A Remark on the Interval Toplogy on Lattices

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Interval Topology

interval topology τ_I on a lattice $\mathcal{L} = \langle L, \wedge, \vee \rangle$:

subbase for the closed sets in τ_I is given by the closed intervals

$$\begin{aligned} [a,b] &:= \{x \in L \mid a \le x \le b\} \\ (a] &:= \{x \in L \mid x \le a\} \\ [a) &:= \{x \in L \mid a \le x\} \end{aligned} \quad , \quad a, b \in L, \tag{1}$$

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Interplay Algebra/Topology

definition of interval topology produces an interplay between algebraic and topological properties:

- the interval topology on $\mathcal{L}$ is quasicompact if and only if $\mathcal{L}$ is complete with respect to the order relation (Frink, 1942)
- if $\mathcal{L}$ is a Boolean lattice, the interval topology is Hausdorff if and only if $\mathcal{L}$ is atomic (Katetov, 1951)
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Representation of Closed Sets

by definition every closed set A in the interval topology is an intersection of finite unions of closed intervals, i.e. is of the form

$$A = \bigcap_{i \in \Lambda} \bigcup_{j=1}^{n_i} I_{ij}, \tag{2}$$

where Λ is an arbitrary index set, n_i a positive integer and I_{ij} a closed interval as in (1)

if $\mathcal{L}$ is complete, then an arbitrary intersection of closed intervals again is a closed interval:

$$A = \bigcup_{f:\Lambda \to \mathbb{N},\ f(i) \leq n_i} \bigcap_{i \in \Lambda} [a_{if(i)}, b_{if(i)}] = \bigcup_{f:\Lambda \to \mathbb{N},\ f(i) \leq n_i} [\sup_{i \in \Lambda} a_{if(i)}, \inf_{i \in \Lambda} b_{if(i)}]$$

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the letter I with or without indices always denotes a closed interval of $\mathcal{L}$ and all intervals occurring are closed

Definition

- A set $\{I_1, I_2, \ldots, I_n\}$ of intervals is called a representation for $A \subseteq L$ if $A = \bigcup_{i=1}^{n} I_i$.
- The representation $\{I_1, \ldots, I_n\}$ is called reduced if for $i_1 \neq i_2$, always $I_{i_1} \not\subseteq I_{i_2}$ holds.
- If $\{I_1, \ldots, I_n\}$ and $\{I'_1, \ldots, I'_m\}$ are representations of the sets A and A', then $\{I_1, \ldots, I_n\}$ is called finer than $\{I'_1, \ldots, I'_m\}$, denoted by

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- omitting the non–maximal intervals of a representation produces a reduced refinement
- the relation $\preceq$ is a partial order on the reduced representations of a fixed set A
- any two representations of a set A have a common refinement with respect to $\preceq$: if $\{I_1, \ldots, I_n\}$ and $\{I'_1, \ldots, I'_m\}$ are two representations of A, then

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the crucial tool for the proof of our main result:

Proposition 1

Let $\mathcal{L}$ be chain–finite. Then every decreasing sequence of reduced representations is finally constant.

Proof: Let $\mathcal{I}_i$, $i \in \mathbb{N}$, be representations of A_i and assume $\mathcal{I}_1 \succ \mathcal{I}_2 \succ \ldots$. We construct a directed graph by induction on i. For every $i \in \mathbb{N}$ a new row of vertices (maybe empty) is added.

$i = 1$: The vertices arising in this step are the elements of $\mathcal{I}_1$.

If the graph is already constructed for $i = 1, \ldots, k$, we arrange the following for $i = k+1$: The new vertices are the elements of $\mathcal{I}_{k+1} \setminus \mathcal{I}_k$, for every $I \in \mathcal{I}_{k+1} \setminus \mathcal{I}_k$ we have an edge from I to those $\hat{I} \in \mathcal{I}_k$ which satisfy $I \subset \hat{I}$.

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Example: Let $A_i = A = \{a, b, c, d\}$ with $a \le b$, $a \le c$, $b \le d$, $c \le d$, b and c incomparable,

and

$$\begin{array}{llll} & \mathcal{I}_1 = \{[a,b],[a,c],[c,d]\} & \succ & \mathcal{I}_2 = \{[a,b],[c,d]\} \quad \succ \\ \succ & \mathcal{I}_3 = \{\{a\},\{b\},[c,d]\} & \succ & \mathcal{I}_4 = \{\{a\},\{b\},\{c\},\{d\}\}, \end{array}$$

then the following graph emerges:

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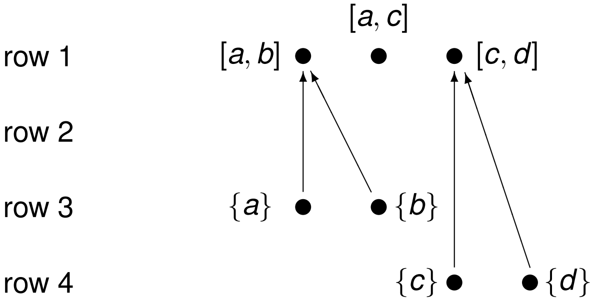

some properties of the graph:

1. The set of vertices coincides with $\bigcup_{i\in\mathbb{N}} \mathcal{I}_i$ and $I \in \bigcup_{i\in\mathbb{N}} \mathcal{I}_i$ occurs as a vertex in that row in which it is an element of the representation for the first time: If $k_1 < k_2 < k_3$ and $I \in \mathcal{I}_{k_i}$, $i = 1, 3$, then there is an $I_2 \in \mathcal{I}_{k_2}$ and an $I_1 \in \mathcal{I}_{k_1}$ with $I \subseteq I_2 \subseteq I_1$. Since $\mathcal{I}_{k_1}$ is reduced it follows $I = I_1 = I_2 \in \mathcal{I}_{k_2}$.
2. It is possible that $\mathcal{I}_{k+1} \setminus \mathcal{I}_k$ is empty for some k but this can happen only finitely often in succession. Hence, according to our assumption the constructed graph is infinite.
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4. Every vertex I has a finite grade of adjacency:

   It is evident from the construction that I is starting point only for finitely many edges.

   Only finitely many edges end in I because all these edges have to start in the same row: Assume I_1 occurring in row k_1 is adjacent to I_2 in row k_2 and I_3 in row k_3 with $k_1 < k_2 < k_3$. The first assumption implies $I_2 \subset I_1$, the second yields $I_1 \in \mathcal{I}_{k_2}$, a contradiction since $\mathcal{I}_{k_2}$ is reduced.

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Proposition 2

In a chain–finite lattice every finite union of closed intervals has a finest representation with respect to $\preceq$.

Proof: Let $A \subseteq L$ be a finite union of closed intervals. If we assume that A has no minimal representation then starting with any representation of A we can find an infinite decreasing sequence of reduced representations of A which contradicts Proposition 1. Since any two representations have a common refinement, all minimal representations have to coincide with the finest one. $\square$

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Let $\{I_1, \ldots, I_n\}$ be the finest representation of A, $\{I'_1, \ldots, I'_m\}$ the finest representation of A' and $A \subseteq A'$. Then $\{I_1, \ldots, I_n\} \preceq \{I'_1, \ldots, I'_m\}$.

Proof: $A \subseteq A'$ implies that $A = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} I_i \cap I'_j$. Because $\{I_1, \ldots, I_n\}$ is the finest representation of A, for all $i \in \{1, \ldots, n\}$ there exists $i' \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$ such that $I_i \subseteq I_{i'} \cap I'_j \subseteq I'_j$. $\square$

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Main Theorem

Let $\mathcal{L}$ be a lattice. Every set closed with respect to the interval topology is a finite union of closed intervals if and only if $\mathcal{L}$ is chain–finite.

Proof: Necessity of the condition: assume that there are elements a_i, $i \in \mathbb{N}$, such that $a_1 < a_2 < \dots$ (if they are ordered dually the proof is analogous). Then

$$\{a_j \mid j \text{ odd}\} = \bigcap_{i \in \mathbb{N}} \left(\bigcup_{j=1}^{i} \{a_{2j-1}\} \cup [a_{2i+1}) \right)$$

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Sufficiency: assume $\mathcal{L}$ is chain–finite and a closed set $A = \bigcap_{i \in \Lambda} \bigcup_{j=1}^{n_i} I_{ij}$ cannot be represented as a finite union of intervals. We choose an arbitrary $i_1 \in \Lambda$, then $A_1 := \bigcup_{j=1}^{n_{i_1}} I_{i_1 j} \supset A$. Hence there exists $i_2 \in \Lambda$ such that

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