Operations on Unambiguous Finite Automata

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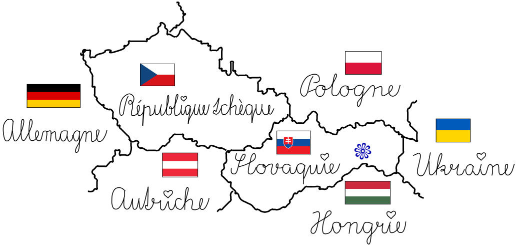


Joint work with Jozef Jirásek, Jr., and Juraj Šebej

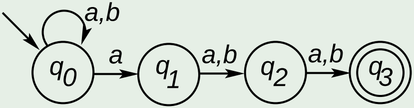
DLT 2016, Montréal, Québec, Canada

Nondeterministic and Deterministic Finite Automata

NFA $N = (Q, \Sigma, \delta, I, F)$:

- $\delta \subseteq Q \times \Sigma \times Q$
- computation on $w = a_1 a_2 \cdots a_k$
  $q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \xrightarrow{a_3} \cdots \xrightarrow{a_k} q_k$
  $q_0 \in I$
- accepting if $q_k \in F$
- rejecting if $q_k \notin F$

Example (An NFA)



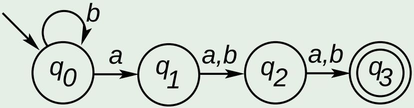
$w = aaa$

- $q_0 \xrightarrow{a} q_1 \xrightarrow{a} q_2 \xrightarrow{a} q_3$ (acc.)
- $q_0 \xrightarrow{a} q_0 \xrightarrow{a} q_0 \xrightarrow{a} q_0$ (rej.)

NFA $N = (Q, \Sigma, \delta, I, F)$ is a DFA:

- $|I| = 1$
- if (q, a, p) and (q, a, r) are in δ, then $p = r$

Example (An incomplete DFA)



- NFAs may have multiple initial states
- DFAs may be incomplete

Subset Automaton and Reverse of NFA

Definition

The (incomplete) subset automaton of NFA $N = (Q, \Sigma, \delta, I, F)$ is the DFA $(2^Q \setminus \{\emptyset\}, \Sigma, \delta', I, F') \ldots$

Proposition

Every n-state NFA can be simulated by an $(2^n - 1)$-state incomplete DFA.

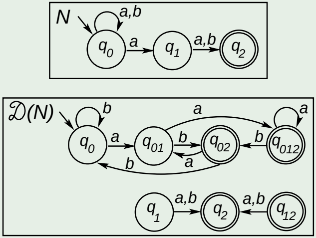
Definition

The reverse of an NFA $N = (Q, \Sigma, \delta, I, F)$ is the NFA

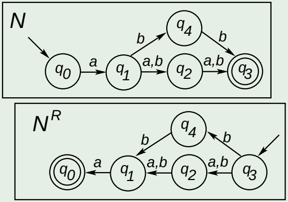
$$N^R = (Q, \Sigma, \delta^R, F, I),$$

where $(p, a, q) \in \delta^R$ iff $(q, a, p) \in \delta$

Example (Subset automaton)



Example (Reverse of NFA)



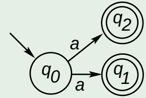
Unambiguous Finite Automata

Definition ($N = (Q, \Sigma, \delta, I, F)$)

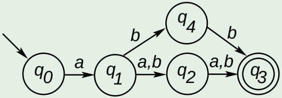
An NFA is unambiguous if it has at most one accepting computation on every input string.

- $S \subseteq Q$ is reachable in N if $S = \delta(I, w)$ for some w
- $S \subseteq Q$ is co-reachable in N if S is reachable in N^R

Example (not unambiguous)



– two accepting computations on a



– two accepting computations on abb

Proposition

An NFA is unambiguous iff

$$|S \cap T| \leq 1$$

for each reachable S and each co-reachable T

Example (unambiguous)

- (in)complete DFA
- NFA N s.t. N^R deterministic
- NFA in the first slide

Why Unambiguous Finite Automata?

Motivation and History

- fundamental notion in the theory of variable-length codes [Bersten, Perrin, Reutenauer: Codes and Automata]
- ambiguity in CF languages: ambiguous, unambiguous, and deterministic CF languages are all different
- ambiguity in finite automata [Schmidt 1978]
  - lower bound method based on ranks of matrices
- elaborated in [Leung 2005]
  - UFA-to-DFA conversion: 2^n
  - NFA-to-UFA conversion: $2^n - 1$
- lower bound method further elaborated in 2002 by Hromkovič, Seibert, Karhumäki, Klauck & Schnitger

Why Operations on Unambiguous Finite Automata?

Motivation for me:-)

- conference trip at DLT 2008 (Kyoto): A. Okhotin - ...
  "What is the complexity of complementation on UFAs?"
- operations on unary UFAs investigated by him in 2012
  - lower bound $n^{2-o(1)}$ for complementation
- the second problem for which
  "give me a large enough alphabet" method didn't work ...

Lower Bounds Methods I

Well known: To prove that a DFA is minimal, show that

- all its states are reachable, and
- no two distinct states are equivalent.

Well known(?): To prove that an NFA is minimal, describe

a fooling set for the accepted language.

For UFAs: rank of matrices [Schmidt 78, Leung 05]:

Let N be an NFA. Let M_N be the matrix in which

- rows indexed by non-empty reachable sets
- columns indexed by non-empty co-reachable sets
- in entry (S, T) we have $0/1$ if S and T are/are not disjoint.

Then every UFA for $L(N)$ has at least $\text{rank}(M_N)$ states.

Lower Bounds Methods II

Lemma (Leung 1998, Lemma 3)

Let M_n be the $(2^n - 1) \times (2^n - 1)$ matrix with

- *rows and columns indexed by non-empty subsets of $\{1, 2, \ldots, n\}$*
- *$M_n(S, T) = 0/1$ iff S and T are/are not disjoint.*

Then $rank(M_n) = 2^n - 1$.

Corollary

If each non-empty set is co-reachable in NFA N,
then every UFA equivalent to N
has $\geq$ |non-empty reachable| states.

Maslov 1970

Dokl. Akad. Nauk SSSR
Tom 194 (1970), No. 6

Soviet Math. Dokl.
Vol. 11 (1970), No. 5

ESTIMATES OF THE NUMBER OF STATES OF FINITE AUTOMATA

519.95

A. N. MASLOV

It is well known that, if $T(A)$ and $T(B)$ are representable in automata A and B with m and n states, respectively $(m \geq 1, n \geq 1)$, then:

1) $T(A) \cup T(B)$ is representable in an automaton with $m \cdot n$ states;

2) $T(A) \cdot T(B)$ is representable in an automaton with $(m-1) \cdot 2^n + 2^{n-1}$ states $(n \geq 3)$;

3) $T(A)^*$ is representable in an automaton with $(3/4) \cdot 2^m - 1$ states $(m \geq 2)$.

Let us construct examples of automata over the alphabet $\Sigma = \{0, 1\}$ for which these estimates are attained.

1. Union. A has states $\{S_0, \cdots, S_{m-1}\}$ and transitions $S_{m-1}1 = S_0$, $S_i1 = S_{i+1}$ for $i \neq m-1$, $S_i0 = S_i$, and S_{m-1} is the terminal state. B has states $\{P_0, \cdots, P_{n-1}\}$ and transitions $P_i1 = P_i$, $P_{n-1}0 = P_0$; $P_i0 = P_{i+1}$ for $i \neq n-1$, P_{n-1} is the terminal state.

2. Product. B has the states $\{P_0, \cdots, P_{n-1}\}$ and transitions $P_{n-1}1 = P_{n-2}$, $P_{n-2}1 = P_{n-1}$, $P_i1 = P_i$ for $i < n-2$, $P_{n-1}0 = P_{n-1}$, $P_i0 = P_{i+1}$ for $i \neq n-1$. P_{n-1} is the terminal state. The automaton A is the same as in the case of the union.

3. Iteration. A has the states $\{S_0, \cdots, S_{m-1}\}$ and transitions $S_{m-1}1 = S_0$, $S_i1 = S_{i+1}$ for $i \neq m-1$, $S_00 = S_0$, $S_i0 = S_{i-1}$ for $i > 0$. S_{m-1} is the terminal state.

Corresponding to A and B we construct automata as in [2,4] and we find the required number of attainable and distinct states, which proves the minimality [3].

A General Formulation of the Problem

Maslov 1970

A general formulation of the problem is as follows: We have events $T(A_i)$ $(1 \leq i \leq k)$ representable in automata A_i with n_i states, respectively, and a k-place operation f on events, preserving representability in finite automata. What is the maximal number of states of a minimal automaton representing $f(T(A_1), \ldots, T(A_k))$, for the given n_i?

"We have languages $L(A_i)$ $(1 \leq i \leq k)$
recognized by automata A_i with n_i states, respectively,
and a k-ary regular operation f.
What is the maximal number of states
of a minimal automaton recognizing $f(L(A_1), \ldots, L(A_k))$,
for the given n_i?"

In this paper:

- automata are unambiguous (UFAs)
- f: intersection, reversal, shuffle, star and positive closure,
  left and right quotients, concatenation,
  complementation, and union

Intersection on Unambiguous Finite Automata

Intersection:
$K \cap L = \{w \mid w \in K \text{ and } w \in L\}$

Known results for intersection:

DFA:	mn	binary	[Maslov 1970]
NFA:	mn	binary	[Holzer & Kutrib 2003]

Our result for intersection on UFAs:

UFA: mn $|\Sigma| \geq 2$

Proof sketch:

- upper bound: given UFAs A and B, construct the direct product automaton $A \times B$; it is a UFA
- lower bound: the witnesses in [HK'03] for NFA intersection are deterministic, so UFAs □

Shuffle on Unambiguous Finite Automata

Shuffle:

$K \sqcup\!\sqcup L = \{u_1 v_1 u_2 v_2 \cdots u_k v_k \mid u_1 u_2 \cdots u_k \in K \text{ and } v_1 v_2 \cdots v_k \in L\}$

Known results for shuffle:

DFA:	???		
in-DFA:	$2^{mn} - 1$	5-letter	[Câmpeanu, Salomaa & Yu 2002]
NFA:	mn	binary	[G. J. & Masopust, DLT 2010]

Our result for shuffle on UFAs:

UFA: $2^{mn} - 1$ $|\Sigma| \geq 5$

Proof sketch for lower bound:

- take the witness incomplete DFAs from [CSY'02]
- in the mn-state NFA for shuffle
  - each non-empty set is reachable [CSY'02]
  - each non-empty set is co-reachable □

Concatenation on Unambiguous Finite Automata

Concatenation:
$KL = \{uv \mid u \in K \text{ and } v \in L\}$

Known results for concatenation:

DFA:	$(m - 1/2) \cdot 2^n$	binary	[Maslov 1970]
NFA:	$m + n$	binary	[Holzer & Kutrib 2003]

Our result for concatenation on UFAs:

UFA: $(3/4) \cdot 2^m \cdot 2^n - 1$ $\quad |\Sigma| \geq 7$

Proof idea for the upper bound:

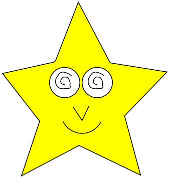
- construct an $(m + n)$-state NFA N for KL
- show that at most $(3/4) \cdot 2^m \cdot 2^n - 1$ subsets are reachable in the subset automaton of N $\square$

Star on Unambiguous Finite Automata

Star:
$L^* = \{u_1 u_2 \cdots u_k \mid k \geq 0 \text{ and } u_i \in L \text{ for all } i\}$

Known results for the star operation:

DFA:	$(3/4) \cdot 2^n$	binary	[Yu, Zhuang & K. Salomaa 1994]
NFA:	$n + 1$	unary	[Holzer & Kutrib 2003]



Our result for star on unambiguous automata:

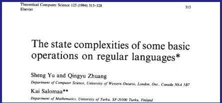
UFA: $(3/4) \cdot 2^n$ $|\Sigma| \geq 3$

Proof idea for the lower bound:

- start with YZS'94 binary witness DFA for star
- define a new symbol c
- compute the rank of the corresponding matrix □

Ternary Witness UFA for Star Meeting the Bound $(3/4) \cdot 2^n$

Yu, Zhuang & K. Salomaa 1994


Theoretical Computer Science 125 (1994) 315–328
Elsevier
315

The state complexities of some basic operations on regular languages*

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Department of Computer Science, University of Western Ontario, London, Ont., Canada N6A 5B7
Kai Salomaa**
Department of Mathematics, University of Turku, SF-20500 Turku, Finland

Theorem 3.3. *For any integer $n \geqslant 2$, there exists a DFA A of n states such that any DFA accepting $(L(A))^*$ needs at least $2^{n-1}+2^{n-2}$ states.*

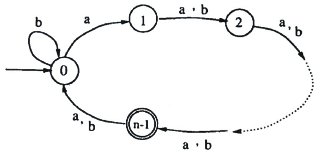
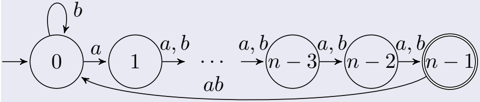
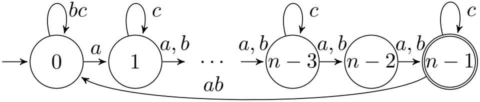


Fig. 4. DFA A_n.



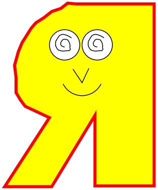


Reversal on Unambiguous Finite Automata

Reversal:
$L^R = \{w^R \mid w \in L\}$, where w^R is the mirror image of w

Known results for the reversal operation:

DFA:	2^n	binary	[Leiss 1981, Šebej 2009]
NFA:	$n+1$	binary	[Holzer & Kutrib 2003, G. J. 2005]



Reversal on UFAs:

UFA: n $|\Sigma| \geq 1$

Proof.

If A is unambiguous, then A^R is unambiguous. □

Complementation on UFAs: Partial Results

Known results for complementation:

DFA:	n	unary	[folklore]
NFA:	2^n	binary	[Birget 1993, G. J. 2005]
UFA:	$\geq n^{2-o(1)}$	unary	[Okhotin 2012]

Our unsuccessful attempts for UFAs:

- the matrix method didn't work:
  $\text{rank}(M_{L^c}) = \text{rank}(M_L) \pm 1$
- the fooling-set method didn't work:
  - if L is accepted by an n-state UFA, then every fooling set for L^c is of size $\leq n^2/2$
  - we only found a fooling set of size $n + \sqrt{n}$
  - conjecture: every fooling set for L^c is of size $\leq 2n$
- large alphabets didn't work either

Complementation on UFAs: Partial Results

Known results for complementation:

DFA:	n	unary	[folklore]
NFA:	2^n	binary	[Birget 1993, G. J. 2005]
UFA:	$\geq n^{2-o(1)}$	unary	[Okhotin 2012]

Our upper bound on complementation for UFAs:

UFA: $\leq 2^{0.79n+\log n}$

Proof sketch for the upper bound:

If L is accepted by an n-state UFA A, then

- $\text{usc}(L^c) \leq |\mathcal{R}|$ (reachable in A)
- $\text{usc}(L^c) \leq |\mathcal{C}|$ (co-reachable in A)
- if $\max\{|S| \mid S \in \mathcal{R}\} \geq n/2$, then $|\mathcal{C}|$ is small
- otherwise, $\min\{|\mathcal{R}|, |\mathcal{C}|\}$ is small $\square$

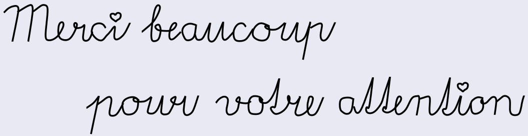
Summary and Open Problems

The complexity of operations on unambiguous finite automata:

	sc	$\|\Sigma\|$	usc	$\|\Sigma\|$	nsc	$\|\Sigma\|$
intersection	mn	2	mn	2	mn	2
left quotient	$2^n - 1$	2	$2^n - 1$	2	$n+1$	2
positive closure	$\frac{3}{4} \cdot 2^n - 1$	2	$\frac{3}{4} \cdot 2^n - 1$	3	n	1
star	$\frac{3}{4} \cdot 2^n$	2	$\frac{3}{4} \cdot 2^n$	3	$n+1$	1
shuffle	?		$2^{mn} - 1$	5	mn	2
reversal	2^n	2	n	1	$n+1$	2
concatenation	$(m - 1/2) \cdot 2^n$	2	$\frac{3}{4} \cdot 2^{m+n} - 1$	7	$m+n$	2
right quotient	n	1	$2^n - 1$	2	n	1
complementation	n	1	$\leq 2^{0.79n + \log n}$		2^n	2
			$\geq n^{2-o(1)}$	1		

Acknowledgments

1. Thank you very much for your attention



2. Many thanks to ...

- "big" Jozko and "small" Jozko
- Maria, Jonas, and Dominik
- ...

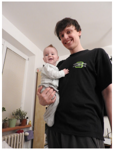
Greetings from Maria, Jonas, and Dominik

Maria 2004

Sept. 2015

3 weeks

3 months



6 months

Easter 2016

last week

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right quotient	n	1	$2^n - 1$	2	n	1
complementation	n	1	$\leq 2^{0.79n + \log n}$		2^n	2
			$\geq n^{2 - o(1)}$	1		