Quantifier Elimination for p-Algebras of the 2nd Lee Class

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Quantifier elimination

Definition: Let $\mathcal{L}$ be a first order language with equality.

- An $\mathcal{L}$-structure $\mathcal{A}$ has **quantifier elimination (q.e.)** if for every $\mathcal{L}$-formula φ there exists a quantifier free $\mathcal{L}$-formula ψ such that $\mathcal{A} \models \varphi \leftrightarrow \psi$.
- A class $\mathbb{K}$ of similar $\mathcal{L}$-structures has q.e. if for every $\mathcal{L}$-formula φ there exists a quantifier free $\mathcal{L}$-formula ψ such that *for every* $\mathcal{A} \in \mathbb{K}$ there holds $\mathcal{A} \models \varphi \leftrightarrow \psi$.

Examples

1. The dense chains with endpoints in the language $\{\leq, 0, 1\}$ have q.e.

2. The real numbers in the language $\{+, -, \cdot, 0, 1\}$ do not have q.e.
   - $\varphi(a) : \exists x(x \cdot x = a)$ has no quantifier free equivalent.

3. The real numbers in the language $\{+, -, \cdot, 0, 1, \leq\}$ have q.e.
   - $\mathbb{R} \models \varphi(a) \leftrightarrow 0 \leq a$.

p-algebras

Definition: Let $\mathcal{L}_p = \{\wedge, \vee, ^*, 0, 1\}$ (similarity type $(2, 2, 1, 0, 0)$).

An $\mathcal{L}_p$-structure $\mathcal{A}$ is called a **p-algebra** if

- $< A; \wedge, \vee, 0, 1 >$ is a bounded distributive lattice and,
- for all $a, b \in A$, $a \leq b^*$ iff $a \wedge b = 0$.

Lee classes of p-algebras

The class of all p-algebras $\mathcal{A}$ satisfying

- $x \vee x^* = 1$ (Boolean algebras) is denoted by $\mathbb{B}_0$,
- $x^* \vee x^{**} = 1$ (Stone algebras) is denoted by $\mathbb{B}_1$,
- $(x_1 \wedge x_2)^* \vee (x_1^* \wedge x_2)^* \vee (x_1 \wedge x_2^*)^* = 1$ is denoted by $\mathbb{B}_2$.
- $\ldots$

$\mathbb{B}_0 \subseteq \mathbb{B}_1 \subseteq \mathbb{B}_2 \subseteq \ldots \subseteq \mathbb{B}_\omega$,
where $\mathbb{B}_\omega$ denotes the class of all p-algebras.

Existential closedness

Definition: Let $\mathbb{K}$ be a class of algebras. An algebra $\mathcal{A} \in \mathbb{K}$ is **existentially closed in** $\mathbb{K}$ (e.c. in $\mathbb{K}$) if every finite set of equations and inequations that has a solution in an extension $\mathcal{B} \in \mathbb{K}$ of $\mathcal{A}$ has a solution in $\mathcal{A}$.

The link between q.e. and e.c.

From a theorem due to Weispfenning (1985) it follows:
If $\mathbb{B}_i$ has the amalgamation property (AP), then the class of p-algebras e.c. in $\mathbb{B}_i$ (denoted by $\mathrm{EC}(\mathbb{B}_i)$) has q.e.

$\mathbb{B}_0$, $\mathbb{B}_1$, $\mathbb{B}_2$ and $\mathbb{B}_\omega$ have AP (Grätzer 1971), so $\mathrm{EC}(\mathbb{B}_i)$ has q.e. ($i = 0, 1, 2, \omega$).

Question: is every infinite q.e. p-algebra e.c. in $\mathbb{B}_i$ ($i = 0, 1, 2, \omega$)?

Answer for Boolean algebras: YES

Theorem (Tarski 1949): A Boolean algebra $\mathcal{A}$ has q.e. iff

- $\mathcal{A}$ is finite and has 1, 2 or 4 elements, or
- $\mathcal{A}$ is e.c. in $\mathbb{B}_0$.

  ($\mathcal{A}$ is e.c. iff $\mathcal{A}$ is atomless.)

Answer for Stone algebras: NO

Theorem (Feuerstein 1989): A non-Boolean Stone algebra $\mathcal{A}$ has q.e. iff

- $\mathcal{A}$ is finite and has three elements, or
- $\mathcal{A}$ is infinite and e.c. in $\mathbb{B}_1$, or
- $\mathcal{A}$ **is a "freak"** (i.e. infinite and has q.e., but not e.c.).

Stonian freaks

A Stone algebra $\mathcal{A}$ is a **freak** if

- $\mathcal{A}$ is contained in the class $\mathbf{C}_{01}$ of dense chains with endpoints,
- $\mathcal{A}$ is contained in the class $\mathbf{1} \oplus \mathbf{B} \oplus \mathbf{1}$, where $\mathbf{B}$ is the class of relatively complemented, dense, distributive lattices without endpoints,
- $\mathcal{A}$ is contained in the class $\mathbf{1} \oplus \mathbf{B}_1$, where $\mathbf{B}_1$ is the class of relatively complemented, dense, distributive lattices with greatest element but without least element.

$\rightarrow$ notice that the center $C(\mathcal{A}) = \{c \in A : c \vee c^* = 1\}$ of all of these algebras is trivial, i.e. $C(\mathcal{A}) = \{0, 1\}$.

Quantifier elimination classes for $\mathbb{B}_2$

Conjecture: A non-Stonian p-algebra $\mathcal{A} \in \mathbb{B}_2$ has q.e. iff

- $\mathcal{A}$ is finite and is isomorphic to $(2 \times 2) \oplus 1$, or
- $\mathcal{A}$ is infinite and e.c. in $\mathbb{B}_2$.

Quantifier elimination classes for $\mathbb{B}_2$

Conjecture: A non-Stonian p-algebra $\mathcal{A} \in \mathbb{B}_2$ has q.e. iff

- $\mathcal{A}$ is finite and is isomorphic to $(2 \times 2) \oplus 1$, or
- $\mathcal{A}$ is infinite and e.c. in $\mathbb{B}_2$.

In other words: If an infinite p-algebra $\mathcal{A} \in \mathbb{B}_2 \setminus \mathbb{B}_1$ has q.e. then $\mathcal{A}$ is e.c. in $\mathbb{B}_2$.

Quantifier elimination classes for $\mathbb{B}_2$ II

Proposition (combine Feuerstein 1989 + Clark/Schmid 1995): If an infinite p-algebra $\mathcal{A} \in \mathbb{B}_2 \setminus \mathbb{B}_1$ has q.e. and $C(\mathcal{A})$ is non-trivial, then $\mathcal{A}$ is e.c. in $\mathbb{B}_2$

Left to prove: There is no infinite p-algebra in $\mathbb{B}_2 \setminus \mathbb{B}_1$ that has q.e. and has a trivial center.

Quantifier elimination classes for $\mathbb{B}_2$ III

Facts:

- If a p-algebra $\mathcal{A}$ has q.e., then so does the subalgebra $D_0(\mathcal{A})$ with carrier set
  $|D_0(\mathcal{A})| = \{d \in A : d^* = 0\} \cup \{0\}$. (a Stone algebra)
- The skeleton $Sk(\mathcal{A}) = \{s \in A : s = s^{**}\}$ with the new join operation $\sqcup$ defined by $a \sqcup b = (a^* \wedge b^*)^*$, forms a Boolean algebra and has q.e.
  (The skeleton is not a subalgebra in general.)

Quantifier elimination classes for $\mathbb{B}_2$ IV

Corollary: If there is a freak $\mathcal{A}$ in $\mathbb{B}_2$, then

- $Sk(\mathcal{A})$ has 2 or 4 elements or is e.c. in $\mathbb{B}_0$,
- $D_0(\mathcal{A})$ has 3 elements or is a Stonian freak or is existentially closed in $\mathbb{B}_1$.

Proposition: If there is an infinite p-algebra $\mathcal{A} \in \mathbb{B}_2 \setminus \mathbb{B}_1$ which has q.e., then it has the following properties:

- $C(\mathcal{A}) = \{0, 1\}$,
- $D_0(\mathcal{A}) \in \mathbf{1} \oplus \mathbf{B}_1$,
- $Sk(\mathcal{A})$ is atomless.

q.e.= e.c. for p-algebras in $\mathbb{B}_\omega$? NO

A p-algebra $\mathcal{A} \in \mathbb{B}_\omega \setminus \mathbb{B}_2$ has q.e. if

- $\mathcal{A}$ is e.c. in $\mathbb{B}_\omega$.
- $\mathcal{A}$ is in the class $\mathbf{B}_{01} \oplus 1$, where B_{01} denotes the class of atomless Boolean algebras.
- (maybe more of them, with non-trivial center.)

References

- S. Feuerstein, *Quantifier Elimination for Stone Algebras*, Arch. Math **28** (1989), 75-89
- J. Schmid, *Model Companions of Distributive p-Algebras*, The Journal of Symbolic Logic **43** (1982), 680-688
- A. Tarski, *Arithmetical Classes and Types of Boolean Algebras (Preliminary report)*, Bull. Am. Soc. Math. **55** (1949), 64
- V. Weispfenning, *Efficient Decision Algorithms for Locally Finite Theories*, In: Lect. Notes Comput.Sci., vol. 229. Proc AAECC-3 Grenoble 1985. Berlin Heidelberg New York Tokyo: Springer 1985, pp.262-273