Extensions of D-posets of fuzzy events

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Abstract

Cieľom je v rámci teórie D-posetov popísať typické situácie (klasické i fuzzy), v ktorých sa kanonickým spôsobom rozširuje základné pole náhodných udalostí pridávaním nových.

$\mathbf{A} \subseteq \sigma(\mathbf{A}) \subseteq \mathbf{M}_{\mathbf{A}}$,

$\mathcal{X} \subseteq \sigma(\mathcal{X})$, $\mathcal{X} \subseteq \mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})})$

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Let **A** be a field of subsets of a set X, let $\mathbf{P}(X)$ be the set of all subsets of X, let $\mathcal{P}(\mathbf{A})$ be the set of all probability measures on **A**, and let $p \in \mathcal{P}(\mathbf{A})$. For each $B \subseteq X$ put

$$p^*(B) = \inf\{\sum_{i=1}^{\infty} p(A_i);\ B \subseteq \cup_{i=1}^{\infty} A_i,\ A_i \in \mathbf{A}\}.$$

The resulting map $p^* : \mathbf{P}(X) \longrightarrow I$ is called **induced outer measure**. A set $M \subseteq X$ is said to be p-**measurable** whenever for each $B \subseteq X$ we have

$$p^*(B) = p^*(B \cap M) + p^*(B \cap M^c),$$

where $M^c = X \setminus M$. Denote $\mathbf{M}_p$ the set of all p-measurable subsets of X.

Let $\mathbf{A}$ be a field of subsets of a set X and let p be a probability measure on $\mathbf{A}$. Then

(i) $\mathbf{M}_p$ is a σ-field, $\mathbf{A} \subseteq \mathbf{M}_p$ and if $B \subseteq X$ and $p^*(B) = 0$, then $B \in \mathbf{M}_p$;

(ii) Define $\overline{p}(B) = p^*(B)$, $B \in \mathbf{M}_p$. Then $\overline{p}$ is a probability measure on $\mathbf{M}_p$ and it is an extension of p over $\mathbf{M}_p$;

(iii) $\mathbf{M}_p$ is the largest σ-field of subsets of X which contains $\mathbf{A}$ and on which p^* defines a probability measure;

(iv) If $B \subseteq X$, then there exists $A \in \sigma(\mathbf{A}) \subseteq \mathbf{M}_p$ such that $B \subseteq A$ and $p^*(B) = \overline{p}(A)$.

Denote

$$\mathbf{M}_{\mathbf{A}} = \bigcap_{p \in \mathcal{P}(\mathbf{A})} \mathbf{M}_p.$$

Clearly, $M \subseteq X$ belongs to $\mathbf{M}_{\mathbf{A}}$ iff it is p-measurable for all $p \in \mathcal{P}(\mathbf{A})$, $\sigma(\mathbf{A}) \subseteq \mathbf{M}_{\mathbf{A}}$, and $\mathbf{M}_{\mathbf{A}}$ is a σ-field of subsets of the set X.

It is known that in general we have $\sigma(\mathbf{A}) \neq \mathbf{M}_{\mathbf{A}}$.

Definition

Let $\mathbf{A}$ be a field of subsets of a set X. Elements of $\mathbf{M}_{\mathbf{A}}$ are said to be **absolutely A-measurable sets**.

Definition

Let $\mathbf{A}$ be a field of subsets of a set X and let p be a probability measure on $\mathbf{A}$. Let $\mathbf{B}$ be a field of subsets of a set X such that $\mathbf{A} \subseteq \mathbf{B}$ and let q be a probability measure on $\mathbf{B}$ such that $p(A) = q(A)$ for all $A \in \mathbf{A}$. If

$$q(B) = \inf\{\sum_{i=1}^{\infty} p(A_i);\ B \subseteq \cup_{i=1}^{\infty} A_i,\ A_i \in \mathbf{A}\}$$

for all $B \in \mathbf{B}$, then q is said to be a **measurable extension** of p.

Let q be a measurable extension of $p \in \mathcal{P}(\mathbf{A})$ on $\mathbf{A}$ over $\mathbf{M}_{\mathbf{A}}$. Clearly, for all $M \in \mathbf{M}_{\mathbf{A}}$ we have $q(M) = p^*(M)$.

Denote **ID** the category having (reduced) D-posets of fuzzy sets as objects and having sequentially continuous D-homomorphisms (preserving constants, order, and the difference) as morphisms. Objects of **ID** are subobjects of the powers I^X.

Essentially, there are two types of extensions. First, we can add functions and then we speak of an ID-**extension**. Second, we can extend the domain of functions and such extensions have stochastic applications

Let $\mathcal{X} \subseteq I^X$ be a D-poset of fuzzy sets. Denote $\mathcal{S}(\mathcal{X})$ the set of all sequentially continuous D-homomorphisms of $\mathcal{X}$ into I; the elements of $\mathcal{S}(\mathcal{X})$ are called **states**. In what follows, each $x \in X$ will be considered as the **evaluation state** on $\mathcal{X}$: $x(u) = u(x)$, $u \in \mathcal{X}$. If $X = \mathcal{S}(\mathcal{X})$, then $\mathcal{X}$ is said to be **sober**. Let Y be a set of states such that $X \subseteq Y \subseteq \mathcal{S}(\mathcal{X})$. For $u \in \mathcal{X}$, put $ev_Y(u) = \{y(u); y \in Y\} \in I^Y$ and denote by ev_Y the corresponding map of $\mathcal{X}$ into I^Y. Put $\mathcal{X}_Y = \{ev_Y(u); u \in \mathcal{X}\}$.

Lemma

(i) *$\mathcal{X}_Y$ is a D-poset of fuzzy sets (with respect to the D-poset structure inherited from I^Y).*

(ii) *ev_Y is an isomorphism.*

Definition

Let $\mathcal{X} \subseteq I^X$ be a D-poset of fuzzy sets and let Y be a set of states such that $X \subseteq Y \subseteq \mathcal{S}(\mathcal{X})$. Then $\mathcal{X}_Y$ is said to be a **domain extension** of $\mathcal{X}$. If $Y = \mathcal{S}(\mathcal{X})$, then $\mathcal{X}_Y$ is said to be the **sobrification** of $\mathcal{X}$.

For $\mathcal{X} \subseteq I^X$ and $Y = \mathcal{S}(\mathcal{X})$, the sobrification $\mathcal{X}_Y$ of $\mathcal{X}$ will be denoted by $\mathcal{X}^*$. Denote **SID** the (full) subcategory of **ID** consisting of sober D-posets of fuzzy sets.

Theorem

The subcategory **SID** *of* **ID** *is epireflective in* **ID**.

Let us recall the notion of an **epireflection**. Let **B** be a full subcategory of a category **A**. Let X be an object of **A**. An object $r(X)$ of **B** is called a **reflection** of X in **B**, or a **B**-reflection, if there exists a morphism $e_X : X \longrightarrow r(X)$ such that for each morphism $f : X \longrightarrow Y$, Y in **B**, there exists a unique morphism $r(f) : r(X) \longrightarrow Y$ such that $r(f) \circ e_X = f$. The functor assigning to each object of **A** its reflection in **B**, is called a reflector. If each object of **A** has a **B**-reflection, then **B** is said to be **reflective** in **A**. A reflective subcategory is called **epireflective** if the canonical morphism $e_X : X \longrightarrow r(X)$ is an epimorphism for every X; in this case we speak of an **epireflector** and **epireflection**.

Let $\mathcal{X} \subseteq I^X$ be a D-poset of fuzzy sets and let $\mathcal{Y} \subseteq I^X$ be an ID-extension of $\mathcal{X}$. Recall that if each state on $\mathcal{X}$ can be extended to a state on $\mathcal{Y}$, i.e. $(\forall s \in \mathcal{S}(\mathcal{X}))(\exists\, t \in \mathcal{S}(\mathcal{Y}))[t{\restriction}\mathcal{X} = s]$, then $\mathcal{X}$ is said to be $\mathcal{S}(\mathcal{X})$-**embedded** in $\mathcal{Y}$.
Let $\mathcal{X} \subseteq I^X$ be a D-poset of fuzzy sets and let $\mathcal{Y} \subseteq I^X$ be an ID-extension of $\mathcal{X}$. Let t be a state on $\mathcal{Y}$ and let $t{\restriction}\mathcal{X}$ be the restriction of t to $\mathcal{X}$. Since $t{\restriction}\mathcal{X} \in \mathcal{S}(\mathcal{X})$, the restriction yields a **restriction map** r of $\mathcal{S}(\mathcal{Y})$ into $\mathcal{S}(\mathcal{X})$ sending t to $r(t) = t{\restriction}\mathcal{X}$.

Definition

Let $\mathcal{X} \subseteq I^X$ be a D-poset of fuzzy sets and let $\mathcal{Y} \subseteq I^X$ be an ID-extension of $\mathcal{X}$. If the restriction map $r : \mathcal{S}(\mathcal{Y}) \longrightarrow \mathcal{S}(\mathcal{X})$ is one-to-one and onto, then $\mathcal{Y}$ is said to be a **state** extension of $\mathcal{X}$. A state extension $\mathcal{Y}$ of $\mathcal{X}$ is said to be **maximal** if there is no proper state extension $\mathcal{Z} \subseteq I^X$ of $\mathcal{Y}$. Let $\mathcal{C}$ be a class of state extensions of $\mathcal{X}$ and let $\mathcal{Y} \in \mathcal{C}$. If there is no proper extension of $\mathcal{X}$ in $\mathcal{C}$, then $\mathcal{Y}$ is said to be **maximal in** $\mathcal{C}$. If $\mathcal{Y}$ is a state extension of $\mathcal{X}$ and $\mathcal{Y}$ is an ID-extension of each state extension $\mathcal{Z}$ of $\mathcal{X}$, then $\mathcal{Y}$ is said to be the **absolute state extension** of $\mathcal{X}$. If $\mathcal{Y} \in \mathcal{C}$ and $\mathcal{Y}$ is an ID-extension of each state extension $\mathcal{Z}$ of $\mathcal{X}$ in $\mathcal{C}$, then $\mathcal{Y}$ is said to be the **absolute state extension** of $\mathcal{X}$ **in** $\mathcal{C}$.

Example

For $X = \{0, 1\}$, let $\mathcal{X} \subseteq I^X$ be the σ-field of all subsets of X. The set $\mathcal{S}(\mathcal{X})$ of all states of $\mathcal{X}$ is the same as the set of all probability measures on the σ-field in question. Each $s \in \mathcal{S}(\mathcal{X})$ can be visualized as an element a_s of the closed unit interval $[0, 1]$, where the number a_s is equal to the corresponding probability of the singleton $\{0\}$ and $(1 - a_s)$ is equal to the corresponding probability of the singleton $\{1\}$; we identify s and the corresponding number a_s and, consequently, we identify $\mathcal{S}(\mathcal{X})$ and $[0, 1]$. Further, the sobrification $\mathcal{X}^*$ can be visualized as the D-poset $\mathcal{X}_{\mathcal{S}(\mathcal{X})} \subseteq I^{[0,1]}$ consisting of the following four functions: constant functions $0_{[0,1]}$ and $1_{[0,1]}$, functions x and $1 - x$, $x \in [0, 1]$.

Example

Consider $\mathcal{X}$ and its sobrification $\mathcal{X}^* \subseteq I^{[0,1]}$ in the previous example. Let **B** be the σ-field of all Borel measurable subsets of $[0,1]$ and let $\mathcal{M}(\mathbf{B})$ be the set of all **B**-measurable functions with values in I. Then $\mathcal{M}(\mathbf{B})$ can be considered as an ID-extension of $\mathcal{X}^*$. Clearly, $\mathcal{X}^*$ is $\mathcal{S}(\mathcal{X}^*)$-embedded in $\mathcal{S}(\mathcal{M}(\mathbf{B}))$. On the other hand, the restriction map r of $\mathcal{S}(\mathcal{M}(\mathbf{B}))$ into $\mathcal{S}(\mathcal{X}^*)$ is far from being none-to-one. Hence $\mathcal{M}(\mathbf{B})$ fails to be a state extension of $\mathcal{X}^*$.