Upgrading probability

Roman Frič, Košice

MÚ SAV, seminar

O čom to bude?

Poznámky o pravdepodobnosti

1. Motivačné príklady - diskrétny model.
2. Prechod ku spojitému modelu.
3. Dualita: náhodná premenná vs pozorovateľná.

Why to upgrade Classical Probability Theory

Three aspects of CPT "calling for" an upgrade:
1. The random events in CPT are black-and-white (Boolean);
2. The random variables in CPT do not model quantum phenomena;
3. The observables and the probability measures (two basic maps in CPT) have very different "mathematical nature".

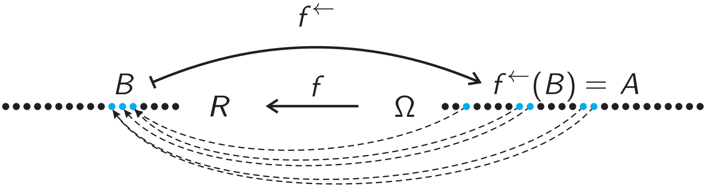
Accordingly, we propose an upgraded probability theory based on Łukasiewicz operations and elementary category theory and comprising CPT as a special case.

Hod dvomi hracími kockami - súčet očiek

Náhodný experiment: pojmy a terminológia
Generický (diskrétny) príklad: súčet očiek
Matematický model: meranie výsledku - číselná hodnota - nezávislé opakovanie - histogram
- náhodná premenná: (náhodná) čierna skrinka a jej otváranie
Matematická štatistika: matematické metódy na otváranie (náhodných) čiernych skriniek

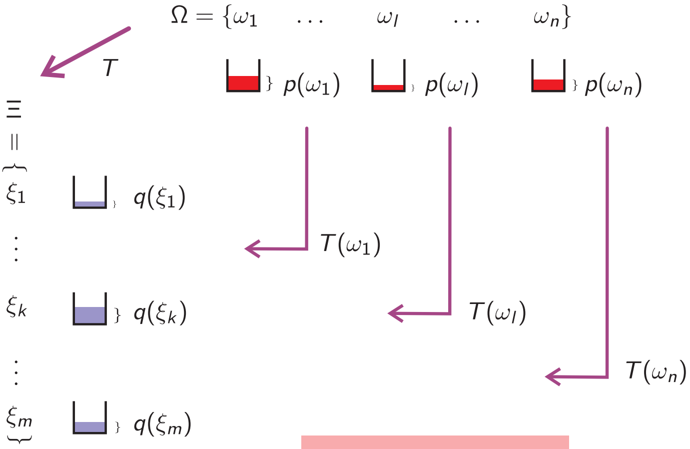
Náhodná premenná

$$\mathbf{B}_R \xrightarrow{f^{\leftarrow}} \mathbf{A}$$



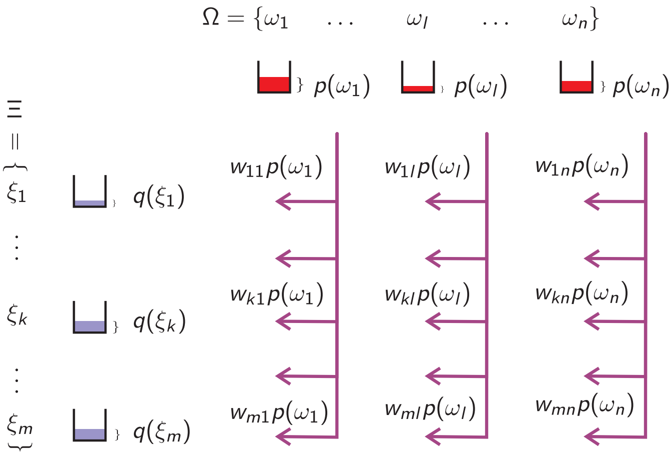
- pozorovaná udalosť $B \in \mathbf{B}_R$, modelovaná udalosť $f^{\leftarrow}(B) = A \in \mathbf{A}$
- $B \longmapsto f^{\leftarrow}(B) = A$
- $f^{\leftarrow} : \mathbf{B}_R \to \mathbf{A}$, "kanál" na transport pravdepodobnosti
- $P_f(B) = P(f^{\leftarrow}(B))$, $P \mapsto P_f$

Classical pipeline



$$q(\xi_k) = \sum_{\omega \in T^{\leftarrow}(\xi_k)} p(\omega)$$

Upgraded pipeline



$$q(\xi_k) = \sum_{l=1}^{n} w_{kl}\, p(\omega_l)$$

"Upgraded" basic notions:

- events ... measurable fuzzy sets, $\mathcal{M}(\mathbf{A}) \subseteq [0,1]^{\Omega}$;
- operations ... Łukasiewicz: for $a, b \in [0,1]^{\Omega}$, $\omega \in \Omega$, put $(a \oplus b)(\omega) = a(\omega) \oplus b(\omega) = \min\{1, a(\omega) + b(\omega)\}$, $(a \odot b)(\omega) = a(\omega) \odot b(\omega) = \max\{0, a(\omega) + b(\omega) - 1\}$, $a^c(\omega) = 1 - a(\omega)$;
- probability measures ... probability integrals on $\mathcal{M}(\mathbf{A})$, $\mathcal{IP}(\mathbf{A})$ ... probability integrals on $\mathcal{M}(\mathbf{A})$ are exactly sequentially continuous D-homomorphisms of $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{T})$;
- observables ... sequentially continuous D-homomorphism $h : \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$, e.g., $\mathbf{B} = \mathbf{B}_R^N$;
- random variables ... statistical maps, "measurable" maps $T : \mathcal{IP}(\mathbf{A}) \longrightarrow \mathcal{IP}(\mathbf{B})$.

Conservative observable

Theorem

Each sequentially continuous D-homomorphism $h_0 : \mathbf{B} \longrightarrow \mathbf{A}$ *can be uniquely extended to a sequentially continuous D-homomorphism* $h : \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$

Definition

A sequentially continuous D-homomorphism $h : \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ is said to be an observable. Moreover, if $h(B) \in \mathbf{A}$ for all $B \in \mathbf{B}$, then h is said to be conservative.

Example

Observe that if $\mathbf{A} = \mathbf{T}$ and $p : \mathbf{B} \longrightarrow [0,1]$ is a nondegenerated probability measure, then the corresponding integral $\int(.)\mathrm{d}p$ as an observable (mapping $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{T}) = [0,1]$) fails to be conservative.

From observables to statistical maps

Probability integrals on $\mathcal{M}(\mathbf{A})$ are also called states and denoted by $\mathcal{S}(\mathcal{M}(\mathbf{A}))$ ($\equiv \mathcal{IP}(\mathbf{A})$). Let s be a state on $\mathcal{M}(\mathbf{A})$, i.e. a probability integral, $s = \int(.)\mathrm{d}p$, $p \in \mathcal{P}(\mathbf{A})$, and let $h : \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ be an observable. Then $\int h(.)\mathrm{d}p$, as a composition $s \circ h$ of two sequentially continuous D-homomorphisms, yields a probability integral $\int(.)\mathrm{d}q$, $q \in \mathcal{P}(\mathbf{B})$ on $\mathcal{M}(\mathbf{B})$. Consequently, the compositions $s \circ h$, $s \in \mathcal{S}(\mathcal{M}(\mathbf{A}))$, define a map

$$T_h : \mathcal{S}(\mathcal{M}(\mathbf{A})) \longrightarrow \mathcal{S}(\mathcal{M}(\mathbf{B})), T_h(\int(.)\mathrm{d}p) \mapsto \int(.)\mathrm{d}q$$

.

Lemma

Let $p \in \mathcal{P}(\mathbf{A})$. If $p(A) \in \{0,1\}$ for all $A \in \mathbf{A}$, then p is said to be degenerated. For $\omega \in \Omega$, denote δ_ω the corresponding degenerated point probability measure: $\delta_\omega(A) = 1$ for $\omega \in A$ and $\delta_\omega(A) = 0$ otherwise.

Lemma

Let $g, h : \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ be observables. If $g \neq h$, then $T_g \neq T_h$

Proof.

Let $g \neq h$. Then there exists $u \in \mathcal{M}(\mathbf{B})$ such that $g(u) \neq h(u)$ and hence there exists $\omega \in \Omega$ such that $(g(u))(\omega) \neq (h(u))(\omega)$. Let δ_ω be the corresponding degenerated point probability on $\mathbf{A}$. Then $T_g(\int(.)\mathrm{d}(\delta_\omega)))(u) = \int g(u)\mathrm{d}(\delta_\omega) = (g(u))(\omega) \neq (h(u))(\omega) = \int h(u)\mathrm{d}(\delta_\omega) = T_h(\int(.)\mathrm{d}(\delta_\omega))(u)$, and hence $T_g \neq T_h$. □

Statistical map

Definition

Let T be a map of $\mathcal{S}(\mathcal{M}(\mathbf{A}))$ into $\mathcal{S}(\mathcal{M}(\mathbf{B}))$. If there exists an observable h of $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$ such that $T = T_h$, then T is said to be a statistical map of $\mathcal{S}(\mathcal{M}(\mathbf{A}))$ into $\mathcal{S}(\mathcal{M}(\mathbf{B}))$. If h is conservative, then the corresponding statistical map T_h is said to be conservative.

Lemma

Let $h : \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ be a conservative observable. If $p \in \mathcal{P}(\mathbf{A})$ is degenerated and if $T_h(\int(.)\mathrm{d}p) = \int(.)\mathrm{d}q$, $q \in \mathcal{P}(\mathbf{B})$, then q is degenerated.

Proof.

Hint: via contradiction. □