Congruence lattices of algebras

Miroslav Ploščica

Slovak Academy of Sciences, Košice

October 16, 2014

Grant

2/0028/2013 Reprezentačné a klasifikačné problémy algebraických štruktúr
Riešitelia: M. Ploščica, E. Halušková, J. Pócs
Ciele:

(1) testovanie kongruenčnej ekvivalentnosti a kongruenčnej maximálnosti variet;
(2) popis zväzov kongruencií algebier v (lokálne konečných) kongruenčne distributívnych varietách s vlastnosťou kompaktného prieniku;
(3) klasifikácia monounárnych algebier a iných štruktúr (retraktové variety, radikálové triedy, konvexity);
(4) direktné a inverzné limity algebier;
(5) aplikácia metód formálnej konceptovej analýzy na niektoré problémy teoretickej informatiky.

Algebras

Algebra = a set with operations:

$$\mathbf{A} = (A, \{f_i \mid i \in I\})$$

n-ary operation on the set A: function $A^n \to A$;

Examples: groups, rings, vector spaces, Boolean algebras, lattices...

Congruences

Congruence on algebra **A** is an equivalence relation θ on the set A, preserved by all basic operations of $\mathbf{A}$, i.e.

$$(a_1, b_1) \in \theta,\ (a_2, b_2) \in \theta, \ldots, (a_n, b_n) \in \theta$$

implies

$$(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n) \in \theta))$$

(for f n-ary).

Congruences

On $\mathbb{Z}$:

$$a \equiv b \pmod{m} \quad \text{if} \quad m|(a-b)$$

Key property:

$$a \equiv b \pmod{m},\ c \equiv d \pmod{m}$$

implies

$$a + c \equiv b + d \pmod{m},$$

$$ac \equiv bd \pmod{m}.$$

Importance of congruences

Congruences enable *quotients*:
On the set of θ-classes we define the operations be means of representatives:

$$f(a_1/\theta, \ldots, a_n/\theta) = f(a_1, \ldots, a_n)/\theta.$$

This gives rise to a new algebra of the same type as $\mathbf{A}$, which is a simplified image of the algebra A.

For instance, $\mathbb{Z}/(\mathrm{mod}\, n) = \mathbb{Z}_n$.

Example: groups

$\mathbf{A}$ ... a commutative group;
Every subgroup $\mathbf{B}$ of $\mathbf{A}$ determines a congruence

$$\theta = \{(a,b) \in A^2 \mid ab^{-1} \in B\}.$$

(That's why we speak about a factorization of a group by a subgroup.)

Similarly: rings, vector spaces

Example: Boolean algebras

$\mathbf{B} = (B; \cup, \cap, ', 0, 1)$
$(B \subseteq \mathcal{P}(X))$;
Ideal: a subset $I \subseteq B$ such that

- if $M \in I$, $N \subseteq M$, then $N \in I$;
- if $M, N \in I$, then $M \cup N \in I$.

Every ideal determines a congruence (and vice versa):

$$\theta = \{(M, N) \in B^2 \mid (M \cap N') \cup (M' \cap N) \in I\}.$$

Example: chains

Consider $(\mathbb{Z}, \max, \min)$ (a distributive lattice)

Fact: Congruences are equivalences, whose all classes are intervals.

Congruence lattices

Congruences on an algebra $\mathbf{A}$ can be ordered by the “refinement" relation (= set inclusion):

$$\varphi \leq \theta \text{ ak } (x\varphi y \text{ implies } x\theta y).$$

We obtain an ordered set $\mathrm{Con}\mathbf{A}$, in which every 2 elements have the largest lower bound (infimum) and the smallest upper bound (supremum) - *lattice*. $\mathrm{Con}\mathbf{A}$ always contains a smallest and a largest element.

Ring of integers

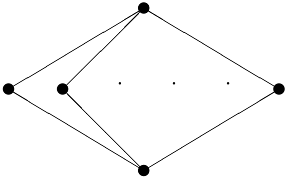
For $(\mathbb{Z}, +, \cdot)$:

$$\pmod{n} \leq \pmod{m} \quad \text{if } m|n.$$

So: the smallest element is $\pmod{0}$, the largest $\pmod{1}$, the infimum is the LCM and the supremum is the GCD.

Vector spaces

Let $\mathbf{A}$ be the 2-dimensional vector space over a field F. Every nontrivial congruence looks the same: its congruence classes are mutually parallel lines. So $\operatorname{Con} \mathbf{A}$ looks as follows. (The number of elements in the middle layer is equal to the number of the lines containing 0. For a finite F it is $n = |F| + 1$.)



Algebraic lattices

Is every lattice isomorphic to the congruence lattice of some algebra?

Theorem

A lattice is isomorphic to the congruence lattice of some algebra if and only if it is algebraic.

What about congruence lattices of special kinds of algebras?

Finite congruence lattices

Open problem: Is every *finite* lattice (isomorphic to) the congruence lattice of some *finite* algebra?

Equivalent group formulation: Is every *finite* lattice (isomorphic to) an interval in the subgroup lattice of a *finite* group?

CLP

Recently solved problem: Is every *distributive* algebraic lattice (isomorphic to) the congruence lattice of some lattice?

Answer: no (F. Wehrung 2005)

Problems

Problem. For a given class $\mathcal{K}$ of algebras describe Con $\mathcal{K}$ =all lattices isomorphic to Con A for some $A \in \mathcal{K}$.

Or, at least,

for given classes $\mathcal{K}$, $\mathcal{L}$ determine if Con $\mathcal{K}$ = Con $\mathcal{L}$
and, if Con $\mathcal{K} \nsubseteq$ Con $\mathcal{L}$, determine

$$\mathrm{Crit}(\mathcal{K}, \mathcal{L}) = \min\{\mathrm{card}(L_c) \mid L \in \mathrm{Con}\,\mathcal{K} \setminus \mathrm{Con}\,\mathcal{L}\}$$

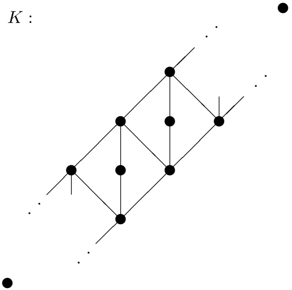
(L_c = compact elements of L)

We are interested in the case when $\mathcal{K}$ and $\mathcal{L}$ are (congruence-distributive) varieties. For instance,
$\mathrm{Crit}(\mathbf{N}_5, \mathbf{M}_3) = 5$,
$\mathrm{Crit}(\mathbf{M}_3, \mathbf{N}_5) = \mathrm{Crit}(\mathbf{M}_3, \mathbf{D}) = \aleph_0$,
$\mathrm{Crit}(\mathbf{M}_4, \mathbf{M}_3) = \aleph_2$,
$\mathrm{Crit}(\mathbf{Maj}, \mathbf{Lat}) = \aleph_2$.
($\mathbf{N}_5$, $\mathbf{M}_3$, $\mathbf{M}_4$ are well-known lattice varieties, $\mathbf{Lat} =$ all lattices, $\mathbf{Maj} =$ all majority algebras.)
P. Gillibert: under some reasonable finiteness conditions, the critical point between two varieties cannot be larger than $\aleph_2$.

Critical points $\aleph_1$

First such example has been discovered by P. Gillibert. We present two more examples.

Let $\mathbf{K}$ be the variety generated by the bounded lattice



Critical points $\aleph_1$

Theorem

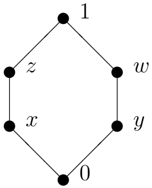
(1) $\mathrm{Crit}(\mathbf{N_5}, \mathbf{K}) = \aleph_1$;

(2) $\mathrm{Crit}(\mathbf{K}, \mathbf{N_5}) = \aleph_0$.

N_6

Let $\mathbf{N}_6^*$ be the variety generated by the bounded lattice N_6 with an additional unary operation of $180°$ rotation ($f(x) = w$...) .

N_6 :



Critical points $\aleph_1$

Theorem

(1) $\mathrm{Crit}(\mathbf{N}_6^*, \mathbf{N}_5) = \aleph_1$;

(2) $\mathrm{Crit}(\mathbf{N}_5, \mathbf{N}_6^*) = \aleph_0$.

Question

What is the mechanism behind these examples?

Con_c functor

For any homomorphism of algebras $f: A \to B$ we define

$$\mathrm{Con}_c f: \mathrm{Con}_c A \to \mathrm{Con}_c B$$

by
$\alpha \mapsto$ congruence generated by $\{(f(x), f(y)) \mid (x,y) \in \alpha\}$.

Fact. $\mathrm{Con}_c f$ preserves $\vee$ and 0, not necessarily $\wedge$.

For every commutative diagram $\mathcal{A}$ of algebras we have a commutative diagram $\mathrm{Con}\,\mathcal{A}$ of $(\vee, 0)$-semilattices.

Let

- $\varphi : S \to T$ be a homomorphism of $(\vee, 0)$-semilattices;
- $f : A \to B$ be a homomorphisms of algebras.

We say that f *lifts* φ, if there are isomorphisms $\psi_1 : S \to \mathrm{Con}_c A$, $\psi_2 : T \to \mathrm{Con}_c B$ such that

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & T \\ \psi_1 \downarrow & & \psi_2 \downarrow \\ \mathrm{Con}_c A & \xrightarrow{\mathrm{Con}_c f} & \mathrm{Con}_c B \end{array}$$

commutes.

Lifting of diagrams

Let P be a poset and let

- $\mathcal{D}: P \to \mathcal{S}$ be a diagram of $(\vee, 0)$-semilattices;
- $\mathcal{A}: P \to \mathcal{K}$ be a diagram of algebras;

We say that $\mathcal{A}$ *lifts* $\mathcal{D}$, if there are isomorphisms $\psi_j: \mathcal{D}(j) \to \operatorname{Con}_c \mathcal{A}(j)$ such that

$$\begin{array}{ccc} \mathcal{D}(j) & \xrightarrow{\mathcal{D}(j,k)} & \mathcal{D}(k) \\ \psi_j \downarrow & & \psi_k \downarrow \\ \operatorname{Con}_c \mathcal{A}(j) & \xrightarrow{\operatorname{Con}_c \mathcal{A}(j,k)} & \operatorname{Con}_c \mathcal{A}(k) \end{array}$$

commutes for every $j \leq k$.

Theorem

(2) implies (1), where

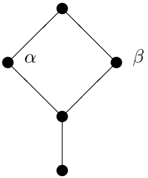
1. $\mathrm{Crit}(\mathcal{K}, \mathcal{L}) \leq \aleph_n$;
2. *there exists a diagram of finite* $(\vee, 0)$*-semilattices indexed by a product of* $n+1$ *finite chains liftable in* $\mathcal{K}$ *but not in* $\mathcal{L}$

If $n = 0$ *then also (1)*$\Longrightarrow$ *(2).*

Especially, if there exists a diagram of finite $(\vee, 0)$-semilattices indexed by a square liftable in $\mathcal{K}$ but not in $\mathcal{L}$, then $\mathrm{Crit}(\mathcal{K}, \mathcal{L}) \leq \aleph_1$.

N6 versus N5

Both N_5 and N_6^* have the same congruence lattice, but N_6^* has an automorphism h (the vertical symmetry), such that $\mathrm{Con}_c\, h$ interchanges α and β:



N6 versus N5

Below: $\mathcal{D}$ is the diagram in $\mathbf{N}_6^*$, so that $\operatorname{Con}\mathcal{D}$ has a lifting in $\mathbf{N}_6^*$ but - no lifting in $\mathbf{N}_5$.

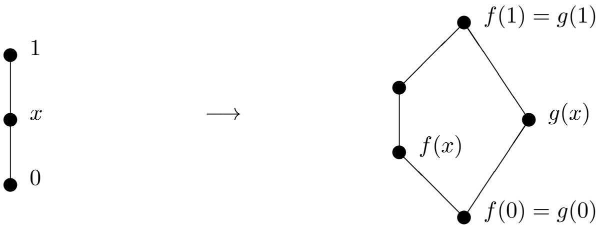
$$
\begin{tikzcd}
 & N_6^* \times N_6^* & \\
N_6^* \arrow[ur, "\mathrm{id}\times\mathrm{id}"] & & N_6^* \arrow[ul, "\mathrm{id}\times h"'] \\
 & \{0,1\} \arrow[ul, "\subseteq"'] \arrow[ur, "\subseteq"] &
\end{tikzcd}
$$

Observation

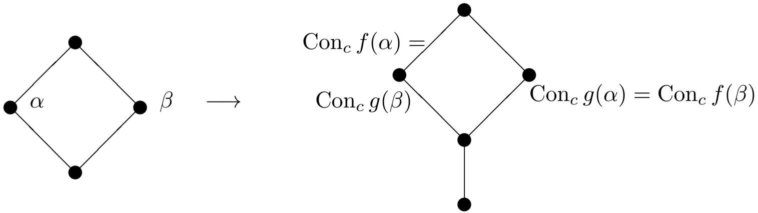
Every automorphism $f\colon A \to A$ induces an automorphism $\mathrm{Con}_c\, f\colon \mathrm{Con}_c\, A \to \mathrm{Con}_c\, A$. These induced automorphisms form a subgroup of the automorphism group of $\mathrm{Con}_c\, A$. And this subgroup has an influence on the class $\mathrm{Con}\,\mathbf{A}$, where A is the variety generated by A.

N5 versus K

The same idea as before, but more subtle. Not only automorphisms are important.
Consider the homomorphisms f, g in $\mathbf{N}_5$:



The maps $\mathrm{Con}_c\, f$ and $\mathrm{Con}_c\, g$:



N5 versus K

If $\mathcal{D}$ is the diagram below, then $\operatorname{Con}\mathcal{D}$ has a lifting in $\mathbf{N}_5$ but not in $\mathbf{K}$.

$$\begin{tikzcd}
& N_5 \times N_5 & \\
N_5 \arrow[ur, "f \times f"] & & N_5 \arrow[ul, "f \times g"'] \\
& \{0,1\} \arrow[ul, "\subseteq"'] \arrow[ur, "\subseteq"] &
\end{tikzcd}$$

Gillibert's example

Different mechanism: a semilattice homomorphism $\varphi : S \to T$ with two liftings $f : A \to B_1$, $g : A \to B_2$ such that $\operatorname{Con} f$ and $\operatorname{Con} g$ have different kernels.

Possible general "theorem":

$\operatorname{Crit}(\mathbf{V}_1, \mathbf{V}_2) = \aleph_1$ occurs when all diagrams indexed by a finite chain liftable in $\mathbf{V}_1$ are also liftable in $\mathbf{V}_2$, but the liftings in $\mathbf{V}_2$ are "less symmetric".