Compatible functions on semilattices

Vladimir Kuchmei

University of Tartu

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An n-ary function f on an algebra $\mathbf{A}$ is called **compatible** if it preserves all congruences of $\mathbf{A}$.

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There are algebras on which polynomial functions are the only compatible functions. Such algebras are called **affine complete**.

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Problem

Given an algebra $\mathbf{A}$, *find a nice generating set for the clone of all compatible functions.*

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An n-ary function f on a semilattice **S** is called **a contraction** if $f(a_1, \ldots, a_n) \leq a_i$ for every $a_1, \ldots, a_n \in S$ whenever f depends on x_i.

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Affine complete semilattices

The following results were proved in

K. Kaarli, L. Márki and E. T. Schmidt, *Affine complete semilattices*, Monatsh. Math. **99** (1985), no. 4, 297–309.

Unary compatible functions

Theorem

A unary function f on a semilattice $\mathbf{S}$ is compatible iff it has one of the following forms:

1. $f = f_I$ *for some almost principal ideal* I *of* $\mathbf{S}$;
2. *there exists an element* $0 \neq a \in S$ *and an almost principal ideal* I *of the subsemilattice* $\uparrow a$ *of* $\mathbf{S}$ *such that the restriction of* f *to* $\uparrow a$ *is* f_I *and* $f(x) = a$ *for* $x \ngeq a$; *moreover, the ideal* I *has the property: if* $u \in I$ *and* $u > a$ *then* $\downarrow u = \downarrow a \cup [a, u]$. *(In this case* f *is order preserving but not a contraction.)*
3. *there are elements* $a, b \in S$ *such that* a *covers* b, $c \wedge a \leq b$ *for all* $c \ngeq a$, *and* $f(x) = b$ *for* $x \geq a$ *and* $f(x) = a$ *otherwise. (In this case* f *is neither order preserving nor a contraction.)*

Affine complete semilattices

Theorem

A semilattice **S** *is affine complete iff it satisfies the following conditions:*

1. *every proper almost principal ideal of* **S** *is principal;*
2. *there is no elements* $a, b \in S$ *with* $0 \neq b < a$ *such that* $\downarrow b \cup [b, a] = \downarrow a$*;*
3. **S** *has no atoms.*

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Clearly such semilattices must satisfy the first two conditions of Theorem and these conditions ensure that all unary order preserving compatible functions are polynomials.

Order affine semilattices are affine complete

Assume that **S** contains an atom a.
Let

$$f(x,y) = \begin{cases} a & \text{if } x \geq a \text{ or } y \geq a; \\ 0 & \text{if } x \not\geq a \text{ and } y \not\geq a \end{cases}.$$

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It is easy to see that f is compatible order preserving but not a contraction, hence is not a polynomial.

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Theorem

A semilattice **S** *is order affine complete iff it is affine complete.*

Compatible contractions

Proposition

Let f be an essentially n-ary compatible contraction on a semilattice $\mathbf{S}$. Then

$$f(x_1, \ldots, x_n) = f(y, \ldots, y) \quad \text{where} \quad y = x_1 \wedge \ldots \wedge x_n \, .$$

Compatible order preserving functions

There are compatible order preserving functions on a semilattice **S** which are not contractions:

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Compatible order preserving functions

Claim

Every compatible order preserving function on a semilattice $\mathbf{S}$ is a join of some unary compatible order preserving functions.

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Lemma

Let f, g be n-ary order preserving compatible function on a semilattice $\mathbf{S}$. If $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ are comparable for any $x_1, \ldots x_n \in S$ then $f \vee g$ is an order preserving compatible function on $\mathbf{S}$.

Compatible order preserving functions

For a semilattice **S** let

$$S^0 = \begin{cases} S & \text{if } S \text{ has the smallest element } 0 \\ S \cup \{0\} & \text{otherwise} \end{cases} .$$

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A unary function f on a semilattice **S** is order preserving compatible iff there exists an almost principal ideal I of **S** and an element $a \in I \cup (S^0 \setminus S)$ which is thin in I such that $f = a \vee f_I$

Binary compatible order preserving functions

Theorem

A binary function f on a semilattice $\mathbf{S}$ is order preserving compatible if and only if there exist almost principal ideals I, J, K of $\mathbf{S}$, $I \subseteq J \subseteq K$ and an element $a \in I \cup (S^0 \setminus S)$ satisfying the following conditions:

1. *a is thin in K;*
2. *$\{x \in I \mid x \geq a\}$ is a chain;*
3. *any element of $\{x \in I \mid x \geq a\}$ is thin in J;*
4. *for any $u \in K$ we have $\downarrow u = \downarrow u_J \cup [u_J, u]$ and $\downarrow u = \downarrow u_I \cup [u_I, u]$*

such that after a suitable reordering of the variables

$$f(x, y) = a \vee x_I \vee y_J \vee (x \wedge y)_K \,.$$