On recent developments in the theory of dynamic Hardy-type inequalities on time scales

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Motivation

- Mathematical inequalities are important in many areas of mathematics.
- Discrete versions of integral inequalities are often harder to prove. We want tools that make them easier.
- A single framework can treat both continuous and discrete cases, so we do not repeat the same ideas twice.
- Time scales calculus lets us define differentiation and integration on any closed subset of $\mathbb{R}$, giving one general inequality that covers many cases.
- With this unified view, we can get new continuous results, new discrete results, and new formulas in quantum calculus.

Hardy-Type Inequalities

- In 1920, Hardy [5] proved the discrete inequality

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a(i)\right)^{p}\leq\left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}a^{p}(n),\qquad p>1, \tag{1}$$

  where $a(n)\geq 0$ for $n\geq 1$ and $a\in\ell^{p}(\mathbb{N})$ (i.e., $\sum_{n=1}^{\infty}a^{p}(n)<\infty$).

- In [6, Theorem A] he proved the integral version of (1): for $f\geq 0$, $f\in L^{p}(0,\infty)$, $p>1$,

$$\int_{0}^{\infty}\left(\frac{1}{x}\int_{0}^{x}f(t)\,\mathrm{d}t\right)^{p}\mathrm{d}x\leq\left(\frac{p}{p-1}\right)^{p}\int_{0}^{\infty}f^{p}(x)\,\mathrm{d}x. \tag{2}$$

- In both continuous and discrete cases the constant $C=\left(\frac{p}{p-1}\right)^{p}$ is sharp.

Generalized Hardy-Type Inequalities

Hardy [6, Theorem B] showed that if $p > 1$, $\lambda(n) > 0$, $a(n) \geq 0$ and $\Lambda(n) = \sum_{i=1}^{n} \lambda(i)$, then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{\Lambda^p(n)} \left(\sum_{i=1}^{n} \lambda(i) a(i) \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} \lambda(n) a^p(n). \tag{3}$$

Copson [4] generalized (3): for $p > 1$, $a(i) \geq 0$, $\lambda(i) > 0$ and $c > 1$,

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{\Lambda^c(n)} \left(\sum_{i=1}^{n} \lambda(i) a(i) \right)^p \leq \left(\frac{p}{c-1} \right)^p \sum_{n=1}^{\infty} \lambda(n) \Lambda^{p-c}(n) a^p(n). \tag{4}$$

If $p > 1$ and $0 \leq c < 1$,

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{\Lambda^c(n)} \left(\sum_{i=n}^{\infty} \lambda(i) a(i) \right)^p \leq \left(\frac{p}{1-c} \right)^p \sum_{n=1}^{\infty} \lambda(n) \Lambda^{p-c}(n) a^p(n). \tag{5}$$

Reversed inequalities

Bennett [1] and Leindler [8] proved reverse forms of (4)–(5). If $c \leq 0 < p < 1$,

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{\Lambda^c(n)} \left(\sum_{i=n}^{\infty} \lambda(i) a(i) \right)^p \geq \left(\frac{p}{1-c} \right)^p \sum_{n=1}^{\infty} \lambda(n) \Lambda^{p-c}(n) a^p(n). \tag{6}$$

If $c > 1 > p > 0$ and $\Lambda_n \to \infty$ as $n \to \infty$,

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{\Lambda_n^c} \left(\sum_{i=1}^{n} \lambda(i) a(i) \right)^p \geq \left(\frac{pL}{c-1} \right)^p \sum_{n=1}^{\infty} \lambda(n) \Lambda_n^{p-c} a^p(n), \tag{7}$$

where $L = \inf \frac{\lambda(n)}{\lambda(n+1)}$.

Weighted Hardy-type inequalities

The first weighted version (Hardy [7, Theorem B]):

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t\right)^p x^\gamma\,\mathrm{d}x \le \left(\frac{p}{p-1-\gamma}\right)^p \int_0^\infty x^\gamma f^p(x)\,\mathrm{d}x, \tag{8}$$

for $p > 1$, $\gamma < p - 1$, and

$$\int_0^\infty \left(\frac{1}{x}\int_x^\infty f(t)\,\mathrm{d}t\right)^p x^\gamma\,\mathrm{d}x \le \left(\frac{p}{1-(p-\gamma)}\right)^p \int_0^\infty x^\gamma f^p(x)\,\mathrm{d}x, \tag{9}$$

for $p > 1$, $p < \gamma + 1$.

Muckenhoupt and mixed norms

Muckenhoupt [9] characterized weights for

$$\left(\int_0^\infty u^p(x)\left(\int_0^x f(t)\,\mathrm{d}t\right)^p \mathrm{d}x\right)^{1/p} \le C\left(\int_0^\infty v^p(x) f^p(x)\,\mathrm{d}x\right)^{1/p}, \tag{10}$$

with $1 < p < \infty$, via

$$\sup_{x>0}\left(\int_x^\infty u^p(t)\,\mathrm{d}t\right)^{1/p}\left(\int_0^x v^{-p^*}(t)\,\mathrm{d}t\right)^{1/p^*} < \infty, \quad p^* = \frac{p}{p-1}.$$

Bradley [3] studied the $L^p \to L^q$ case $(1 \le p \le q \le \infty)$:

$$\left(\int_0^\infty u^q(x)\left(\int_0^x f(t)\,\mathrm{d}t\right)^q \mathrm{d}x\right)^{1/q} \le C\left(\int_0^\infty v^p(x) f^p(x)\,\mathrm{d}x\right)^{1/p}, \tag{11}$$

via

$$\sup_{x>0}\left(\int_x^\infty u^q(t)\,\mathrm{d}t\right)^{1/q}\left(\int_0^x v^{-p^*}(t)\,\mathrm{d}t\right)^{1/p^*} = A < \infty,$$

and $A \le C \le p^{1/q}(p^*)^{1/p^*}A$, for $1 < p < q < \infty$ and $A = C$ if $p = 1$ and $q = \infty$.

Finite-interval versions

Opic–Kufner [10]: for $-\infty \le a < b < \infty$, $1 < p \le q < \infty$,

$$\left(\int_a^b u(x)\left(\int_a^x f(t)\,\mathrm{d}t\right)^q \mathrm{d}x\right)^{1/q} \le C\left(\int_a^b v(x) f^p(x)\,\mathrm{d}x\right)^{1/p}, \tag{12}$$

iff

$$B = \sup_{a<x<b}\left(\int_x^b u(t)\,\mathrm{d}t\right)^{1/q}\left(\int_a^x v^{1-p^*}(t)\,\mathrm{d}t\right)^{1/p^*} < \infty,$$

with estimates

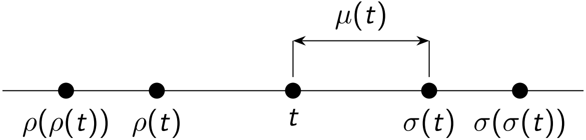
$$B \le C \le \left(1+\frac{q}{p^*}\right)^{1/q}\left(1+\frac{p^*}{q}\right)^{1/p^*} B.$$

Sinnamon [12]: for $0 < q < 1 < p < \infty$,

$$D = \left[\int_a^b \left(\int_x^b u(t)\,\mathrm{d}t\right)^{r/p}\left(\int_a^x v^{1-p^*}(t)\,\mathrm{d}t\right)^{r/p^*} u(x)\,\mathrm{d}x\right]^{1/r} < \infty, \tag{13}$$

where $1/r = 1/q - 1/p$, and $(p^*)^{1/p^*} q^{1/p} D \le C \le (p^*)^{1/r} q^{1/p} D$.

Time scale at a glance



Basic concepts on time scales

- A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$.
- Forward/backward jumps $\sigma, \rho : \mathbb{T} \to \mathbb{T}$:

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

- Graininess $\mu(t) := \sigma(t) - t$.
- The set

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \sup \mathbb{T} = \infty. \end{cases}$$

Differentiation on time scales

Theorem (see [2, Theorem 1.20]): If $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^\kappa$ and $\alpha, \beta \in \mathbb{R}$:

1. $(\alpha f + \beta g)^\Delta(t) = \alpha f^\Delta(t) + \beta g^\Delta(t)$.
2. $(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t))$.
3. If $g(t)g(\sigma(t)) \neq 0$, then
$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.$$

Chain rule (two forms)

Theorem (see [2, Theorem 1.87]): if $v : \mathbb{R} \to \mathbb{R}$ is continuous, $v : \mathbb{T} \to \mathbb{R}$ is delta differentiable, and $u : \mathbb{R} \to \mathbb{R}$ is C^1, then

$$(u \circ v)^{\Delta}(t) = u'(v(c))\, v^{\Delta}(t) \quad \text{for some } c \in [t, \sigma(t)]. \tag{14}$$

Lemma (see [2, Theorem 1.87]): if $f : \mathbb{R} \to \mathbb{R}$ is C^1 and $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable, then

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t) + h\, \mu(t)\, g^{\Delta}(t))\, \mathrm{d}h \right\} g^{\Delta}(t). \tag{15}$$

Integration on time scales

Theorem (see [2, Theorem 1.77]): for $a, b, c \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$, and $u, v \in C_{\mathrm{rd}}([a, b]_{\mathbb{T}}, \mathbb{R})$:

1. $\displaystyle\int_a^b (\alpha u + \beta v)\,\Delta t = \alpha \int_a^b u\,\Delta t + \beta \int_a^b v\,\Delta t.$
2. $\displaystyle\int_a^b u\,\Delta t = \int_a^c u\,\Delta t + \int_c^b u\,\Delta t.$
3. $\displaystyle\int_a^a u\,\Delta t = 0.$
4. $\displaystyle\left|\int_a^b u\,\Delta t\right| \leq \int_a^b |u|\,\Delta t.$
5. If $u \geq 0$ on $[a, b]_{\mathbb{T}}$ then $\displaystyle\int_a^b u\,\Delta t \geq 0.$
6. Integration by parts:
$$\int_a^b u\,v^{\Delta}\,\Delta t = [uv]_a^b - \int_a^b u^{\Delta} v^{\sigma}\,\Delta t.$$

Hölder and Jensen on time scales

Hölder ([2, Theorem 6.13]): if $f, g \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}^+)$ and $\gamma > 1$, $1/\gamma + 1/\nu = 1$,

$$\int_a^b f(t)g(t)\,\Delta t \leq \left(\int_a^b f^{\gamma}(t)\,\Delta t\right)^{1/\gamma} \left(\int_a^b g^{\nu}(t)\,\Delta t\right)^{1/\nu}. \tag{16}$$

The inequality reverses (with sign change) for either $p < 0$ or $0 < p < 1$.
Jensen ([2, Theorem 6.17]): if $f \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}}, (c,d))$ and F is convex on (c,d),

$$F\left(\frac{\int_a^b f(t)\,\Delta t}{b-a}\right) \leq \frac{\int_a^b F(f(t))\,\Delta t}{b-a}. \tag{17}$$

Main results

- Recent developments in Hardy-type inequalities:
  - A new reformulation valid for $p = 1$.
  - Using different time scales $\mathbb{T}$ and $\tilde{\mathbb{T}} = \nu(\mathbb{T})$, with ν strictly increasing.
  - Integration by substitution from $\mathbb{T}$ to $\tilde{\mathbb{T}}$ and the derivative of the inverse.
  - Dynamic Hardy-type inequalities with negative parameters.
  - A global measure of the dispersion of f around its integral mean $\dfrac{1}{x}\displaystyle\int_0^x f(t)\,\mathrm{d}t$.

Main result: reformulation valid for $p = 1$

If $\mathbb{T}$ is a time scale with $a, \ell \in \mathbb{T}$, $f \in C_{\mathrm{rd}}([a,\ell]_{\mathbb{T}}, \mathbb{R}^+)$, and either $p \geq 1$ or $p < 0$, then

$$\int_a^\ell \left(\frac{\int_a^x f(y)\,\Delta y}{x-a} \right)^p \frac{\Delta x}{\sigma(x)-a} \leq \int_a^\ell \left(1 - \frac{\sigma(x)-a}{\ell - a} \right) f^p(x)\, \frac{\Delta x}{\sigma(x)-a}. \tag{18}$$

Change of time scale and a Beta-type form

Using substitution from $\mathbb{T}$ to $\tilde{\mathbb{T}} = \nu(\mathbb{T})$ with $\nu(x) = x/y$ $(x, y \in \mathbb{T},\ y > \ell)$: assume $\ell \in \mathbb{T}$, $\ell \geq 0$, $p \geq 1$, $\alpha > 0$, and $f \in C_{\mathrm{rd}}((\ell,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ is nonincreasing. Then

$$\int_\ell^\infty \left(\int_x^\infty f(y)\,\Delta y \right)^p x^{\alpha-1}\,\Delta x \geq \frac{p}{\alpha} \int_\ell^\infty y^{p+\alpha-1} f^p(y)\, T(y)\,\Delta y, \tag{19}$$

where $T(y) = \alpha\,\beta_{\ell/y}(\alpha, p)$ and

$$\beta_\lambda(u,v) := \int_\lambda^1 s^{u-1}(1-s)^{v-1}\,\tilde{\Delta} s, \qquad 0 \leq \lambda < 1,$$

with $\beta_0(u,v) = \beta(u,v)$.

Hardy-type with negative parameters

As in [11]: let $a, b \in \mathbb{T}$, $0 \le a < b < \infty$, $p < 0$, $r > 1$, and $f, g \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}^+)$ with $x/g(x)$ nondecreasing on (a, b). Then

$$\int_a^b g^{-r}(x) F^p(x)\,\Delta x \le \left(\frac{p}{1-r}\right)^{p-1} \int_a^b x^{\frac{1+p-r}{p^*}} f^p(x) \left(1 - \left(\frac{x}{b}\right)^{\frac{1-r}{p}}\right)^{p-1} \left(\frac{x}{g(x)}\right)^r \left(\int_a^{\sigma(x)} t^{\frac{1-r}{p}-1}\,\Delta t\right) \tag{20}$$

where $F(x) = \int_x^b f(t)\,\Delta t$.

Dispersion inequality

Assumptions: $a, b \in \mathbb{T}$ with $b > a$, $p > 1$, $f \in C_{\mathrm{rd}}([a, b]_{\mathbb{T}}, \mathbb{R}^+)$ differentiable, nondecreasing, and $\lim_{t \to a}(t - a)f(t) = 0$.

Dispersion inequality

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$$\int_a^b \left(f(\sigma(x)) - \frac{1}{\sigma(x)-a} \int_a^{\sigma(x)} f(t)\,\Delta t \right)^p \Delta x \le \left(\frac{p}{p-1} \right)^p \int_a^b \left((\sigma(x)-a) f^{\Delta}(x) \right)^p \times \left[\left(\frac{\sigma(x)-a}{x-a} \right)^{\frac{p-1}{p}} - \left(\frac{\sigma(x)-a}{b-a} \right)^{\frac{p-1}{p}} \right] \Delta x. \quad (21)$$

Two-function form

If $a, b \in \mathbb{T}$, $b > a$, $p > 1$, and $f, g \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}^+)$ are differentiable, nondecreasing with $\lim_{t \to a}(t-a)f(t) = \lim_{t \to a}(t-a)g(t) = 0$, then

$$\begin{aligned}
&\int_a^b \left(f(\sigma(x)) - \frac{1}{\sigma(x)-a} \int_a^{\sigma(x)} f(t)\,\Delta t \right) \left(g(\sigma(x)) - \frac{1}{\sigma(x)-a} \int_a^{\sigma(x)} g(t)\,\Delta t \right) \Delta x \\
&\leq \frac{p^2}{p-1} \left(\int_a^b ((\sigma(x)-a) f^{\Delta}(x))^p \left[\left(\frac{\sigma(x)-a}{x-a} \right)^{\frac{p-1}{p}} - \left(\frac{\sigma(x)-a}{b-a} \right)^{\frac{p-1}{p}} \right] \Delta x \right)^{1/p} \\
&\quad \times \left(\int_a^b ((\sigma(x)-a) g^{\Delta}(x))^{\frac{p}{p-1}} \left[\left(\frac{\sigma(x)-a}{x-a} \right)^{\frac{1}{p}} - \left(\frac{\sigma(x)-a}{b-a} \right)^{\frac{1}{p}} \right] \Delta x \right)^{\frac{p-1}{p}}. \qquad (22)
\end{aligned}$$

Product version (bounded case)

If also $f(b), g(b) < \infty$, then

$$\begin{aligned}
&\int_a^b \left| f(\sigma(x))g(\sigma(x)) - \frac{1}{(\sigma(x)-a)^2}\left(\int_a^{\sigma(x)} f(t)\,\Delta t\right)\left(\int_a^{\sigma(x)} g(t)\,\Delta t\right)\right|^p \Delta x \\
&\leq 2^{p-1}\max\{f^p(b), g^p(b)\}\left(\frac{p}{p-1}\right)^p \int_a^b (\sigma(x)-a)^p\left[(f^\Delta(x))^p + (g^\Delta(x))^p\right] \\
&\qquad \times \left[\left(\frac{\sigma(x)-a}{x-a}\right)^{\frac{p-1}{p}} - \left(\frac{\sigma(x)-a}{b-a}\right)^{\frac{p-1}{p}}\right]\Delta x. \qquad (23)
\end{aligned}$$

Specializations of time scales

Time scale $\mathbb{T}$	$\sigma(t)$	Interpretation
$\mathbb{R}$	t	classical integral
$\mathbb{N}$	$t+1$	discrete sums/series
$q^{\mathbb{N}_0}$, $q > 1$	qt	q-calculus analogue

Already published works

- E. Awwad and **A. I. Saied**, Some weighted dynamic inequalities of Hardy type with kernels on time scales nabla calculus. Journal of Mathematical Inequalities, 18, 457-475, (2024).
- **A. I. Saied**, A study on reversed dynamic inequalities of Hilbert-type on time scales nabla calculus. Journal of Inequalities and Applications, 2024(1), 75, 1-16, (2024).
- M. Bohner, I. Jadlovská and **A. I. Saied**, Some new Hardy-type inequalities with negative parameters on time scales. Qualitative Theory of Dynamical Systems, 24(2). 72, 1-23, (2025).
- **A. I. Saied**, N. K. Al-Oushoush and M. Krnić, The impact of monotone functions on dynamic inequalities of Hardy type with a negative parameter on time scales. Filomat, (13)39, 4299-4324, (2025).

- **A. I. Saied**, I. Jadlovska and M. Krnić, Some new characterizations of weights for Hardy-type inequalities with kernels on time scales. Mathematica Slovaca (2025) **(accepted)**.
- **A. I. Saied** and Irena Jadlovská, Novel Dynamic inequalities of Hilbert-Pachpatte-type for a class of non-homogeneous kernels on time scales, Journal of inequalities and application **(accepted)**.
- **A. I. Saied** and Irena Jadlovská, A study on generalized dynamic inequalities of Hilbert-type on time scales, Journal of Mathematics and Computer Science **(accepted)**.
- **A. I. Saied** and Mario Krnic, Generalized dynamic inequalities similar to Hardy's inequality involving a convex function on time scales, Ricerche di Matematica **(submitted)**.

- Martin Bohner, Irena Jadlovská and **Ahmed I. Saied**, A new formulation of Hardy-type dynamic inequalities on time scales, AIMS Mathematics **(submitted)**.
- Masar Faseeh jabbar, Azhar Ali Abbas, Nizar Kh. Al-Oushoush, Mario Krnić and **Ahmed I. Saied**, Novel dynamic inequalities involving Steklov operator on time scales, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemática **(submitted)**.
- **Ahmed I. Saied** and Christophe Chesneau, Dynamic Hardy-Hilbert's inequality involving homogeneous absolute-max kernel functions on time scales, Functiones et Approximatio, Commentarii Mathematici **(submitted)**.
- Christophe Chesneau and **Ahmed I. Saied**, Improvement of Dynamic Hardy-Hilbert-type inequality involving the power, maximum, and absolute difference functions of the variables on time scales, Results in Mathematics **(submitted)**.

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Thank You