A}})_{\varrho \in R})$ be a relational structure. Then the relationships presented in the following diagram hold:



Minimal witness criterion

A witness. . .

. . . is a quadruple $(\mathbf{B_1}, \mathbf{B_2}, f, c)$, such that $\mathbf{B_1}$ is a finite substructure of **A**, c is a weak center of $\mathbf{B_1}$ in A, $\mathbf{B_2}$ is a substructure of **A**, and $f : \mathbf{B_1} \to \mathbf{B_2}$ is surjective, but f cannot be extended to $\mathbf{B_1} \cup \{c\}$.

- It is possible to introduce a quasi-order on witness.
- A minimal element in this quasi-order is called **a minimal witness**.

Minimal witness criterion:

A structure **A** has the weak one-point extension property if and only if it has no minimal witnesses.

MH relational structures

A rel. structure A is an MH structure ...

... if every local monomorphism of **A** can be extended to an endomorphism of **A**

Bijective minimal witness

A minimal witness $(\mathbf{B_1}, \mathbf{B_2}, f, c)$, where f is bijective.

Proposition

Let $\mathcal{K}$ be a class of countable relational structures. If all minimal witnesses for elements of $\mathcal{K}$ are bijective, then

HH if and only if MH.

Examples

Proposition

Let $\mathbf{A} = (A, (\varrho_{\mathbf{A}})_{\varrho \in R})$ *be such that exactly one relation in* R *is binary and symmetric, and all others are unary. If* $(\mathbf{B_1}, \mathbf{B_2}, f, c)$ *is a minimal witness, then* f *is bijective.*

Proposition

Let $\mathbf{A} = (A, (\varrho_{\mathbf{A}})_{\varrho \in R})$ *be such that* R *consists of exactly one binary antisymmetric transitive relation* σ*. If* $(\mathbf{B_1}, \mathbf{B_2}, f, c)$ *is a minimal witness, then* f *is bijective.*

Weakly homogeneous graphs

Definition

A graph G is called **weakly homogeneous** if whenever two induced subgraphs are isomorphic, then at least one isomorphism between them extends to an automorphism of G.

Theorem (Gardiner 1976, Ronse 1978, Enomoto 1981)

Every weakly homogeneous finite graph is homogeneous, and vice versa.

Question

What can be said about weakly homomorphism-homogeneous and homomorphism-homogenous relational structures?

Weakly homomorphism-homogeneous structures

Definition

A structure **A** is called **weakly homomorphism-homogeneous** if whenever there exists an epimorphism between two finite substructures of **A**, then at least one epimorphism between them extends to an endomorphism of **A**.

Observation

If **A** is a weakly HH countable structure that is not an HH structure, then there exists a witness $(\mathbf{B_1}, \mathbf{B_2}, f, c)$ such that there exists a $g : \mathbf{B_1} \twoheadrightarrow \mathbf{B_2}$ such that $(\mathbf{B_1}, \mathbf{B_2}, g, c)$ is not a witness.

Proposition

If in a class of countable structures all possible minimal witnesses are independent of the particular epimorphism, then and only then the subclass of weakly HH structures coincides with the subclass of HH structures.

Example

Such classes are e.g.

1. countable graphs (loops allowed);
2. countable structures with precisely one transitive binary relations.

HH transitive relations

HH-relations

A relation ϱ on the given set A is homomorphism-homogeneous if (A, ϱ) is an HH-structure.

Antisymmetric transitive relations

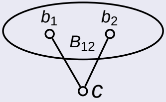
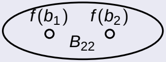
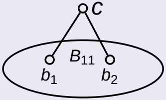
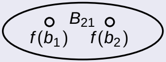
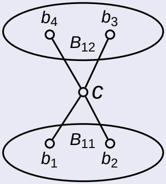
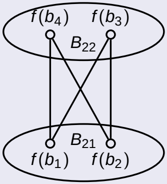
There are 48 types of minimal witnesses.

Transitive relations

A transitive relation has no minimal witness if and only if its retract to an antisymmetric transitive relation has no minimal witness.

Example

Minimal witnesses for posets

	B_1	**B_2**
$B_{11} = \emptyset$		 no lower bound
$B_{12} = \emptyset$		no upper bound 
$B_{11}, B_{12} \neq \emptyset$		 no $d: B_{21} \leq_\varrho d \leq_\varrho B_{22}$

HH tolerance relations

Good news

If a tolerance relation ϱ is HH, then

- either ϱ is an equivalence relation,
- or the graph of ϱ is connected and of diameter at most 2.

Bad news

Let $\mathbf{D} = (D, \sigma)$ be such that σ is a noncentral tolerance relation and D is finite. Then there exists a structure $\mathbf{A}$ with a minimal witness $(\mathbf{B_1}, \mathbf{B_2}, f, c)$ such that $\mathbf{B_2} \cong \mathbf{D}$.

Nevertheless...

- There is a catalogue of all noncentral HH tolerance relations on a basic set of cardinality at most 9.
- It is possible to determine some infinite families of HH tolerance relations.

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Some infinite families of HH tolerance relations

Example

Let G be a simple graph such that all connected components of G are cycles of length 5, i.e. every component is isomorphic to C_5. Let A be the vertex set of G and let ϱ be the complement of the adjacency relation of G. Then $\mathbf{A} = (A, \varrho)$ is homomorphism-homogeneous.

Example

Let $V = \{v_1, \ldots, v_n\}$ and let $W = \{w_1, \ldots, w_n\}$, $n \geq 2$. Let $A = V \cup W$ and let

$$\varrho = V^2 \cup W^2 \cup \{(v_i, w_i) \mid i \in \{1, \ldots, n\} \cup \{(w_i, v_i) \mid i \in \{1, \ldots, n\}\}.$$

Then $\mathbf{A} = (A, \varrho)$ is homomorphism-homogeneous.