A note on congruences and ideals in pseudo effect algebras and in their total algebra counterparts

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Introduction

- effect algebras (EA) were introduced by Foulis and Bennett in 1994 for modeling unsharp measurement in quantum mechanical system;
  they generalize many structures which arise in quantum physics, noncommutative probability and fuzzy logic (orthomodular lattices, orthomodular posets, MV-algebras etc.)
- equivalently there have been introduced:
  D-posets by Kôpka and Chovanec
  weak orthoalgebras by Giuntini and Greuling
- in 2001 Dvurečenskij and Vetterlein introduced pseudo effect algebras (PEA) as a noncommutative generalization of EA
- attempts to study EA as total algebraic structures were made, most recently by Chajda, Halaš and Kühr and then also adapted for PEA

The aim

ChHK : if $(E;+,0,1)$ is a lattice effect algebra and we define $x \oplus y := (x \wedge y') + y$ and $\neg x := x'$, then $(E;\oplus,\neg,0,1)$ is a basic algebra satisfying the quasi-identity

$$(*) \qquad x \leq \neg y, x \oplus y \leq \neg z \;\Rightarrow\; x \oplus (z \oplus y) = (x \oplus y) \oplus z$$

on the other hand - every basic algebra satisfying (*) after restriction of $\oplus$ to + operation for $a \leq \neg b$ becomes a lattice effect algebra.

PuVi : an ideal of the basic algebra (EBA) $\leftrightarrow$ a Riesz ideal of the effect algebra

ChK : totalization of pseudo effect algebras

$\hookrightarrow$ question: how to define ideals on total algebras corresponding to pseudo effect algebras?

Poset with sectional antiautomorphism (PSA)

- $\mathcal{P} = (A; (\gamma_p, \delta_p)_{p\in A}, 0, 1)$
  $(A, \leq)$ is a poset with $0 \leq x \leq 1$,
  γ_p and δ_p are antiautomorphisms on the section $[p, 1]$ inverse to each other
- we may define an operation $\sqcup$ on a PSA like this:

$$x \sqcup y = y \sqcup x := \begin{cases} x \vee y & \text{if } x \vee y \text{ exists} \\ \text{some upper bound of \{x,y\}} & \text{otherwise} \end{cases}$$

- the grupoid $(A, \sqcup)$ thus obtained is then a commutative directoid (idempotent grupoid satisfying the identity $x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$); we may further define two total binary operations:
  $x \to y := \gamma_y(x \sqcup y)$ and $x \rightsquigarrow y := \delta_y(x \sqcup y)$
  ...

PSA and total algebras

- $\mathcal{P}^A = (A; \to, \rightsquigarrow, 0, 1)$ of type $(2, 2, 0, 0)$ satisfies special conditions
- $\mathcal{A} = (A; \to, \rightsquigarrow, 0, 1)$ satisfying the conditions

  define $x \leq y$ iff $x \to y = 1$ iff $x \rightsquigarrow y = 1$
  and mappings γ_a, δ_a on $[a, 1]$ such that
  $\gamma(x) = x \to a$ and $\delta(x) = x \rightsquigarrow a$
- $\mathcal{A}^P = (A; \leq, (\gamma_a, \delta_a)_{a \in A}, 0, 1)$ is a PSA
- $(\mathcal{A}^P)^A = \mathcal{A}$ if we use $x \sqcup y = (x \to y) \rightsquigarrow y$
  also $(\mathcal{P}^A)^P = \mathcal{P}$, but the correspondence is clearly not one-to-one until the PSA is a lattice
- the operations $\to$ and $\rightsquigarrow$ may be alternatively replaced also by another (total) operations $\oplus, \boxplus, ^-, ^\sim$ under the following definition:
  $x^- := x \to 0 = \gamma_0(x) \qquad x \boxplus y := y^- \rightsquigarrow x = \delta_x(x \sqcup y^-)$
  $x^\sim := x \rightsquigarrow 0 = \delta_0(x) \qquad x \oplus y := x^\sim \to y = \gamma_y(x^\sim \sqcup y)$

Basic definitions

- A pseudo effect algebra is a partial algebra $(E; +, 0, 1)$ of the type $(2, 0, 0)$ where the following axioms hold for any $a, b, c \in E$:
  - $a + b$ and $(a + b) + c$ exist iff $b + c$ and $a + (b + c)$ exist and then $(a + b) + c = a + (b + c)$
  - $\exists! d \in E$ and $\exists! e \in E$ such that $a + d = e + a = 1$
  - if $a + b$ exists, there exist elements $d, e \in E$ such that $a + b = d + a = b + e$
  - if $a + 1$ or $1 + a$ exists, then $a = 0$
- partial order on PEA:
  $a \leq b$ iff $c + a = b$ for some $c \in E$ iff $a + d = b$ for some $d \in E$
- determined subtractions
  $b \diagdown a : \quad b = b \diagdown a + a$ $\qquad a \diagup b : \quad b = a + a \diagup b$
  left complement: $a^- := 1 \diagdown a$ right complement: $a^\sim := a \diagup 1$

Congruences and ideals on PEA

- Riesz congruence on a PEA is an equivalence $\sim$ where:
  - if $a \sim a_1, b \sim b_1$ and if $a+b, a_1+b_1$ exist, then $a+b \sim a_1+b_1$
  - if $a+b\,\exists$, then for any $a_1 \sim a$ there is $b_1 \sim b$ such that $a_1+b_1\,\exists$ and for any $b_2 \sim b$ there is $a_2 \sim a$ such that $a_2+b_2\,\exists$
  - if $a \sim b$, then $\exists\, c_1, c_2 : c_1+a, c_1+b, a+c_2, b+c_2$ are defined and $c_1+a \sim 1 \sim c_1+b$ and $a+c_2 \sim 1 \sim b+c_2$
- $E/\sim$: $[a]+[b]=[a_1+b_1]$ where $a_1 \sim a, b_1 \sim b$ (and $a_1+b_1\,\exists$)
- normal Riesz ideal I on a PEA E:
  - $\forall\, a \in E, r \in I$ with $a \leq r \;\Rightarrow\; a \in I$
  - $\forall\, r, s \in I$ such that $r+s$ is defined $\Rightarrow\; r+s \in I$
  - $\forall\, a, r, s \in E$ such that $r+a$ and $a+s$ exist and are equal $\Rightarrow\; r \in I$ iff $s \in I$
  - $\forall a, b \in E$ with $a+b$ defined and for any $r \in I$ such that $r \leq a+b \;\Rightarrow\; i, j \in I$ such that $i \leq a, j \leq b$ and $r \leq i+j$

Congruences and ideals on PEA

- following Dvurečenskij, Gudder, Pulmannová, Avallone and Vitolo . . .
- if I is an ideal on a PEA E, we may define for $a, b \in E$ a relation $\sim_I$ like this:

  $a \sim_I b$ iff $\exists i, j \in I : i \leq a, j \leq b, a \diagdown i = b \diagdown j$

LiLi :

- $\sim_I$ is a congruence on E iff I is a normal Riesz ideal on E
- If I is a normal Riesz ideal on a PEA, then $\sim_I$ is a Riesz congruence and $[0]_{\sim_I} = I$.
- If $\sim$ is a Riesz congruence on a PEA, then $I = [0]_\sim$ is a normal Riesz ideal and $\sim_I = \sim$.

PEA as total algebra

- now equip a PEA E with a quasi-supremum operation as before ($a \sqcup b = a \vee b$ or some upper bound)
  so that we can consider a PEA ($(E; \sqcup)$) to be a commutative directoid
  and define a total operations $\to$ and $\rightsquigarrow$ similarly as for PSA:
  $$(*) \qquad x \to y := (x \sqcup y)^{-} + y \qquad\qquad x \rightsquigarrow y := y + (x \sqcup y)^{\sim}$$
- in fact, if $\mathcal{E} = (E; +, 0, 1)$ is a PEA,
  defining $\gamma_e(x) = x^{-} + e$ and $\delta_e(x) = e + x^{\sim}$
  we obtain a PSA $\mathcal{E}^P = (E; \leq, (\gamma_e, \delta_e)_{e \in E}, 0, 1)$
  with some special conditions, where $\leq$ is the partial order of the PEA

  (Chajda, Kühr)

PEA as total algebra

(Chajda, Kühr):

- Let $\mathcal{E} = (E; +, 0, 1)$ be a PEA and let the operations $\rightarrow, \rightsquigarrow$ be defined by (*). Then the total algebra $\mathcal{E}^A = (E; \rightarrow, \rightsquigarrow, 0, 1)$ satisfies the following identities:
  - (a) $1 \rightarrow x = x = 1 \rightsquigarrow x$
  - (b) $x \sqcup y = y \sqcup x = (y \rightsquigarrow x) \rightarrow x = (x \rightsquigarrow y) \rightarrow y$
  - (c) $x \rightarrow ((x \sqcup y) \sqcup z) = 1$
  - (d) $(((x \sqcup y) \sqcup z) \rightarrow x) \rightarrow (y \rightarrow x) = 1$
  - (e) $(((x \sqcup y) \sqcup z) \rightsquigarrow x) \rightsquigarrow (y \rightsquigarrow x) = 1$
  - (f) $0 \rightarrow x = 1$
  - (g) $x \leq y \leq z \;\Rightarrow\; (y \rightarrow x) \rightsquigarrow (z \rightarrow x) = z \rightarrow y$
  - (h) $x \leq y \leq z \;\Rightarrow\; (y \rightsquigarrow x) \rightarrow (z \rightsquigarrow x) = z \rightsquigarrow y$
  - (i) $x \leq y$ and $y \rightarrow x \leq z \;\Rightarrow\; z \rightsquigarrow (y \rightarrow x) = y \rightarrow (z \rightsquigarrow x)$
- Let a total algebra $(A; \rightarrow, \rightsquigarrow, 0, 1)$ satisfies the previous conditions and define: $x + y$ exists iff $y \leq x^{\sim}$ and $x + y = x^{\sim} \rightarrow y$. Then the obtained structure $\mathcal{A}^E = (A; +, 0, 1)$ is a PEA.

Congruence in PEA and in corresponding total algebra

- Theorem: Let $(A, \to, \rightsquigarrow, 0, 1)$ be an algebra satisfying the conditions (a) - (i). Let $\sim$ be a congruence on this algebra, that is, an equivalence relation which preserves $\to$ and $\rightsquigarrow$. Then $\sim$ is a Riesz congruence in the corresponding pseudo effect algebra $\mathcal{A}^E$.

- Theorem: Let $\mathcal{E} = (E; +, 0, 1)$ be a pseudo-effect algebra and I a normal Riesz ideal in E. Then there is a total algebra $\mathcal{E}^A = (E; \to, \rightsquigarrow, 0, 1)$ such that the congruence $\sim_I$ induced by I in $\mathcal{E}$, is also a congruence of $\mathcal{E}^A$.

Lattice pseudo effect algebras

- in the case of a lattice PEA $\mathcal{E} = (E; +, 0, 1)$, the algebra $\mathcal{E}^A = (E; \to, \rightsquigarrow, 0, 1)$ has uniquely defined operations:
  $$x \to y = (x \vee y)^- + y \qquad x \rightsquigarrow y = y + (x \vee y)^\sim$$
- the total algebra thus obtained satisfies another identities:
  $$((y \sqcup z) \to x) \to (y \to x) = 1$$
  $$((y \sqcup z) \rightsquigarrow x) \rightsquigarrow (y \rightsquigarrow x) = 1$$
- and we have one-to-one correspondence between total algebras $\mathcal{A} = (A; \to, \rightsquigarrow, 0, 1)$ with these two properties and lattice PEA $\mathcal{A}^E = (A; +, 0, 1)$
- now we may 'switch' between the operations by these rules:
  $$x^- = x \to 0 \qquad x^\sim = x \rightsquigarrow 0$$
  $$x \to y = x^- \oplus y \qquad x \rightsquigarrow y = y \boxplus x^\sim$$
  $$x \oplus y = (x^\sim \vee y)^- + y \qquad x \boxplus y = x + (x \vee y^-)^\sim$$
  $x + y = x \oplus y = x \boxplus y$ iff $x \leq y^-$ iff $y \leq x^\sim$

Lattice PEAs - ideals

- Definition: A subset I of A is an ideal if the following is satisfied:
  - $x, t, s \in A : x + s = t + x \;\Rightarrow\; s \in I$ iff $t \in I$
  - $x, y \in I \;\Rightarrow\; x \oplus y \in I$ and $x \boxplus y \in I$
  - $x \in I, y \in A \;\Rightarrow\; (x^- \oplus y)^\sim \in I$ (also $(y \boxplus x^\sim)^- \in I$)
- if $\sim$ is a congruence of A, then the zero class of this congruence is such ideal in A

$\hookrightarrow$
  - (id2) follows from definition
  - (id3): if $x \sim 0$ then $(x^- \oplus y)^\sim \sim (0^- \oplus y)^\sim = (1 \oplus y)^\sim = 1^\sim = 0$
  - (id1): let $s \sim 0$; then $x + s \sim x + 0 = x$ and so $t + x \sim x$, therefore $[x] = [t] + [x] = [t]^\sim \to [x] = \gamma_{[x]}([t]^\sim \vee [x]) = \gamma_{[x]}([t]^\sim)$ because $[x] \leq [t]^\sim$ and so $[t]^\sim = [1]$ and $t \sim 0$

Lattice PEAs - ideals

- an ideal I on A also satisfies:
  $x \in I, y \leq x \;\Rightarrow\; y \in I$ and $x, y \in I \;\Rightarrow\; x \vee y \in I$
- Theorem: A subset I of A is an ideal if and only if I is a normal Riesz ideal in the pseudo effect algebra $\mathcal{A}^E$.
- we may express the relation $\sim_I$ in terms of the operations on total algebra:
- Theorem: Let I be an ideal of $\mathcal{A}$. Define $x \sim y$ iff $((x \wedge y) + (x \vee y)^\sim))^- \in I$ (or $(x \vee y)^- + (x \wedge y)^\sim \in I$). Then $\sim$ is a congruence of $\mathcal{A}$ which coincides with the congruence $\sim_I$ of the pseudo effect algebra $\mathcal{A}^E$.

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