Bounded lattices with an antitone involution the complemented elements of which form a sublattice

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*-lattices (these are bounded lattices with an involution, denoted by *, satisfying De Morgan's laws) often serve as models for logics. *-complemented elements of such logics can be considered as sharp assertions corresponding to classical logic. The natural question arises when these elements form a sublogic. The problem of characterizing the structure of bounded lattices with an antitone involution the complemented elements of which form a sublattice seems to be very hard.

We start with the definition of a bounded lattice with an antitone involution and of a complemented element.

Definition

A **bounded lattice with an antitone involution** is an algebra $\mathcal{L} = (L, \vee, \wedge, ^*, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and

$$(x \vee y)^* = x^* \wedge y^*,$$
$$(x \wedge y)^* = x^* \vee y^* \text{ and}$$
$$(x^*)^* = x$$

hold for all $x, y \in L$. An element a of L is called **complemented** if $a \vee a^* = 1$ and $a \wedge a^* = 0$. Let $\mathrm{CE}(\mathcal{L})$ denote the set of all complemented elements of $\mathcal{L}$.

It is evident that if $\mathcal{L}$ is moreover, distributive, i.e. a De Morgan algebra, then $\mathrm{CE}(\mathcal{L})$ is the set of its Boolean elements and hence a sublattice of $\mathcal{L}$. Further, let us mention that $0, 1 \in \mathrm{CE}(\mathcal{L})$ in each case.

Denote by **K** the class of all *-lattices $\mathcal{L}$ for which $\mathrm{CE}(\mathcal{L})$ is a sublattice of $\mathcal{L}$.

A *-lattice $\mathcal{L}_2$ is called a **0-1-homomorphic image** of the *-lattice $\mathcal{L}_1$ if there exists a homomorphism f from $\mathcal{L}_1$ onto $\mathcal{L}_2$ satisfying $f^{-1}(\{0\}) = \{0\}$ and $f^{-1}(\{1\}) = \{1\}$.

Further, for every class $\mathbf{K_1}$ of *-lattices let $\mathbf{H}_{01}(\mathbf{K_1})$ denote the class of all 0-1-homomorphic images of algebras of $\mathbf{K_1}$.

First we state some conditions which are equivalent to the fact that a *-lattice belongs to **K**:

Lemma

For a **-lattice* $\mathfrak{L} = (L, \vee, \wedge, ^*, 0, 1)$ *the following are equivalent:*
(i) $\mathfrak{L} \in \mathbf{K}$
(ii) $\mathrm{CE}(\mathfrak{L})$ *is a subuniverse of* $\mathfrak{L}$
(iii) $\mathrm{CE}(\mathfrak{L})$ *is a subuniverse of* $(L, \vee)$
(iv) $\mathrm{CE}(\mathfrak{L})$ *is a subuniverse of* $(L, \wedge)$.

Now we provide some examples. In the following, Hasse diagrams of *-lattices are drawn in such a way that * is the reflection on the central point of the Hasse diagram.

Example

The modular non-distributive *-lattice $\mathcal{M}_{3,3} = (M_{3,3}, \vee, \wedge, ^*, 0, 1)$ having the Hasse diagram

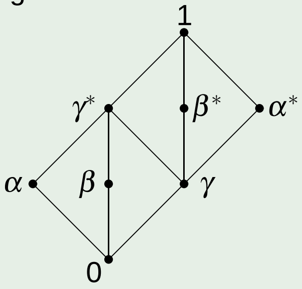


Fig. 1

does not belong to **K** since $\mathrm{CE}(\mathcal{M}_{3,3}) = \{0, \alpha, \beta, \beta^*, \alpha^*, 1\}$ is not a subuniverse of $\mathcal{M}_{3,3}$.

Example

The modular non-distributive *-lattice $\overline{\mathcal{M}}_{3,3} = 1 \oplus \mathcal{M}_{3,3} \oplus 1$ (ordinal sum) having the Hasse diagram

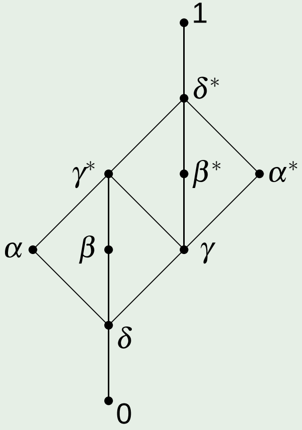


Fig. 2

belongs to **K** since $\mathrm{CE}(\overline{\mathcal{M}}_{3,3}) = \{0,1\}$ is a subuniverse of $\overline{\mathcal{M}}_{3,3}$.

Example

The non-modular *-lattice $\mathcal{L}_0$ having the Hasse diagram

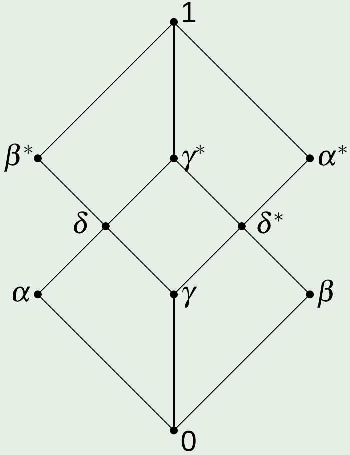


Fig. 3

does not belong to **K** since $\mathrm{CE}(\mathcal{L}_0) = \{0, \alpha, \beta, \beta^*, \alpha^*, 1\}$ is not a subuniverse of $\mathcal{L}_0$.

Remark. It is easy to see that $\mathcal{M}_{3,3} \notin \mathbf{H}_{01}(\mathcal{L}_0)$.

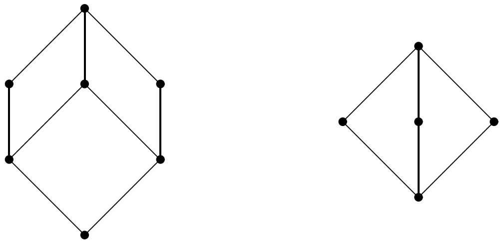
Remark. **K** is not a variety since $\overline{\mathcal{M}}_{3,3}$ belongs to **K** but its homomorphic image $\mathcal{M}_{3,3}$ does not.

Next we prove necessary respectively sufficient conditions for *-lattices to belong to **K**. In the following theorem a necessary condition for *-lattices to belong to **K** is given:

Theorem 1

For a *-lattice $\mathcal{L} = (L, \vee, \wedge, ^*, 0, 1)$ the condition $\mathcal{M}_{3,3}, \mathcal{L}_0 \notin \mathbf{H}_{01}(\mathbf{S}(\{\mathcal{L}\}))$ is necessary for $\mathcal{L} \in \mathbf{K}$.

Next we state some sufficient conditions for *-lattices to belong to **K**. First we define two join-semilattices. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ denote the join-semilattices with 1 with Hasse diagrams



respectively.

Theorem 2

For a *-lattice $\mathcal{L} = (L, \vee, \wedge, ^*, 0, 1)$ any single one of the following conditions is sufficient for $\mathcal{L} \in \mathbf{K}$:

(i) $x \vee y \vee (x^* \wedge y^*) \geq (x \vee x^*) \wedge (y \vee y^*)$ for all $x, y \in L$

(ii) $(x \vee x^*) \wedge y = (x \wedge y) \vee (x^* \wedge y)$ for all $x, y \in L$

(iii) $x \vee y \vee (x^* \wedge y^*) = (x \vee y \vee x^*) \wedge (x \vee y \vee y^*)$ for all $x, y \in L$

(iv) $\mathcal{S}_1, \mathcal{S}_2 \notin \mathbf{H}(\mathbf{S}(\{(L, \vee, 1)\}))$.

Remark. Since **K** is defined completely symmetric with respect to $\vee$ and $\wedge$, the dual assertions also hold.

Corollary. According to Theorem 2 every De Morgan algebra, i.e. every distributive *-lattice belongs to **K**.

Corollary. From the proof of Theorem 2 it follows that every *-lattice containing at most seven elements belongs to **K**.

Remark. Though $\overline{\mathcal{M}}_{3,3}$ belongs to **K**, it does not satisfy (i) of Theorem 2 since

$$\alpha \vee \beta \vee (\alpha^* \wedge \beta^*) = \gamma^* \not\geq \delta^* = (\alpha \vee \alpha^*) \wedge (\beta \vee \beta^*)$$

and hence also (ii) and (iii) of Theorem 2 are not satisfied. Moreover, $\overline{\mathcal{M}}_{3,3}$ does not satisfy (iv) of Theorem 2. This shows that any single one of the conditions (i)–(iv) of Theorem 2 is not necessary for $\mathcal{L} \in$ **K**.

$\mathcal{L}$ is called a **near-chain** if for all $a, b \in L$ either a and b or a and b^* (or both) are comparable. Let us note that the lattice $\mathcal{L}_0$ depicted in Fig. 3 is a near-chain which is not modular. Of course, every chain is a near-chain.

Now, we characterize near-chains belonging to **K**:

Theorem 3

For a near-chain $\mathcal{L} = (L, \vee, \wedge, ^*, 0, 1)$ the condition $\mathcal{L}_0 \notin \mathbf{H_{01}}(\mathbf{S}(\{\mathcal{L}\}))$ is necessary and sufficient for $\mathcal{L} \in \mathbf{K}$.

Corollary. Every near-chain containing at most nine elements belongs to **K**.

Corollary. Every modular near-chain belongs to **K**.

From now on, we consider bounded lattices with an antitone involution whose complemented elements do not form a sublattice. First, we get three technical lemmas.

Lemma

If $a, b \in \mathrm{CE}(\mathcal{L})$ *and either* $a \vee b \notin \mathrm{CE}(\mathcal{L})$ *or* $a \wedge b \notin \mathrm{CE}(\mathcal{L})$ *or both then* $a \wedge b \not\geq a^* \vee b^*$ *and* $a^* \wedge b^* \not\geq a \vee b$.

Lemma

Let $a, b \in \mathrm{CE}(\mathcal{L})$.

(i) *If* $a \vee b \notin \mathrm{CE}(\mathcal{L})$ *then* $0, 1, a, a^*, b, b^*, a \vee b, a^* \wedge b^*$ *are pairwise distinct.*

(ii) *If* $a \wedge b \notin \mathrm{CE}(\mathcal{L})$ *then* $0, 1, a, a^*, b, b^*, a \wedge b, a^* \vee b^*$ *are pairwise distinct.*

(iii) *If* $a \vee b, a \wedge b \notin \mathrm{CE}(\mathcal{L})$ *then* $0, 1, a, a^*, b, b^*, a \vee b, a^* \wedge b^*$, $a \wedge b, a^* \vee b^*$ *are pairwise distinct.*

Lemma

If $a, b \in \mathrm{CE}(\mathcal{L})$, $a \vee b, a \wedge b \notin \mathrm{CE}(\mathcal{L})$, $a \wedge b < a^* \vee b^*$ *and* $a^* \wedge b^* < a \vee b$ *then* (i) – (iii) *hold:*

(i) $0, 1, a, a^*, b, b^*, a \vee b, a^* \wedge b^*, a \wedge b, a^* \vee b^*, (a \wedge b) \vee (a^* \wedge b^*)$ *are pairwise distinct.*

(ii) $0, 1, a, a^*, b, b^*, a \vee b, a^* \wedge b^*, a \wedge b, a^* \vee b^*, (a \vee b) \wedge (a^* \vee b^*)$ *are pairwise distinct.*

(iii) $(a \wedge b) \vee (a^* \wedge b^*) \leq (a \vee b) \wedge (a^* \vee b^*)$

Using the previous lemmas, we can prove the last theorem characterizing minimal forbidden sublattices.

Theorem 4

Let $\mathfrak{L} = (L, \vee, \wedge, ^*, 0, 1)$ be a bounded lattice with an antitone involution the set $\mathrm{CE}(\mathfrak{L})$ of all complemented elements of which does not form a sublattice. Then there exist $a, b \in \mathrm{CE}(\mathfrak{L})$ such that either $a \vee b \notin \mathrm{CE}(\mathfrak{L})$ or $a \wedge b \notin \mathrm{CE}(\mathfrak{L})$ or both and, up to symmetry, the following cases are possible:

(i) $a \vee b, a \wedge b \notin \mathrm{CE}(\mathfrak{L})$, $a \wedge b < a^* \vee b^*$ and $a^* \wedge b^* < a \vee b$

(ii) $a \vee b, a \wedge b \notin \mathrm{CE}(\mathfrak{L})$, $a \wedge b < a^* \vee b^*$ and $a^* \wedge b^* \parallel a \vee b$

(iii) $a \vee b, a \wedge b \notin \mathrm{CE}(\mathfrak{L})$, $a \wedge b \parallel a^* \vee b^*$ and $a^* \wedge b^* < a \vee b$

(iv) $a \vee b, a \wedge b \notin \mathrm{CE}(\mathfrak{L})$, $a \wedge b \parallel a^* \vee b^*$ and $a^* \wedge b^* \parallel a \vee b$

(v) $a \vee b \in \mathrm{CE}(\mathfrak{L})$, $a \wedge b \notin \mathrm{CE}(\mathfrak{L})$, $a \vee b = 1$ and $a \wedge b < a^* \vee b^*$

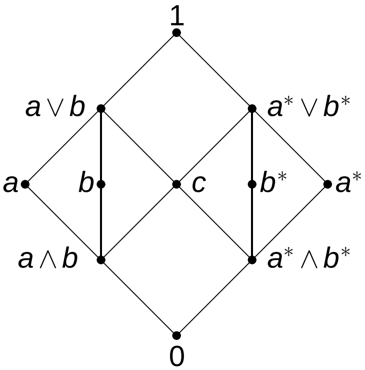
(vi) $a \vee b \in \mathrm{CE}(\mathfrak{L})$, $a \wedge b \notin \mathrm{CE}(\mathfrak{L})$, $a \vee b = 1$ and $a \wedge b \parallel a^* \vee b^*$

(vii) $a \vee b \in \mathrm{CE}(\mathfrak{L})$, $a \wedge b \notin \mathrm{CE}(\mathfrak{L})$, $a \vee b \neq 1$ and $a \wedge b < a^* \vee b^*$

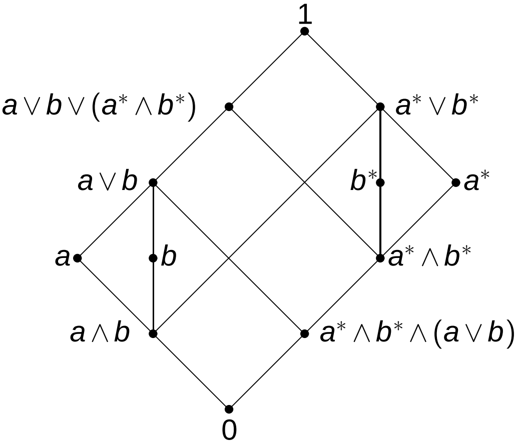
(viii) $a \vee b \in \mathrm{CE}(\mathfrak{L})$, $a \wedge b \notin \mathrm{CE}(\mathfrak{L})$, $a \vee b \neq 1$ and $a \wedge b \parallel a^* \vee b^*$

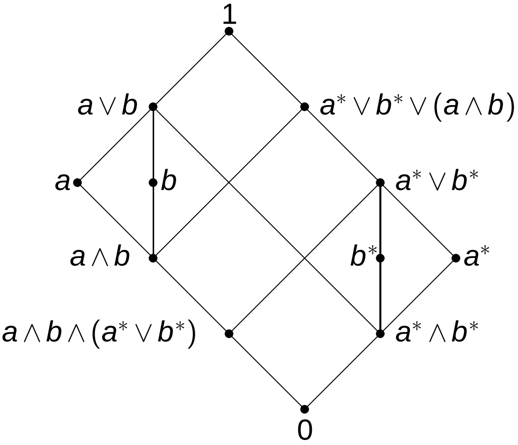
In the listed cases the following minimal (with respect to the cardinality) lattices exist:

(i):

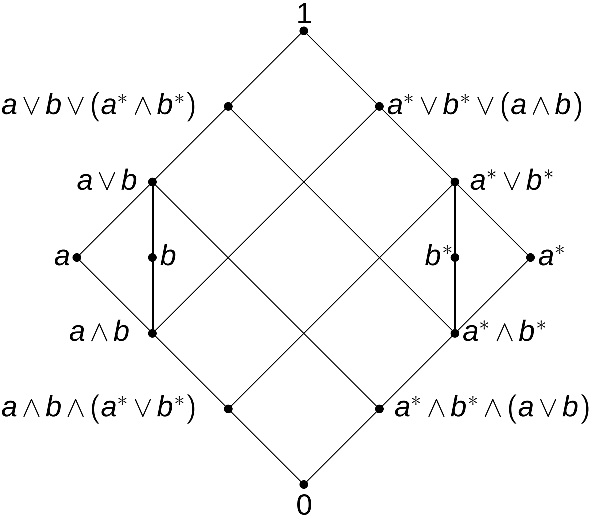


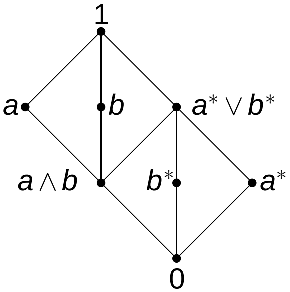
Here $c := (a \wedge b) \vee (a^* \wedge b^*) = (a \vee b) \wedge (a^* \vee b^*)$.

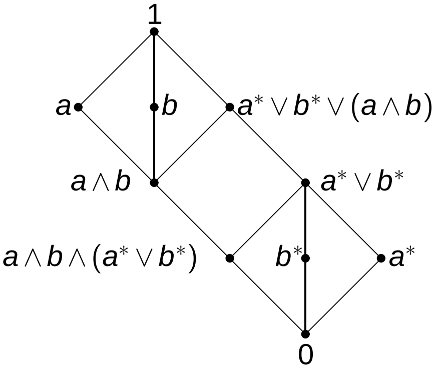

1
a ∨ b ∨ (a* ∧ b*)
a* ∨ b*
a ∨ b
b*
a*
a
b
a* ∧ b*
a ∧ b
a* ∧ b* ∧ (a ∨ b)
0


1
$a \vee b$
$a^* \vee b^* \vee (a \wedge b)$
a
b
$a^* \vee b^*$
$a \wedge b$
b^*
a^*
$a \wedge b \wedge (a^* \vee b^*)$
$a^* \wedge b^*$
0

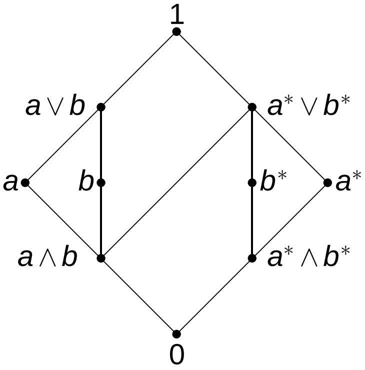
(iv):


1
$a \vee b \vee (a^* \wedge b^*)$
$a^* \vee b^* \vee (a \wedge b)$
$a \vee b$
$a^* \vee b^*$
a
b
b^*
a^*
$a \wedge b$
$a^* \wedge b^*$
$a \wedge b \wedge (a^* \vee b^*)$
$a^* \wedge b^* \wedge (a \vee b)$
0

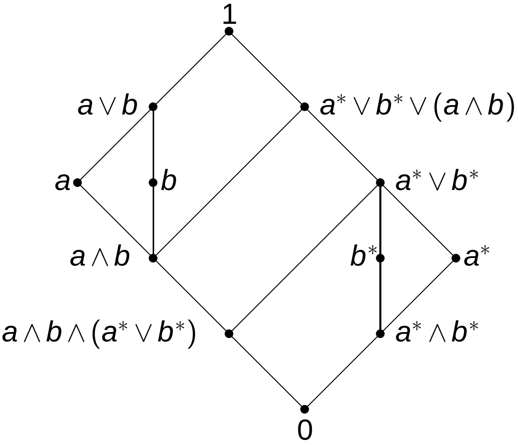

1
a
b
a* ∨ b*
a ∧ b
b*
a*
0


1
a
b
a* ∨ b* ∨ (a ∧ b)
a ∧ b
a* ∨ b*
a ∧ b ∧ (a* ∨ b*)
b*
a*
0

(vii):



(viii):


1
$a \vee b$
$a^* \vee b^* \vee (a \wedge b)$
a
b
$a^* \vee b^*$
$a \wedge b$
b^*
a^*
$a \wedge b \wedge (a^* \vee b^*)$
$a^* \wedge b^*$
0

Remark. The remaining case $a \vee b \notin \mathrm{CE}(\mathcal{L})$, $a \wedge b \in \mathrm{CE}(\mathcal{L})$ need not be considered since in this case a^*, b^* satisfies one of the conditions (v) – (viii).