Some Aspects of Quasi-Stone Algebras Part I

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QSAs – a generalisation of Stone algebras

Recall:

Definition

A *Stone algebra* is a pseudocomplemented distributive lattice $(L; \wedge, \vee, ^*, 0, 1)$ where for all $a \in L$

$$a^* \vee a^{**} = 1 \quad \text{(Stone identity).}$$

Variety generated by the following three-element algebra

$1 \quad 1^* = 0$

$a \quad a^* = 0$

$0 \quad 0^* = 1$

Definition (Sankappanavar and Sankappanavar, 1993)

An algebra $(L; \wedge, \vee, ', 0, 1)$ is a *quasi-Stone algebra* (QSA) if $(L; \wedge, \vee, 0, 1)$ is a bounded distributive lattice and the unary operation $'$ satisfies the following conditions:

(QS1) $0' = 1$ and $1' = 0$,

(QS2) $(a \vee b)' = a' \wedge b'$ (the $\vee$-DeMorgan law),

(QS3) $(a \wedge b')' = a' \vee b''$ (the weak $\wedge$-DeMorgan law),

(QS4) $a \wedge a'' = a$,

(QS5) $a' \vee a'' = 1$ (Stone identity),

for all $a, b \in L$.

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- Stone $\Longrightarrow$ QSA
- QSA + $(a \wedge b)' = a' \vee b' \Longrightarrow$ Stone

Q-distributive lattices (Cignoli, 1991)

Bounded distributive lattice $(L; \wedge, \vee, \nabla, 0, 1)$ with a unary operation ∇, called *quantifier*

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- QSA $\Longrightarrow$ Q-distributive lattice with $\nabla a := a''$
- Q-distributive lattice + $\nabla(L) = \{\nabla y \mid y \in L\}$ complemented $\Longrightarrow$ QSA with $a' := (\nabla a)^c$

Duality for QSAs

Priestley Duality:

$(L; \wedge, \vee, 0, 1)$ bounded distributive lattices	$\longleftrightarrow$	$(X; \tau, \leq)$ compact, totally order-disconnected spaces
L	$\longrightarrow$	$D(L)$ prime filters
$E(X)$ clopen increasing sets	$\longleftarrow$	X
$f : L_1 \to L_2$ $\{0, 1\}$-lattice homomorphisms	$\longleftrightarrow$	$\varphi : X_2 \to X_1$ continuous, order preserving functions

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A *quasi-Stone space* (*QS-space*) is a pair $(X;\mathcal{E})$ such that X is a Priestley space and $\mathcal{E}$ is an equivalence relation on X satisfying the following three conditions:

(1) The equivalence classes of $\mathcal{E}$ are closed in X,

(2) $\mathcal{E}(U) \in E(X)$ for each $U \in E(X)$,

(3) $X \setminus \mathcal{E}(U) \in E(X)$ for each $U \in E(X)$.

$\mathcal{E}(Y) := \bigcup_{x\in Y}[x]_{\mathcal{E}}$, for each $Y \subseteq X$

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- Let $(X; \mathcal{E})$ be a QS-space. Then $(E(X);')$ is a QSA with $U' = X \setminus \mathcal{E}(U)$ for each $U \in E(X)$.

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- Let $(X;\mathcal{E})$ be a QS-space. Then $(E(X);')$ is a QSA with $U' = X \setminus \mathcal{E}(U)$ for each $U \in E(X)$.
- Let $(L;')$ be a QSA. Then $(D(L);\mathcal{E})$ is a QS-space with $\mathcal{E} = \{(P,Q) \in D(L) \times D(L) \mid P \cap B(L) = Q \cap B(L)\}$. $B(L) = \{x' \mid x \in L\}$ is the skeleton of L.

Lemma (Gaitán, 2000)

If $(X, \mathcal{E})$ *is a QS-space and* $x, y \in X$ *are non-equivalent, then they are incomparable.*

QS-maps:

Let $(L_1,')$, $(L_2,^*)$ be QSAs and let $(X_1,\mathcal{E}_1)$, $(X_2,\mathcal{E}_2)$ be their corresponding QS-spaces.
Let $f : L_1 \to L_2$ be a $\{0,1\}$-lattice homomorphism and $\varphi : X_2 \to X_1$ its dual map.
Then f is a QSA homomorphism (i.e. $f(a') = f(a)^*$ for each $a \in L$) if and only if

$$\mathcal{E}_2(\varphi^{-1}(U)) = \varphi^{-1}(\mathcal{E}_1(U))$$

for each clopen increasing set $U \subseteq X_1$.

Lemma (finite case: Gaitán, 2000; general: J.D.F., S.F.)

If $(X_1, \mathcal{E}_1)$ *and* $(X_2, \mathcal{E}_2)$ *are QS-spaces, then* $\varphi : X_1 \to X_2$ *is a QS-map if and only if*

1. $(x, y) \in \mathcal{E}_1$ *implies* $(\varphi(x), \varphi(y)) \in \mathcal{E}_2$*, and*
2. z *maximal in* $[\varphi(x)]_{\mathcal{E}_2}$ *implies* $z = \varphi(y)$ *for some* $y \in [x]_{\mathcal{E}_1}$.

Subvarieties of **QSA**

Finite subdirectly irreducible QSA's:

Q_0 denotes the one-element QSA and
$Q_{m,n} := (\widehat{B}_m \times B_n; \wedge, \vee, ', (0,0), (u_m, 1))$, where B_n denotes the Boolean lattice with n atoms, $\widehat{B}_m := B_m \oplus \{u_m\}$, and the operation $'$ is *special*, i.e.

$$(x,y)' = \begin{cases} (0,0) & \text{if } (x,y) \neq (0,0)\,, \\ (u_m, 1) & \text{if } (x,y) = (0,0)\,. \end{cases}$$

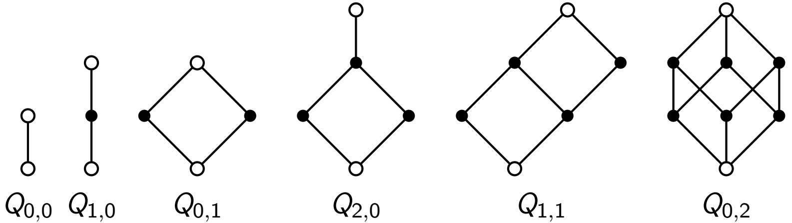
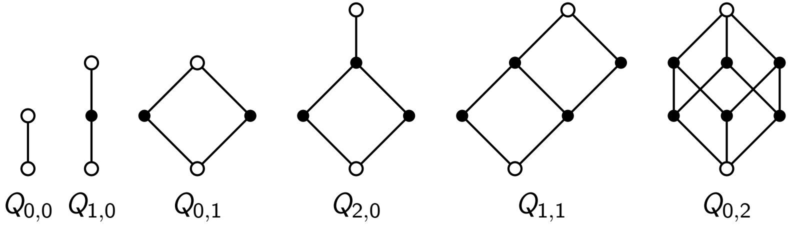


Figure: Some subdirectly irreducible QSAs.



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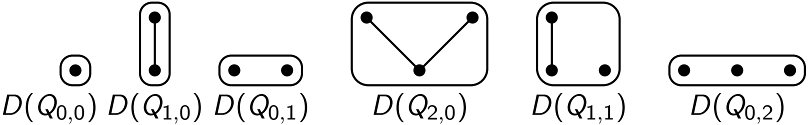

Figure: The corresponding QS-spaces.

$(\omega + 1)$-chain of subvarieties:

$$
\begin{array}{lllllllll}
\mathbb{V}(Q_0) & \subset & & & & & & & \\
\mathbb{V}(Q_{0,0}) & \subset & & & & & & & \\
\mathbb{V}(Q_{1,0}) & \subset & \mathbb{V}(Q_{0,1}) & & & & & & \\
\mathbb{V}(Q_{2,0}) & \subset & \mathbb{V}(Q_{1,1}) & \subset & \mathbb{V}(Q_{0,2}) & \subset & & & \\
\mathbb{V}(Q_{3,0}) & \subset & \mathbb{V}(Q_{2,1}) & \subset & \mathbb{V}(Q_{1,2}) & \subset & \mathbb{V}(Q_{0,3}) & \subset & \\
\ldots & & & & & & & & \\
\ldots & & & & & & & & \subset \mathbf{QSA}
\end{array}
$$

Lemma ($\Leftarrow$: Sankappanavar, 1993; $\Rightarrow$: J.D.F., S.F.)

Let $Q_{j,k}$ and $Q_{m,n}$ be finite subdirectly irreducible Quasi-Stone algebras. Then there exists an embedding $f : Q_{j,k} \hookrightarrow Q_{m,n}$ if and only if $j + k \leq m + n$ and $k \leq n$.

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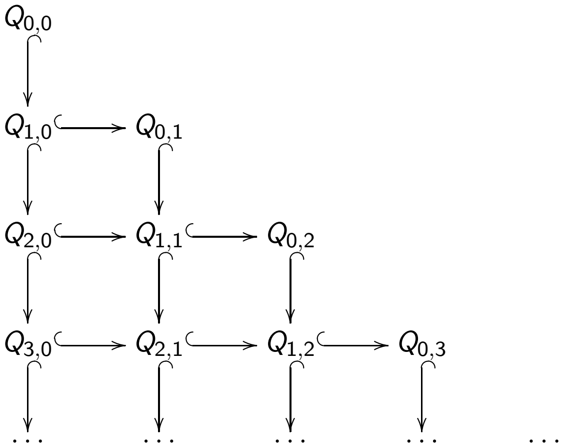
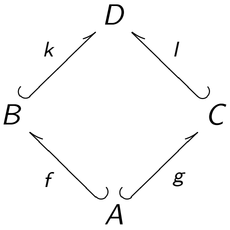


Amalgamation property

Definition

A class $\mathbb{K}$ of algebras has the *amalgamation property* (AP) if, for all A, B, $C \in \mathbb{K}$ and all embeddings $f : A \hookrightarrow B$ and $g : A \hookrightarrow C$, there is $D \in \mathbb{K}$ and embeddings $k : B \hookrightarrow D$ and $l : C \hookrightarrow D$ such that $k \circ f = l \circ g$.



Amalgamation property in the subvarieties of **QSA**

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  Do not have CEP (Congruence extension property), since $Q_{1,0} \hookrightarrow Q_{0,1}$ and $Q_{0,1}$ is simple but $Q_{1,0}$ is not.
  $\Rightarrow$ no AP (Bergman and McKenzie (1988): congruence distributivity, residually smallness, and AP imply CEP)

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- **QSA**: AP for $\mathbf{QSA}_{\text{fin}}$ claimed by Gaitán (2000)

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Strategy:

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Duality:

Coproduct of QSAs $\longleftrightarrow$ Product of QS-spaces

Problem: Products in the category of QS-spaces do not correspond to cartesian products!

QSA congruences

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$\theta_Y = \{(U,V) \mid U \cap Y = V \cap Y\}$	$\longleftarrow$	$Y \subseteq X$

Proposition (S.F.)

Let $(X, \mathcal{E})$ be a QS-space, $(L, ')$ its dual quasi-Stone algebra, and $Y \subseteq X$ a closed subset. Then θ_Y is a QSA congruence on L (i.e. $(a,b) \in \theta_Y \Rightarrow (a',b') \in \theta_Y$ for $a, b \in L$) if and only if $\mathcal{E}(Y) = \downarrow Y$.

($\downarrow Y = \{x \in X \mid x \leq y$ for some $y \in Y\}$)

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