From MV-algebras to GMV-algebras

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- **Aim:** to give an algebraic proof of the completeness of the Łukasiewicz infinite-valued sentential calculus.

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- 1. $x\oplus(y\oplus z)=(x\oplus y)\oplus z$
  2. $x\oplus y=y\oplus x$
  3. $x\oplus 0=x$
  4. $x^{**}=x$
  5. $x\oplus 0^*=0^*$
  6. $(x^*\oplus y)^*\oplus y=(y^*\oplus x)^*\oplus x.$

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- $(\Gamma(G, u), \oplus, ^*, 0)$ - prototypical example of MV-algebras

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- **GMV-algebra** is an algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z;$

(A2) $x \oplus 0 = 0 \oplus x = x;$

(A3) $x \oplus 1 = 1 \oplus x = 1;$

(A4) $1^{\sim} = 0; \; 1^{-} = 0;$

(A5) $(x^{-} \oplus y^{-})^{\sim} = (x^{\sim} \oplus y^{\sim})^{-};$

(A6) $x \oplus (x^{\sim} \odot y) = y \oplus (y^{\sim} \odot x) = (x \odot y^{-}) \oplus y = (y \odot x^{-}) \oplus x;$

(A7) $x \odot (x^{-} \oplus y) = (x \oplus y^{\sim}) \odot y;$

(A8) $(x^{-})^{\sim} = x.$

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- GMV-algebra M is an MV-algebra iff $x \oplus y = y \oplus x$ for all $x, y \in M$.

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- GMV-algebra M is an MV-algebra iff $x \oplus y = y \oplus x$ for all $x, y \in M$.
- a partial operation $+$ on M: $a + b = a \oplus b$ iff $a \odot b = 0$ iff $a \leq b^{-}$ iff $b \leq a^{\sim}$.

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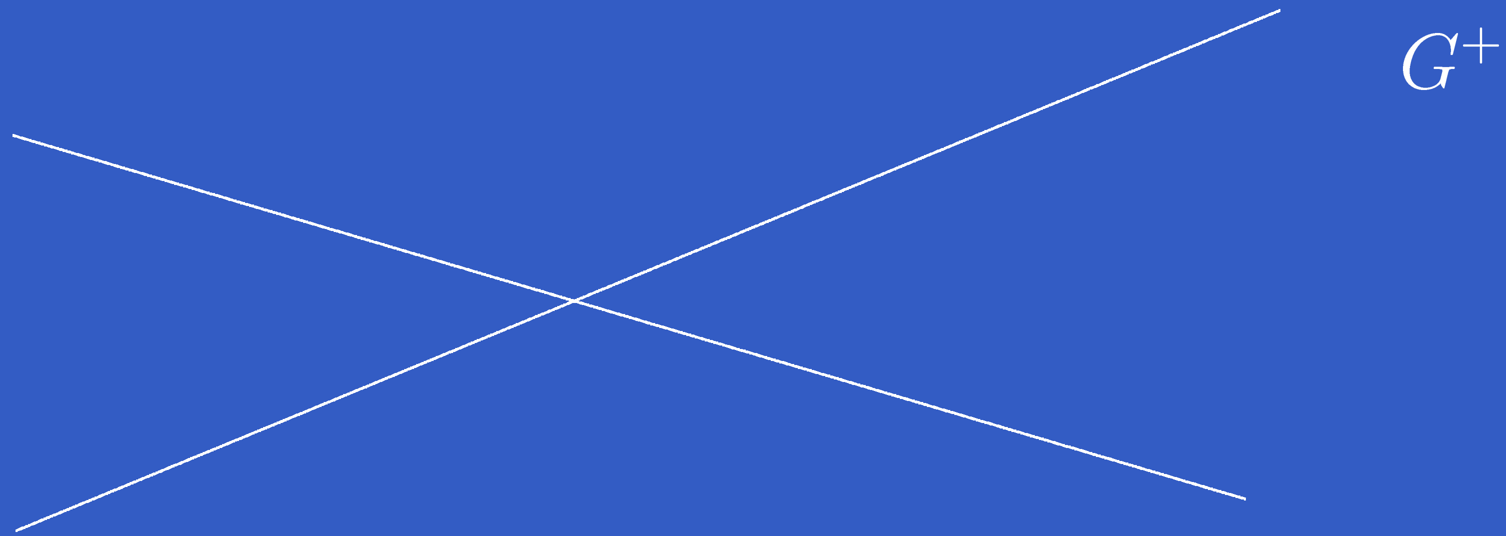
$(\Gamma(G, u); \oplus, ^{-}, ^{\sim}, 0, u)$ is a GMV-algebra.

Theorem 0.1 [Dvu 2002] *For any GMV-algebra* M*, there exists a unique* $($*up to isomorphism*$)$ *unital* ℓ*-group* G *with a strong unit* u *such that* $M \cong \Gamma(G,u)$.
The functor Γ *defines a categorical equivalence between the category of GMV-algebras and the category of unital* ℓ*-groups.*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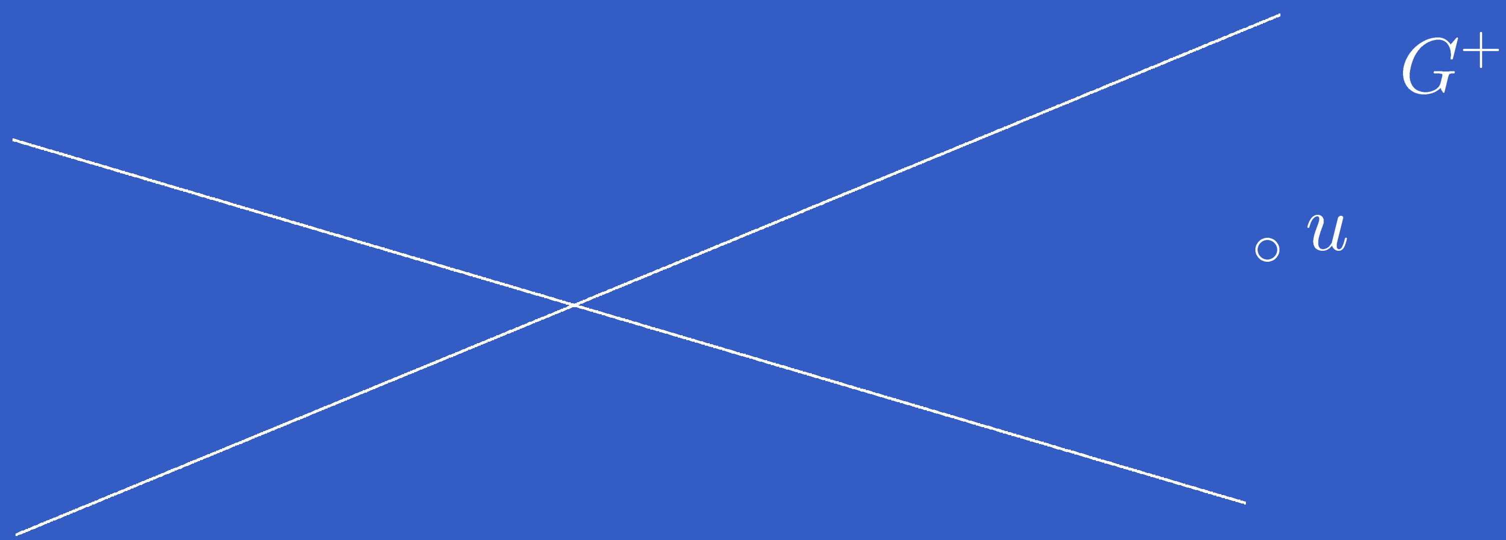
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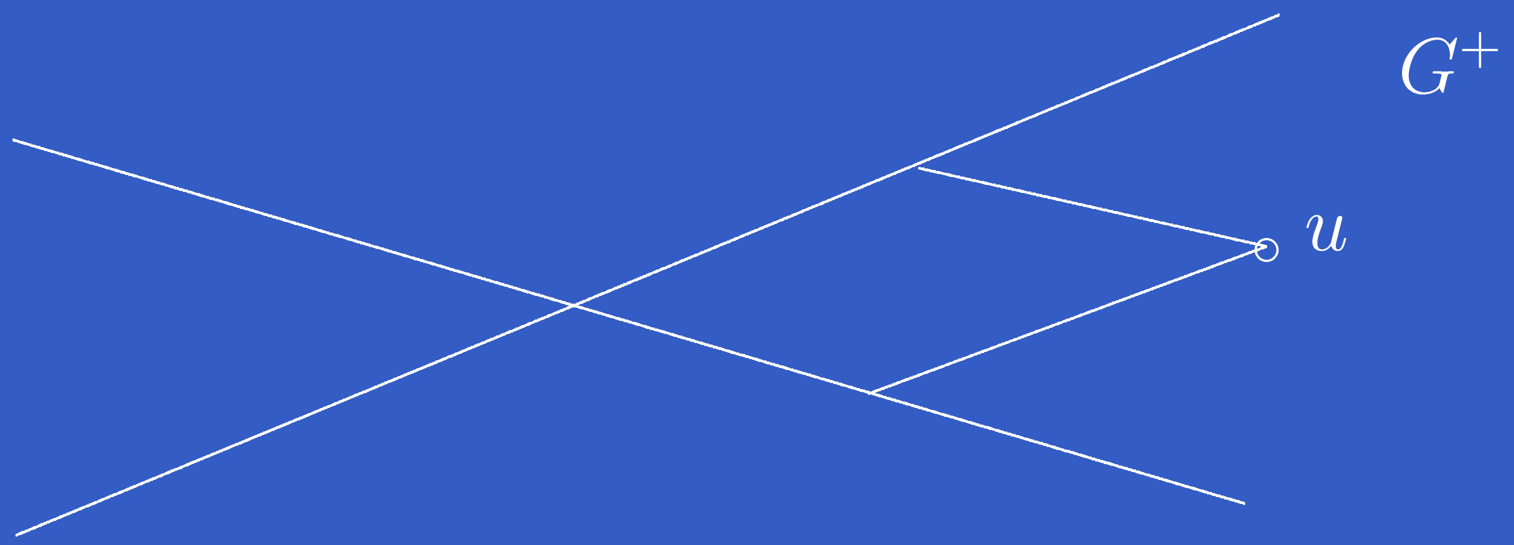
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- $\mathcal{V}(\mathcal{PGMV})$

Theorem 0.2 *The functor $\mathcal{E}$ defines a categorical equivalence between the category, $\mathcal{PGMV}$, of perfect GMV-algebras and the category, $\mathcal{L}$, of ℓ-groups.*

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- Every ℓ-group $G \hookrightarrow$ Aut(Ω) for some chain Ω. (G, Ω) permutation group
- (G, Ω) is **transitive** $\alpha, \beta \in \Omega\ \exists f \in G\ \alpha f = \beta$
- (G, Ω) is **doubly transitive** given $\alpha_1 < \alpha_2, \beta_1 < \beta_2 \in \Omega\ \exists f \in G\ \alpha_1 f = \beta_1$ $\alpha_2 f = \beta_2$

- Let $u \in$ Aut$(\mathbb{R})$ be the translation $tu = t + 1$, $t \in \mathbb{R}$, and let

$$\mathrm{BAut}(\mathbb{R}) = \{g \in \text{Aut}(\mathbb{R}) : \exists\, n \in \mathbb{N},\ \ u^{-n} \leq g \leq u^{n}$$

Then $($BAut$(\mathbb{R}), u)$ is a doubly transitive unital ℓ-permutation group

Theorem 0.8 [DiNola, A.D., Tsinakis] *If* $\mathcal{G}$ *is a variety of* ℓ*-groups, then* $\Phi(\mathcal{G}) := \mathcal{V}(\{\mathcal{E}(G) : G \in \mathcal{G}\})$ *is a subvariety of* $\mathcal{V}(\mathcal{PGMV})$ *and the mapping* $\mathcal{G} \mapsto \Phi(\mathcal{G})$ *is set-inclusion-preserving. Consequently, the variety of symmetric GMV-algebras has continuum many subvarieties.*

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- Komory: there is only countably many subvarieties of the variety of MV-algebras, $\mathcal{MV}$

n-perfect GMV-algebras

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(c) $M_i^- = M_{n-i} = M_i^\sim$ for any $i = 0, 1, \ldots, n$,

(d) if $x \in M_i$ and $y \in M_j$, then $x \oplus y \in M_{(i+j)\wedge n}$.

- prototypical example of an n-perfect symmetric GMV-algebra

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- Characterization of n-perfect is more complicated than that of perfect - partial result using cyclic elements
- $\Phi_n(\mathcal{G}) = \mathcal{V}(\{\mathcal{E}_n(G) : \; G \in \mathcal{G}\}, \mathcal{G}$ a subvariety of the variety ℓ-groups. Φ_n is a set-inclusion preserving mapping.

Top varieties

For any value V of (G, u), we set

$$K(V) = \bigcap_{g \in G} g^{-1} V g$$

(we momentarily employ multiplicative notation for (G, u)). Then $K(V)$ is a normal convex ℓ-subgroup of (G, u) contained in V, and $(G/K(V), G/V)$ is a primitive transitive ℓ-permutation group, called a **component** of G.

Let

$$\mathcal{T}(\mathcal{V}) =$$

$$\{\Gamma(G,u) : \Gamma(G/K(V), u/K(V)) \in \mathcal{V}, \forall\, V \in \Gamma(u)\}.$$

By [DvHo], $\mathcal{T}(\mathcal{V})$ is a variety, referred to as a **top variety** of $\mathcal{V}$.

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$$\mathcal{T}(\mathcal{S}_n) = \mathcal{T}(\mathcal{S}_n^\omega) =$$
$$=: \mathcal{BP}_n = \mathcal{T}(\mathcal{BP}_n).$$

Komori characterization

$$\delta(i) = \{n \in \mathbb{N} : \ 1 \leq n, \ n \text{ is a divisor of } i\},$$

$J \subset \mathbb{N}$ and $i \geq 2$ we let

$$\Delta(i, J) = \{d \in \delta(i) \setminus \bigcup_{j \in J} \delta(j)\}.$$

In case $J = \emptyset$, we define $\Delta(i, \emptyset) = \delta(i)$.

Lettieri Di Nola: $\mathcal{V}$ there exist finite sets I and J, $M \in \mathcal{V}$ iff M satisfies the identities

$$((n+1)\odot x^n)^2 = 2\odot x^{n+1}, \quad n = \max\{I \cup J\}, \tag{3.2}$$

$$(p \odot x^{p-1})^{n+1} = (n+1) \odot x^p, \tag{3.3}$$

for every integer p, $1 < p < n$, such that p is not a divisor of any $i \in I \cup J$, and

$$(n+1) \odot x^q = (n+2) \odot x^q \tag{3.4}$$

for every $q \in \bigcup_{i\in I} \Delta(i, J)$.

If $I = \emptyset$, we rewrite (3.2)–(3.3) as follows: Let J be a nonempty finite set of positive integers, and consider the identity

$$((n+1) \odot x^n)^2 = 2 \odot x^{n+1}, \quad n = \max J, \tag{3.6}$$

as well as all identities of the form

$$(p \odot x^{p-1})^{n+1} = (n+1) \odot x^p \tag{3.7}$$

where p is an integer with $1 < p < n$ and p is not a divisor of any $i \in J$.

Theorem 0.10 *Let $\mathcal{W}$ be the variety of GMV-algebras from $\mathcal{M}$ satisfying identities* (3.6)–(3.7) *for some finite nonempty set J of natural numbers and for all integers p, $1 < p < n$, such that p is not a divisor of any $j \in J$. Then*

$$\mathcal{W} = \bigvee_{j \in J} \mathcal{BP}_j = \bigvee_{j \in J} \mathcal{T}(\mathrm{V}(S_j^{\omega} : \ j \in J)).$$

Theorem 0.11 *Let $\mathcal{W}$ be a proper top subvariety of the variety $\mathcal{M}$. Then there exists a finite set J of natural numbers such that*

$$\mathcal{W} = \bigvee_{n \in J} \mathcal{BP}_n. \tag{3.8}$$

In addition, a GMV-algebra $M \in \mathcal{M}$ belongs to $\mathcal{W}$ if and only if M satisfies identities (3.6)–(3.7) *for any $n \in J$ and for any integer p, $1 < p < n$, such that p is not a divisor of any $n \in J$.*

- **Theorem 0.12** *There are only countably many proper top subvarieties of GMV-algebras within the variety* $\mathcal{M}$.

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Covers of the Variety of MV-algebras

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- Holland 2004 found some non-commutative covers of the variety of Boolean subalgebras, i.e., generated by $\Gamma(\mathbb{Z}, 1)$.

Covers where at least one element has a noncommutative radical

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Theorem 0.15 *If* $\mathcal{G}$ *is a cover of the variety of Abelian* ℓ*-groups,* $\mathcal{A}$*, then the variety* $\mathcal{MV} \vee \Phi(\mathcal{G}) \subseteq \mathcal{SYM} \cap \mathcal{M}$ *is a cover of the variety of MV-algebras,* $\mathcal{MV}$*, such that at least one its element has a non-commutative radical.*

Covers where at least one element has a noncommutative radical

Theorem 0.16 *If $\mathcal{G}$ is a cover of the variety of Abelian ℓ-groups, $\mathcal{A}$, then the variety $\mathcal{MV} \vee \Phi(\mathcal{G}) \subseteq \mathcal{SYM} \cap \mathcal{M}$ is a cover of the variety of MV-algebras, $\mathcal{MV}$, such that at least one its element has a non-commutative radical.*
And conversely.

- Holland-Medvedev 1994 – there is uncountably many covers of the variety of Abelian ℓ-groups.

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Corollary 0.18 *$\mathcal{MV}$ has uncountably many covers in $\mathcal{SYM} \cap \mathcal{M}$.*

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- p - prime, the **Scrimger group** S_p

$$\mathbb{Z} \overrightarrow{\rtimes}_\phi \left(\Pi_{i=0}^{p-1} \mathbb{Z}_i\right)$$

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Covers where each element has a commutative radical

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- $\mathcal{S}_p$ the variety of ℓ-groups generated by S_p. If $p \neq q$, then $\mathcal{S}_p \neq \mathcal{S}_q$. $\mathcal{S}_p$ is a cover of $\mathcal{A}$.

Theorem 0.19 *Let p be any prime number, $n \geq 1$ an integer, S_p be the Scrimger ℓ-group with a fixed strong unit $u_n = (p^n; 0, \ldots, 0)$, and let $\Sigma(S_p, n)$ be the variety of symmetric GMV-algebras in $\mathcal{M}$ generated by the p^n-perfect GMV-algebra $\Gamma(S_p, u_n)$. Then $\mathcal{MV} \vee \Sigma(S_p, n)$ is a cover of $\mathcal{MV}$ such that* **every element of** $\Sigma(S_p, n)$ **has a commutative radical.**

Theorem 0.20 *Let p be any prime number, $n \geq 1$ an integer, S_p be the Scrimger ℓ-group with a fixed strong unit $u_n = (p^n; 0, \ldots, 0)$, and let $\Sigma(S_p, n)$ be the variety of symmetric GMV-algebras in $\mathcal{M}$ generated by the p^n-perfect GMV-algebra $\Gamma(S_p, u_n)$. Then $\mathcal{MV} \vee \Sigma(S_p, n)$ is a cover of $\mathcal{MV}$ such that* **every element of $\Sigma(S_p, n)$ has a commutative radical.**

And conversely

Remarks

(1) If $\mathcal{V}$ is a non-commutative cover of the variety of Boolean algebras, $\mathcal{B}$, then $\mathcal{MV} \cap \mathcal{V} = \mathcal{B}$, and $\mathcal{MV} \vee \mathcal{V}$ is a cover of $\mathcal{MV}$. (The converse is not true)

(2) Holland's example

$$T = \{\sum m_i t^{n_i} \ : \ m_i, n_i \in \mathbb{Z}\}$$

$$(r,n)(s,m) = (r + t^n s, n+m)$$

$$S_t = T \overleftarrow{\times} \mathbb{Z}, \quad \mathcal{C}_t = \mathrm{V}(\Gamma(S_t,(1,0)))$$

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$\mathcal{C}_t \vee \mathcal{MV}$, $t \in \mathbb{R}^+$, are non-commutative covers of $\mathcal{B}$.

We have continuum many covers, $\mathcal{MV} \vee \mathcal{C}_t$, of $\mathcal{MV}$ which are not symmetric but they are from $\mathcal{M}$.

Problem What about other covers of $\mathcal{MV}$ outside of $\mathcal{SYM} \cap \mathcal{M}$?

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- $S_n \sqcup^{\mathcal{GMV}} S_n$ is non-commutative

Thank you for your attention

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