Shadowing, Hyers–Ulam stability and hyperbolicity for nonautonomous linear delay differential equations

Mihály Pituk

Department of Mathematics,
University of Pannonia,
Veszprém, Hungary

Mathematical Institute of the
Slovak Academy of Sciences
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Joint work with Lucas Backes (right) (Univ. Federal do Rio Grande do Sul, Brazil) and Davor Dragičević (left) (University of Rijeka, Croatia)

Introduction

Given $r \geq 0$, let $C = C([-r,0],\mathbb{R}^d)$ be the Banach space of all continuous maps $\phi\colon [-r,0] \to \mathbb{R}^d$ equipped with the supremum norm, $\|\phi\| := \sup_{\theta\in[-r,0]} |\phi(\theta)|$ for $\phi \in C$, where $|\cdot|$ is any norm on $\mathbb{R}^d$.

The symbol $\mathcal{L}(C,\mathbb{R}^d)$ denotes the space of bounded linear operators from C into $\mathbb{R}^d$ equipped with the operator norm.

Consider the nonautonomous linear delay differential equation

$$x'(t) = L(t)x_t, \tag{1}$$

where $L\colon [0,\infty) \to \mathcal{L}(C,\mathbb{R}^d)$ is continuous and $x_t \in C$ is defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r,0]$.

Given $s \geq 0$, by a *solution of* (1) *on* $[s,\infty)$, we mean a continuous function $x\colon [s-r,\infty) \to \mathbb{R}^d$ which is differentiable on $[s,\infty)$ and satisfies (1) for all $t \geq s$. (By the derivative at $t = s$, we mean the right-hand derivative.)

For every $s \geq 0$ and $\phi \in C$, Eq. (1) has a unique solution x on $[s, \infty)$ with initial value $x_s = \phi$.

For $t \geq s \geq 0$, the *solution operator* $T(t,s)\colon C \to C$ is defined by $T(t,s)\phi = x_t$, where x is the unique solution of (1) with $x_s = \phi$.

Definition

We say that Eq. (1) is *shadowable* if for every $\varepsilon > 0$ there exists $\delta > 0$ with the following property: for every continuous function $y\colon [-r, \infty) \to \mathbb{R}^d$ which is continuously differentiable on $[0, \infty)$ and satisfies

$$\sup_{t \geq 0} |y'(t) - L(t)y_t| \leq \delta,$$

there exists a solution x of (1) on $[0, \infty)$ such that

$$\sup_{t \geq 0} \|x_t - y_t\| \leq \varepsilon.$$

Definition

We say that Eq. (1) is *Hyers–Ulam stable* if there exists $\kappa > 0$ such that for every continuous function $y\colon [-r,\infty) \to \mathbb{R}^d$ which is continuously differentiable on $[0,\infty)$ and satisfies

$$\sup_{t\geq 0} |y'(t) - L(t)y_t| \leq \delta \qquad \text{for some } \delta > 0,$$

there exists a solution x of (1) on $[0,\infty)$ such that

$$\sup_{t\geq 0} \|x_t - y_t\| \leq \kappa\delta.$$

Definition

We say that Eq. (1) admits an *exponential dichotomy* if there exist a family of projections $(P(t))_{t\geq 0}$ on C and constants $D, \lambda > 0$ with the following properties:

- for $t \geq s \geq 0$,
$$P(t)T(t,s) = T(t,s)P(s)$$
and $T(t,s)|_{\ker P(s)} : \ker P(s) \to \ker P(t)$ is invertible;
- for $t \geq s \geq 0$,
$$\|T(t,s)P(s)\| \leq De^{-\lambda(t-s)},$$
- for $0 \leq t \leq s$,
$$\|T(t,s)Q(s)\| \leq De^{-\lambda(s-t)},$$
where $Q(s) = I - P(s)$ for $s \geq 0$ and $T(t,s) := \left(T(s,t)|_{\ker P(t)}\right)^{-1}$ for $0 \leq t \leq s$.

- S.Yu. Pilyugin, *Shadowing in Dynamical Systems*, Lecture Notes in Mathematics, Vol. 1706, Springer–Verlag, Berlin, 1999.
- K. Palmer, *Shadowing in Dynamical Systems*, Theory and Applications, Kluwer, Dordrecht (2000)
- J. Brzdek, D. Popa, I. Rasa, and B. Xu, Ulam Stability of Operators. Mathematical Analysis and its Applications, Academic Press, London, 2018.

The following result is a corollary of a more general theorem on weighted shadowing from the paper

L. Backes, D. Dragičević, M. Pituk, and L. Singh, *Weighted shadowing for delay differential equations*, *Arch. Math. (Basel)*, **119** (2022), 539–552.

Theorem

If Eq. (1) *has an exponential dichotomy, then it is Hyers–Ulam stable and hence shadowable.*

In a recent paper

L. Barreira and C. Valls, *Stability of delay equations*, *Electron. J. Qual. Theory Differ. Equ.*, **45** (2022), no. 45, 1–24.

have proved the converse result in the special case when Eq. (1) is autonomous and periodic and its spectrum is simple.

Main Result

Our main result is the following theorem which shows that the converse result is true if we merely assume that coefficient operators $L(t)\colon C \to \mathbb{R}^d$, $t \geq 0$, are uniformly bounded.

The importance of the uniform boundedness of the coefficients will be illustrated by an example.

Theorem

Suppose that

$$\sup_{t\geq 0} \|L(t)\| < \infty.$$

If Eq. (1) is shadowable, then it has an exponential dichotomy.

As a consequence, we have that for delay equations with uniformly bounded coefficients all three notions in the title are equivalent.

Theorem

If

$$\sup_{t\geq 0} \|L(t)\| < \infty,$$

then the following statements are equivalent:

(i) *Eq.* (1) *is Hyers–Ulam stable;*

(ii) *Eq.* (1) *is shadowable;*

(iii) *Eq.* (1) *admits an exponential dichotomy.*

Remark

The previous theorem improves the recent results by Barreira and Valls (2022), where the equivalence of Hyers–Ulam stability and the existence of an exponential dichotomy has been proved only in the special case of autonomous and periodic linear delay differential equations under the additional assumption that their spectrum is “simple”.

Remark

The uniform boundedness of the coefficients in the theorems cannot be omitted. This can be shown by the following example.

Example

Let $v\colon [0,\infty) \to (0,\infty)$ be a positive, continuously differentiable function satisfying

$$\int_0^t v(s)\,ds \le v(t), \qquad t \ge 0, \tag{2}$$

$$v(t) \to \infty, \qquad t \to \infty, \tag{3}$$

and

$$\frac{v(n-\alpha_n)}{v(n)} > n, \qquad n \in \mathbb{N}, \tag{4}$$

where $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers such that $\alpha_n \to 0$ as $n \to \infty$. For an explicitly given v and α with the above properties, see the monograph by

Ju.L. Daleckiĭ and M.G. Kreĭn,
Stability of Solutions of Differential Equations in Banach Space, Translations of Mathematical Monographs Vol. 43, American Mathematical Society, Providence, R.I., 1974.

Example (continuation)

Consider the scalar ordinary differential equation

$$x'(t) = a(t)x(t), \tag{5}$$

where

$$a(t) = -\frac{v'(t)}{v(t)}, \qquad t \geq 0.$$

Eq. (5) is a special case of (1) when $r = 0$, $d = 1$ and $L(t)\phi = a(t)\phi(0)$ for $t \geq 0$ and $\phi \in C$. The solutions of Eq. (5) have the form

$$x(t) = \frac{v(s)}{v(t)}x(s), \qquad t, s \in [0, \infty). \tag{6}$$

Hence

$$[T(t,s)\phi](0) = \frac{v(s)}{v(t)}\phi(0), \qquad t \geq s \geq 0, \quad \phi \in C,$$

Example (continuation)

which implies that

$$\|T(t,s)\| = \frac{v(s)}{v(t)}, \qquad t \geq s \geq 0. \tag{7}$$

From this and (2), we find for $t \geq 0$,

$$\int_0^t \|T(t,s)\|\, ds \leq \int_0^t \frac{v(s)}{v(t)}\, ds \leq 1.$$

By the application of Theorem 2.2 from

L. Backes, D. Dragičević, M. Pituk, and L. Singh, *Weighted shadowing for delay differential equations*, *Arch. Math. (Basel)* , **119** (2022), 539–552.

we conclude that Eq. (5) is Hyers–Ulam stable and hence shadowable.

Example (continuation)

We will show that Eq. (5) has no exponential dichotomy. Suppose, for the sake of contradiction, that Eq. (5) has an exponential dichotomy. It follows from the definition that if Eq. (1) has an exponential dichotomy and all solutions of (1) are bounded, then $Q(s) = 0$ and hence $P(s) = \mathrm{Id}$ for all $s \geq 0$, which implies that the exponential dichotomy is an exponential contraction, i.e

$$\|T(t,s)\| \leq De^{-\lambda(t-s)}, \qquad t \geq s \geq 0. \tag{8}$$

Indeed, if $Q(s)\phi \neq 0$ for some $s \geq 0$ and $\phi \in C$, then (3) implies that the norm of the solution $x_t = T(t,s)\phi$ of Eq. (1) tends to infinity exponentially as $t \to \infty$, which contradicts the boundedness of x. Since Eq. (5) has an exponential dichotomy and (3) and (6) imply that all solutions of (5) are bounded, we have that (8) holds. From (7) and (8), we obtain for $n \in \mathbb{N}$,

$$\frac{v(n-\alpha_n)}{v(n)} = \|T(n, n-\alpha_n)\| \leq De^{-\lambda\alpha_n}.$$

Example (continuation)

From this, letting $n \to \infty$ and taking into account that $\alpha_n \to 0$ as $n \to \infty$, we conclude that

$$\limsup_{n\to\infty} \frac{v(n-\alpha_n)}{v(n)} \leq D.$$

On the other hand, (4) implies that $\lim_{n\to\infty} \frac{v(n-\alpha_n)}{v(n)} = \infty$, which yields a contradiction. Thus, Eq. (5) is Hyers–Ulam stable and shadowable, but it has no exponentially dichotomy.

Key arguments of the proof

- The shadowing property of Eq. (1) implies the Perron type property. Namely, the nonhomogeneous equation associated with Eq. (1) has at least one bounded solution for every bounded and continuous nonhomogeneity.
- The Perron property, combined with the eventual compactness of the solution operator and Schäffer's result about regular covariant sequences corresponding to compact linear operators in a Banach space, implies that the stable subspace of Eq. (1) is closed and has finite codimension.
- The fact that the stable subspace is complemented yields a direct sum decomposition of the phase space into stable and unstable subspaces at each time instant in a standard manner.
- The required exponential estimates along the stable and unstable directions are obtained by adaptation of a technique from the admissibility theory of ordinary differential equations to delay equations.

Claim 1

Eq. (1) has the following Perron type property: for each bounded and continuous function $z\colon [0,\infty) \to \mathbb{R}^d$, there exists a bounded and continuous function $x\colon [-r,\infty) \to \mathbb{R}^d$ which is differentiable on $[0,\infty)$ and satisfies

$$x'(t) = L(t)x_t + z(t), \qquad t \geq 0.$$

Claim 1 is related to the admissibility theory of delay equations. For admissibilty results for delay equations, see

G. Pecelli,
Functional differential equations: Dichotomies, perturbations, and admissibility,
J. Differential Equations , **16** (1974), 72–102.

For each $s \geq 0$, define the *stable subspace of Eq.* (1) *at time* s by

$$\mathcal{S}(s) = \Big\{\phi \in C : \sup_{t \geq s} \|T(t,s)\phi\| < \infty\Big\}.$$

Claim 2

The stable subspace $\mathcal{S}(0)$ of Eq. (1) is closed and has finite codimension in C.

Let X be a Banach space. A subspace S of X is called *subcomplete in* X if there exist a Banach space Z and a bounded linear operator $\Phi \colon Z \to X$ such that $\Phi(Z) = S$.

Let $A \colon \mathbb{N}_0 \to \mathcal{L}(X)$ be an operator-valued map. For $n \geq m \geq 0$, the corresponding *transition operator* $U(n,m) \colon X \to X$ is defined by

$$U(n,m) = A(n-1)A(n-2)\cdots A(m) \qquad \text{for } n > m \geq 0$$

and $U(m,m) = I$ for $m \geq 0$.

A sequence $Y = (Y(n))_{n\in\mathbb{N}_0}$ of subspaces in X is called a *covariant sequence for A* if

$$[\,A(n)\,]^{-1}(Y(n+1)) = Y(n) \qquad \text{for all } n \in \mathbb{N}_0.$$

A covariant sequence $Y = (Y(n))_{n\in\mathbb{N}_0}$ for A is called *algebraically regular* if

$$U(n,0)X + Y(n) = X \qquad \text{for each } n \in \mathbb{N}_0.$$

Finally, a covariant sequence $Y = (Y(n))_{n\in\mathbb{N}_0}$ for A is called *subcomplete* if the subspace $Y(n)$ is subcomplete in X for all $n \in \mathbb{N}_0$.

We will need the following result by

J.J. Schäffer,
Linear difference equations: Closedness of covariant sequences
Math. Ann., **187** (1970), 69–76.

Lemma (Schäffer)

Let X be a Banach space and $A\colon \mathbb{N}_0 \to \mathcal{L}(X)$. Suppose that $Y = (Y(n))_{n\in\mathbb{N}_0}$ is a subcomplete algebraically regular covariant sequence for A. If the transition operator $U(n,m)\colon X \to X$ is compact for some $n, m \in \mathbb{N}_0$, $n \geq m$, then the subspaces $Y(n)$, $n \in \mathbb{N}_0$, are closed and have constant finite codimension in X.

Claim 3

For each $t \geq s \geq 0$, we have that

$$[\,T(t,s)\,]^{-1}(\mathcal{S}(t)) = \mathcal{S}(s).$$

Claim 4

For $t \geq s \geq 0$, we have the algebraic sum decomposition

$$C = T(t,s)C + \mathcal{S}(t).$$

Claim 5

For each $s \geq 0$, $\mathcal{S}(s)$ is the image of a Banach space under the action of a bounded linear operator.

The stable subspaces $Y(n) := \mathcal{S}(nr) \subset C$ of Eq. (1) form a subcomplete algebraically regular covariant sequence for $A\colon \mathbb{N}_0 \to \mathcal{L}(C)$ defined by

$$A(n) := T((n+1)r, nr), \qquad n \in \mathbb{N}_0.$$

For each $n \in \mathbb{N}_0$, $A(n)\colon C \to C$ is compact. By the application of Schäffer's Lemma, we conclude that $Y(0) = \mathcal{S}(0)$ is closed and has finite codimension in C.

Since $\mathcal{S}(0)$ is closed and has finite codimension in C, it is complemented in C, i.e., there exists a subspace $\mathcal{U}$ of C such that $\dim \mathcal{U} = \operatorname{codim} \mathcal{S}(0) < \infty$ and

$$C = \mathcal{S}(0) \oplus \mathcal{U}.$$

Claim 6

For each bounded and continuous function $z\colon [0,\infty) \to \mathbb{R}^d$, there exists a unique bounded and continuous function $x\colon [-r,\infty) \to \mathbb{R}^d$ with $x_0 \in \mathcal{U}$ which is differentiable on $[0,\infty)$ and satisfies

$$x'(t) = L(t)x_t + z(t), \qquad t \geq 0.$$

Moreover, there exists a constant $A > 0$, independent of z, such that

$$\sup_{t \geq -r} |x(t)| \leq A \sup_{t \geq 0} |z(t)|.$$

For $s \geq 0$, define

$$\mathcal{U}(s) = T(s,0)\mathcal{U}$$

so that $\mathcal{U}(0) = \mathcal{U}$.

It is easily seen that

$$T(t,s)\mathcal{S}(s) \subset \mathcal{S}(t) \quad \text{and} \quad T(t,s)\mathcal{U}(s) = \mathcal{U}(t)$$

whenever $t \geq s \geq 0$.

Claim 7

For $t \geq s \geq 0$, $T(t,s)|_{\mathcal{U}(s)} : \mathcal{U}(s) \to \mathcal{U}(t)$ is invertible.

Claim 8

For each $t \geq 0$, C can be decomposed into the direct sum

$$C = \mathcal{S}(t) \oplus \mathcal{U}(t).$$

Claim 9

There exists $Q > 0$ such that

$$\|T(t,s)\phi\| \leq Q\|\phi\|,$$

for $t \geq s \geq 0$ and $\phi \in \mathcal{S}(s)$.

Claim 10

There exist $D, \lambda > 0$ such that

$$\|T(t,s)\phi\| \leq De^{-\lambda(t-s)}\|\phi\|,$$

for $t \geq s \geq 0$ and $\phi \in \mathcal{S}(s)$.

Proof of Claim 9

The uniform boundedness of the coeffiecients imply that the evolution family $(T(t,s))_{t\geq s\geq 0}$ is exponentially bounded, i.e., there exist $K, a > 0$ such that

$$\|T(t,s)\| \leq Ke^{a(t-s)}, \quad t \geq s \geq 0.$$

Fix $s \geq 0$, $\phi \in \mathcal{S}(s)$ and let $u\colon [s-r,\infty) \to \mathbb{R}^d$ be the solution of the equation $x'(t) = L(t)x_t$ on $[s,\infty)$ such that $u_s = \phi$. Choose a continuously differentiable function $\psi\colon [-r,\infty) \to [0,1]$ such that $\operatorname{supp}\psi \subset [s,\infty)$, $\psi \equiv 1$ on $[s+1,\infty)$ and $|\psi'| \leq 2$. Define $x\colon [-r,\infty) \to \mathbb{R}^d$ and $z\colon [0,\infty) \to \mathbb{R}^d$ by

$$x(t) = \psi(t)u(t), \qquad t \geq -r,$$

and

$$z(t) = \psi'(t)u(t) + \psi(t)L(t)u_t - L(t)(\psi_t u_t), \qquad t \geq 0.$$

Clearly, x and z are continuous, x is differentiable on $[0,\infty)$ and $x'(t) = L(t)x_t + z(t)$ for $t \geq 0$. Since $\phi \in \mathcal{S}(s)$, the solution u and hence x is bounded on $[-r,\infty)$. Moreover, $x_0 = 0 \in \mathcal{U}$. Note that $\psi \equiv 0$ on $[-r,s]$ and hence $z \equiv 0$ on $[0,s]$. Furthermore, $\psi \equiv 1$ on $[s+1,\infty)$, which implies that $\psi' \equiv 0$ on $[s+1,\infty)$, $\psi_t \equiv 1$ for $t \geq s+r+1$ and hence $z(t) = 0$ for $t \geq s+r+1$. From this, we find that

$$\begin{aligned}
\sup_{t\geq 0}|z(t)| &= \sup_{t\in[s,s+r+1]}|z(t)| \\
&\leq 2\sup_{t\in[s,s+1]}|u(t)| + \sup_{t\in[s,s+r+1]}(\psi(t)|L(t)u_t| + |L(t)(\psi_t u_t)|) \\
&\leq 2\sup_{t\in[s,s+1]}\|u_t\| + 2M\sup_{t\in[s,s+r+1]}\|u_t\| \\
&= 2\sup_{t\in[s,s+1]}\|T(t,s)\phi\| + 2M\sup_{t\in[s,s+r+1]}\|T(t,s)\phi\| \\
&\leq 2Ke^{a}\|\phi\| + 2MKe^{a(r+1)}\|\phi\|.
\end{aligned}$$

From (1), taking into account that $\psi \equiv 1$ on $[s+1, \infty)$, we conclude that

$$\sup_{t \geq s+1} |u(t)| \leq \sup_{t \geq -r} |x(t)| \leq A \sup_{t \geq 0} |z(t)| \leq 2A(Ke^a + MKe^{a(r+1)})\|\phi\|.$$

Hence,

$$\|T(t,s)\phi\| = \|u_t\| \leq 2A(Ke^a + MKe^{a(r+1)})\|\phi\|, \quad t \geq s+r+1.$$

On the other hand, the exponential estimate on the growth of the evolution family implies that

$$\|T(t,s)\phi\| \leq Ke^{a(r+1)}\|\phi\|, \qquad t \in [s, s+r+1].$$

Consequently, the conclusion of the claim holds with

$$Q := \max\{Ke^{a(r+1)}, 2A(Ke^a + MKe^{a(r+1)})\} > 0.$$

Claim 11

There exists $Q' > 0$ such that

$$\|T(t,s)\phi\| \leq Q'\|\phi\|,$$

for $0 \leq t \leq s$ and $\phi \in \mathcal{U}(s)$.

Claim 12

There exist $D', \lambda' > 0$ such that

$$\|T(t,s)\phi\| \leq D' e^{-\lambda'(s-t)}\|\phi\|,$$

for $0 \leq t \leq s$ and $\phi \in \mathcal{U}(s)$.

For each $t \geq 0$, let $P(t)$ denote the projection of C onto $\mathcal{S}(t)$ along $\mathcal{U}(t)$ associated with the decomposition

$$C = \mathcal{S}(t) \oplus \mathcal{U}(t).$$

Claim 13

The projections $P(t)$, $t \geq 0$, are uniformly bounded, i.e.

$$\sup_{t \geq 0} \|P(t)\| < \infty.$$

Claims 10, 12 and 13 imply that Eq. (1) has an exponential dichotomy. $\square$

This talk was based on the paper

L. Backes, D. Dragičević, and M. Pituk,
Shadowing, Hyers–Ulam stability and hyperbolicity for nonautonomous linear delay differential equations,
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Veszprém, Hungary