A new family of distance regular covers of complete graphs

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Introduction

- In 2008 we (i.e. Klin and P.), using a computer, discovered two new antipodal distance regular graphs on 108 and 135 vertices, respectively (with new parameters).
- After long efforts it became possible to embed the example on 108 vertices to a potentially wide infinite class of distance regular graphs.
- Also progress with the understanding of the example on 135 vertices was achieved.

Metric Decompositions of Graphs

Given:

- A graph Γ,
- a vertex u of Γ.

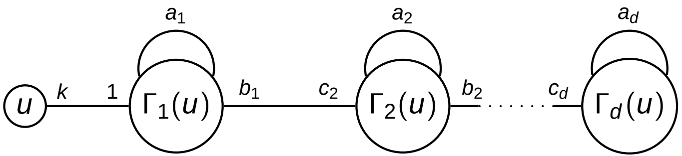
Metric Decomposition:

- Cells of the metric partition of Γ with respect to u are the vertices on the same distance i from u.
- If the diameter $d = d(\Gamma)$ of Γ is finite, we have $d + 1$ cells.
- We denote by $\Gamma_i(u)$ the subgraph of Γ induced by the vertices on distance i from u.

A connected regular graph Γ of valency k and diameter d is called distance regular (briefly DRG) if for each vertex u the metric partition

$$\{\{u\}, \Gamma_1(u), \ldots, \Gamma_d(u)\}$$

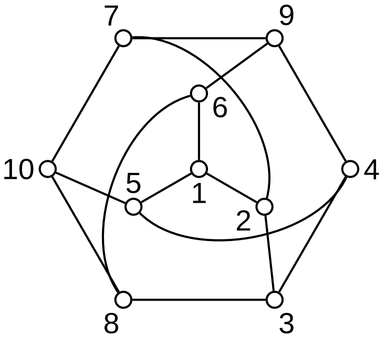
is equitable with the set of intersection numbers which does not depend on the selection of u.



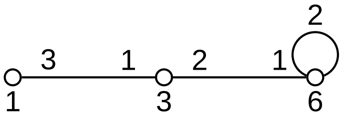
Intersection diagram of a DRG

- A DRG of diameter $d = 2$ is called a strongly regular graph (briefly SRG).
- A DRG Γ is called primitive if all distance i graphs Γ_i for $1 \leq i \leq d$ are connected. Otherwise Γ is called imprimitive.
- Note that $\{x, y\}$ is an edge in Γ_i if and only if $d(x, y) = i$ in the graph Γ.

Example 1



Metric decomposition of the Petersen graph and its intersection diagram:

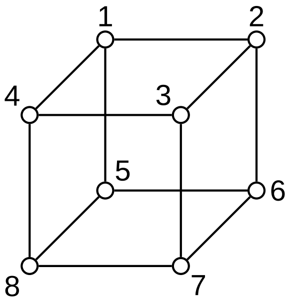


- An imprimitive DRG Γ of diameter d is called antipodal if its distance graph Γ_d is disconnected.
- In this case Γ_d is a disjoint union of n copies of the complete graph K_r.
- The partition formed by the vertices of these n copies is called the antipodal partition of Γ.

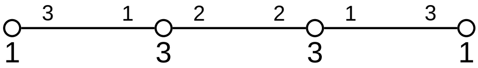
Theorem (D.H.Smith, A.Gardiner)

An imprimitive DRG is bipartite or antipodal (here "or" is not exclusive).

Example 3: the 3-dimensional cube Q_3



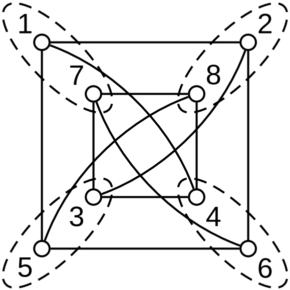
$n = 4, r = 2$



Q_3 is bipartite and antipodal.

Example 3 (cont.)

Another glance onto Q_3:



- Antipodal cells are “metavertices”.
- The quotient graph is K_4.
- Each edge of K_4 is represented by 1-factor between two metavertices.

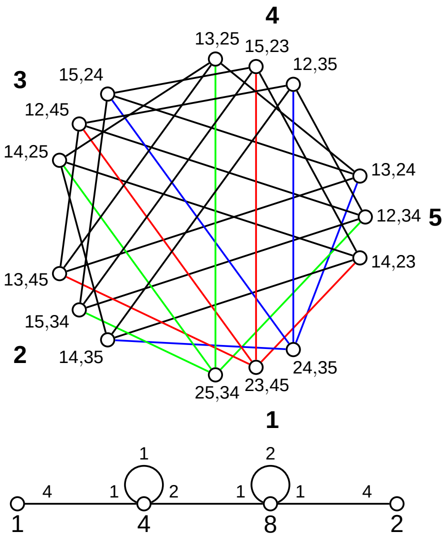
- A graph Γ is called a cover of another graph Δ if there is a surjection $h : V(\Gamma) \to V(\Delta)$ that maps edges of Γ to edges of Δ which is locally an isomorphism.
- The function h is called a covering of Δ.
- Preimages of vertices from Δ are called the fibres of the covering.

- Each fibre induces an empty subgraph.
- Between two fibres there are either no edges, or the edges between the two fibres form a perfect matching.
- For the covers of a connected graph all fibres have the same size r.
- Let $\ker h$ be the equivalence relation defined by the fibres.
- Clearly, $\Gamma / \ker h$ is isomorphic to Δ.

- If the cover Γ of the graph Δ is distance regular, then Γ is called antipodal distance regular cover of Δ.
- Note that in this case Δ is also a DRG.

- From now on and onwards the complete graph will serve as the quotient graph Δ.
- The antipodal distance regular covers in this case have diameter d equal to 3.

Example 7: Line graph of the Petersen graph P



3-fold cover of K_5

Lemma

An antipodal r-fold cover of K_n is antipodal distance regular if and only if there exists a constant c_2 such that any two non-adjacent vertices from different fibres of the cover have exactly c_2 common neighbors.

Thus we will call an antipodal distance regular cover of K_n an (n, r, c_2)-cover.

Main known infinite series

Construction	Parameters	Conditions
Mathon	$(q+1, r, c)$	$q = rc + 1$ is a prime power
Bondy	$(n, n-2, 1)$	Projective plane of order $n-1$ exists
Thas-Somma	(q^{2j}, q, q^{2j-1})	q is a prime power
Brouwer	$(st+1, s+1, t-1)$	spread in pseudo GQ
Godsil-Hensel	$(p^{2i}, p^{i-k}, p^{i+k})$	p is prime, $0 \leq k < i$
de Caen-Mathon-Moorhouse	$(2^{2t}, 2^{2t-1}, 2)$	
de Caen-Fon Der Flaass	(q^{d+1}, q^{d}, q)	$q = 2^t$

- Let Γ be a connected antipodal distance regular cover of K_n with index r.
- $G = \mathrm{Aut}(\Gamma)$.
- Let us consider the subgroup $T \leq G$ which stabilizes each of the fibres of Γ (that is, T preserves each fibre as a set).

Lemma

Every element $\sigma \in T$, $\sigma \neq e$ is fixed point free.

Corollary

- $|T| \leq r$,
- *T acts semiregularly on the fibres.*

If the group T has order r and thus acts regularly on each fibre, then we say that Γ is a regular cover.

If in addition T is abelian or cyclic, then Γ is called abelian or cyclic cover, respectively.

The group T will be called the voltage group.

Godsil-Hensel matrices

“Matrix-representation of symmetric arc functions”:

Let T be a voltage group, $A = (a_{i,j})$ be a square matrix of order n, where

- $a_{i,j} \in \overline{T}$,
- $\overline{T} = T \cup \{0\}$, where 0 is an additional element distinct from any element of T.

A will be called a formal matrix over T.

We call $A = (a_{i,j})$ a covering matrix if

- $a_{i,j} = (a_{j,i})^{-1}$ for all $i, j \in \{1, \ldots, n\}$,
- $a_{i,i} = 0$ for all $i \in \{1, \ldots, n\}$,

Godsil-Hensel matrices (cont.)

We associate to the covering matrix A two graphs:

- the underlying graph $\Delta = \Delta_A$ with the vertex set

$$V(\Delta_A) = \{1, 2, \ldots, n\},$$

  and the edge set

$$E(\Delta_A) = \{\{i, j\} \mid a_{i,j} \neq 0\};$$

- the cover of $\Gamma = \Gamma^A$ with the vertex set

$$V(\Gamma^A) = \{1, 2, \ldots, n\} \times T,$$

  and the edge set

$$E(\Gamma^A) = \{\{(i, g), (j, h)\} \mid a_{i,j} \neq 0, g \cdot a_{i,j} = h\}.$$

Godsil-Hensel matrices (cont.)

It is easy to observe that the function

$$h : V(\Gamma^A) \to V(\Delta_A) \text{ defined as } (i, g) \mapsto i$$

is a covering function, thus the graph Γ^A is a cover of the graph Δ_A.

Moreover, each regular cover of Δ_A (up to isomorphism) with the voltage group T can be obtained in this way.

If Γ^A is an antipodal distance regular cover of Δ_A, then we call A the Godsil-Hensel matrix of this cover (briefly GH-matrix).

Theorem (Godsil-Hensel, 1992)

Let T be a voltage group and let A be a covering matrix of order n over T.
Then A is a GH-matrix of a regular antipodal (n, r, c_2)-cover of K_n with the voltage group T if and only if

$$A^2 = (n-1)I + \delta A + c_2\underline{T}(J - I), \qquad (*)$$

where $\delta = n - 2 - rc_2$.

(Here I and J are natural modifications of classical notations to formal matrices. Moreover, matrix multiplication is performed in the matrix-ring over the group-ring of T, and $\underline{T}$ denotes the sum of all elements of T in the group-ring of T.)
The class of all such matrices which satisfy $(*)$ will be denoted by

$$\mathrm{GHM}(T, n, r, c_2).$$

We extend the concept of a conjugate transpose matrix A^* onto formal matrices.

Definition

Let T be a finite group, and let $A = (a_{i,j})$ be a formal matrix of order n over T such that A does not contain the entry 0.
Then we call A a generalized Hadamard matrix if for $c = {}^{n}/_{|T|}$ we have:

$$AA^* = A^*A = nI + c\underline{T}(J - I).$$

We denote by $\mathrm{gH}(T, n)$ the set of all gH-matrices of order n over T.

Lemma

Let T be a finite group and let A be a covering matrix over T with $\Delta_A = K_n$. Then the graph Γ^A is a regular (n, r, c_2)-cover if and only if

$$(A + I)^2 = nI + (n - rc_2)A + c_2\underline{T}(J - I).$$

Proof.

$$\begin{aligned}(A + I)^2 &= A^2 + 2A + I \\ &= (n - 1)I + (n - 2 - rc_2)A + c_2\underline{T}(J - I) + 2A + I \\ &= nI + (n - rc_2)A + c_2\underline{T}(J - I).\end{aligned}$$

□

This slight modification turns out to be helpful for the case $\delta = -2$ (that is $n - rc_2 = 0$).

Corollary

Let T be a finite group with neutral element e and let A be a covering matrix over T with $\Delta_A = K_n$.
Then the graph Γ^A is a regular (n, r, c_2) cover with $\delta = -2$ if and only if $A + I$ is a self-adjoint $\mathrm{gH}(T, n)$-matrix (note that this generalized Hadamard matrix has everywhere on its diagonal the element e).

Remarks

(1) It is convenient to call self-adjoint gH-matrices with identity diagonal skew gH-matrices.

(2) If T is a cyclic group of order 2, then we obtain the equivalence of distance regular double covers of K_n to regular two-graphs and in turn to classical skew Hadamard matrices.

The following construction leads to new infinite series of DRGs.

Theorem

Let T be a finite group, let $H = (h_{i,j})$ be any $\mathrm{gH}(T, n)$. Let $\psi : \{1, 2, \ldots, n\}^2 \to \{1, 2, \ldots, n^2\}$ be any bijection.
Define

$$R_H = (r_{(i,j)^\psi,(k,l)^\psi})$$

according to

$$r_{(i,j)^\psi,(k,l)^\psi} = h_{k,j} \cdot h_{i,l}^{-1}.$$

Then R_H is a skew $\mathrm{gH}(T, n^2)$.

Corollary

If there exists a gH(T, n) *over a finite group* T*, then for all* $t \in \mathbb{N} \setminus \{0\}$ *there exists a skew* gH(T, n^{2^t})*.*

Therefore, starting from any gH(T, n)-matrix we obtain an infinite series of regular covers of complete graphs.

- Consider the following gH-matrix A of order 6 over $\mathbb{Z}_3$:

$$A = \begin{pmatrix} 0_3 & 0_3 & 0_3 & 0_3 & 0_3 & 0_3 \\ 0_3 & 0_3 & 1_3 & -1_3 & -1_3 & 1_3 \\ 0_3 & 1_3 & 0_3 & 1_3 & -1_3 & -1_3 \\ 0_3 & -1_3 & 1_3 & 0_3 & 1_3 & -1_3 \\ 0_3 & -1_3 & -1_3 & 1_3 & 0_3 & 1_3 \\ 0_3 & 1_3 & -1_3 & -1_3 & 1_3 & 0_3 \end{pmatrix}$$

- Consider, for example, the function $\psi : (i,j) \mapsto 6(i-1)+j$

Get a skew gH-matrix B of order 36:

$$B = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
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\end{pmatrix}$$

- Matrix B is a skew gH($\mathbb{Z}_3, 36$).
- Hence $B - I$ is a $GHM(\mathbb{Z}_3, 36, 3, 12)$,
- We obtain a regular $(36, 3, 12)$-cover.
- To the best of our knowledge, the series of DRGs on $3 \cdot 6^{2^k}$ vertices is new.

#	parameters of H	parameters of R_H	(n, r, c_2) of Γ^{R_H}	$\lvert V(\Gamma^{R_H})\rvert$
1	gH$(E_2, 2)$	gH$(E_2, 4)$	$(4, 2, 2)$	8
2	gH$(E_3, 3)$	gH$(E_3, 9)$	$(9, 3, 3)$	27
3	gH$(E_4, 4)$	gH$(E_4, 16)$	$(16, 4, 4)$	64
4	gH$(E_2, 4)$	gH$(E_2, 16)$	$(16, 2, 8)$	32
5	gH$(E_5, 5)$	gH$(E_5, 25)$	$(25, 5, 5)$	125
6	**gH$(\mathbf{E_3}, \mathbf{6})$**	**gH$(\mathbf{E_3}, \mathbf{36})$**	**$(\mathbf{36}, \mathbf{3}, \mathbf{12})$**	**108**
7	gH$(E_7, 7)$	gH$(E_7, 49)$	$(49, 7, 7)$	343
8	gH$(E_8, 8)$	gH$(E_8, 64)$	$(64, 8, 8)$	512
9	gH$(E_4, 8)$	gH$(E_4, 64)$	$(64, 4, 16)$	256
10	gH$(E_2, 8)$	gH$(E_2, 64)$	$(64, 2, 32)$	128
11	gH$(E_9, 9)$	gH$(E_9, 81)$	$(81, 9, 9)$	729
12	gH$(E_3, 9)$	gH$(E_3, 81)$	$(81, 3, 27)$	243
13	**gH$(\mathbf{E_5}, \mathbf{10})$**	**gH$(\mathbf{E_5}, \mathbf{100})$**	**$(\mathbf{100}, \mathbf{5}, \mathbf{20})$**	**500**
14	gH$(E_{11}, 11)$	gH$(E_{11}, 121)$	$(121, 11, 11)$	1331
15	**gH$(\mathbf{E_3}, \mathbf{12})$**	**gH$(\mathbf{E_3}, \mathbf{144})$**	**$(\mathbf{144}, \mathbf{3}, \mathbf{48})$**	**432**
16	**gH$(\mathbf{E_4}, \mathbf{12})$**	**gH$(\mathbf{E_4}, \mathbf{144})$**	**$(\mathbf{144}, \mathbf{4}, \mathbf{36})$**	**576**
17	**gH$(\mathbf{E_2}, \mathbf{12})$**	**gH$(\mathbf{E_2}, \mathbf{144})$**	**$(\mathbf{144}, \mathbf{2}, \mathbf{72})$**	**288**

Table: Small regular covers obtained from our construction