Hardy-type inequalities

Ahmed Ibrahim Mohammed Abo Saied

Mathematical Institute, Slovak Academy of Sciences, Grešákova 6, 040 01, Košice, Slovakia
Mathematical Department-Faculty of Science, Benha University, Benha, Egypt

05/02/2025

Contents

In 1920, Hardy [7] proved the discrete inequality

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a(i)\right)^{p}\leq\left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}a^{p}(n),\quad p>1, \tag{1}$$

where $a(n)\geq 0$ for $n\geq 1$, $a(n)\in l^{p}(\mathbb{N})$ (i.e. $\sum_{n=1}^{\infty}a^{p}(n)<\infty$).
In 1925, Hardy [8] proved the integral version of (1), by using the calculus of variations, which states that for $f\geq 0$ and integrable over any finite interval $(0,x)$, where $x\in(0,\infty)$ and $f\in L^{p}(0,\infty)$ and $p>1$, then

$$\int_{0}^{\infty}\left(\frac{1}{x}\int_{0}^{x}f(t)dt\right)^{p}dx\leq\left(\frac{p}{p-1}\right)^{p}\int_{0}^{\infty}f^{p}(x)dx. \tag{2}$$

The constant $(p/\,(p-1))^{p}$ in (1) and (2) is the best possible.

In 1927, Hardy and Littlewood [11] showed that the inequality (2) is reversed for $0 < p < 1$, provided that the integral $\int_0^x f(t)dt$ is replaced by $\int_x^\infty f(t)dt$ and the constant $(p/(p-1))^p$ is replaced by $(p/(1-p))^p$, then

$$\int_0^\infty \left(\frac{1}{x}\int_x^\infty f(t)dt\right)^p dx \geq \left(\frac{p}{1-p}\right)^p \int_0^\infty f^p(x)dx, \qquad (3)$$

where the constant $(p/(1-p))^p$ is the best possible value. We say that the constant $(p/(1-p))^p$ is best possible (sharp) if it can not be replaced by a smaller one without the affecting the validity of the inequality (3) for all possible functions.

The first weight version of the classical Hardy inequality (2) was proved by Hardy himself [9] and given by

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^\gamma dx \le \left(\frac{p}{p-1-\gamma}\right)^p \int_0^\infty x^\gamma f^p(x)dx, \quad (4)$$

provided that $p > 1, \gamma < p-1$, and

$$\int_0^\infty \left(\frac{1}{x}\int_x^\infty f(t)dt\right)^p x^\gamma dx \le \left(\frac{p}{1-(p-\gamma)}\right)^p \int_0^\infty x^\gamma f^p(x)dx, \quad (5)$$

provided that $p > 1, p < \gamma + 1$, where f is a measurable and nonnegative function on $(0, \infty)$.

In 1990, Ariňo and Muckenhoupt [1] generalized (4), (5) and characterized the weighted function w, such that the inequality

$$\int_0^\infty w(t)\left(\frac{1}{t}\int_0^t f(x)dx\right)^p dt \leq C\int_0^\infty w(t)f^p(t)dt, \tag{6}$$

holds for all nonnegative nonincreasing measurable functions f on $(0,\infty)$ with a constant $C>0$ independent of f (here $1\leq p<\infty$). The characterization reduces to the condition that the nonnegative function w satisfies

$$\int_t^\infty \frac{w(x)}{x^p}dx \leq \frac{B}{t^p}\int_0^t w(x)dx, \forall t\in(0,\infty) \text{ and } B>0. \tag{7}$$

For the discrete case, in 2006, Bennett and Gross-Erdmann [3] proved that the inequality

$$\sum_{n=1}^{\infty} w_n \left(\frac{1}{n} \sum_{k=1}^{n} g_k \right)^p \leq C \sum_{n=1}^{\infty} w_n g_n^p, \ 1 \leq p < \infty, \tag{8}$$

holds for all nonnegative nonincreasing sequences $(g_n)_{n\geq 1}$ and $C > 0$. The characterization reduces to the condition that the nonnegative sequence $(w_n)_{n\geq 1}$ satisfies

$$\sum_{k=n}^{\infty} \frac{w_k}{k^p} \leq \frac{B}{n^p} \sum_{k=1}^{n} w_k, \ \forall n \in \mathbb{N} \text{ and } B > 0. \tag{9}$$

In 2014, Gao [6] extended the results of Bennett and Gross-Erdmann [3] and characterized the weights such that the inequality

$$\sum_{n=1}^{\infty} \frac{w_n}{A_n^p} \left(\sum_{k=1}^{n} a_k g_k \right)^p \leq C \sum_{n=1}^{\infty} w_n g_n^p, \; p \geq 1, \tag{10}$$

holds for all nonnegative nonincreasing sequences $(g_n)_{n\geq 1}$ and $(a_n)_{n\geq 1}$ is a nonnegative and nonincreasing sequence with $a_1 > 0$ and the constant $C > 0$ is independent of a_n and g_n. The characterization reduces to the condition that the nonnegative sequences $(a_n)_{n\geq 1}$ and $(w_n)_{n\geq 1}$ satisfy

$$\sum_{k=n}^{\infty} \frac{w_k}{A_k^p} \leq \frac{B}{A_n^p} \sum_{k=1}^{n} w_k, \; \text{ for all } n \in \mathbb{N}, \text{ and } B > 0, \tag{11}$$

where $A_k = \sum_{s=1}^{k} a_s$.

In 1972, Muckenhoupt [12] generalized (6) and characterized the weights such that the inequality

$$\left(\int_0^\infty u\,(x)\left(\int_0^x f(t)dt\right)^p dx\right)^{1/p} \leq C\left(\int_0^\infty v\,(x) f^p(x)dx\right)^{1/p}, \tag{12}$$

holds for all measurable functions $f \geq 0$ and the constant $C > 0$ is independent of f (here $1 < p < \infty$). The characterization reduces to the condition that the nonnegative functions u and v satisfy

$$\sup_{x>0}\left(\int_x^\infty u(t)dt\right)^{1/p}\left(\int_0^x v^{\frac{-p^*}{p}}(t)dt\right)^{1/p^*} = K < \infty, \quad p^* = \frac{p}{p-1},$$

and $K \leq C \leq p^{1/p}(p^*)^{1/p^*}K$.

In 1978, Bradley [5] studied (12) in the different spaces $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$ when $1 \leq p \leq q \leq \infty$, and gave new characterizations of weights such that the general inequality

$$\left(\int_0^\infty u(x)\left(\int_0^x f(t)dt\right)^q dx\right)^{1/q} \leq C\left(\int_0^\infty v(x) f^p(x)dx\right)^{1/p}, \tag{13}$$

holds for all measurable functions $f \geq 0$ and the constant $C > 0$ is independent of f (here $1 \leq p \leq q \leq \infty$). The characterization reduces to the condition that the nonnegative functions u and v satisfy

$$A = \sup_{x>0}\left(\int_x^\infty u(t)dt\right)^{1/q}\left(\int_0^x v^{\frac{-p^*}{p}}(t)dt\right)^{1/p^*} < \infty,$$

and $A \leq C \leq p^{1/q}(p^*)^{1/p^*}A$, for $1 < p < q < \infty$ and $A = C$ if $p = 1$ and $q = \infty$.

In 1990, Opic and Kufner [13] proved that the inequality

$$\left(\int_a^b u(x)\left(\int_a^x f(t)dt\right)^q dx\right)^{1/q} \leq C\left(\int_a^b v(x) f^p(x)dx\right)^{1/p}, \tag{14}$$

holds for all nonnegative functions f and u, v are measurable positive functions in (a, b), $-\infty < a < b < \infty$ and $1 < p \leq q < \infty$, if and only if the following condition holds

$$B = \sup_{a<x<b}\left(\int_x^b u(t)dt\right)^{1/q}\left(\int_a^x v^{1-p^*}(t)dt\right)^{1/p^*} < \infty.$$

Moreover, the estimate for the constant $C > 0$ in (14) is given by

$$B \leq C \leq \left(1 + \frac{q}{p^*}\right)^{1/q}\left(1 + \frac{p^*}{q}\right)^{1/p^*} B.$$

In 1995, Heinig and Maligranda [10] proved that if $0 < p \leq 1 \leq q$ and u, v are positive functions, then there exists a constant $C > 0$ such that the inequality

$$\begin{aligned} &\left[\int_0^\infty u(x)\left(\int_0^\infty k_1(x,t)f(t)dt\right)^q dx\right]^{\frac{1}{q}} \\ \leq\ & C\left[\int_0^\infty v(x)\left(\int_0^\infty k_2(x,t)f(t)dt\right)^p dx\right]^{\frac{1}{p}}, \qquad (15) \end{aligned}$$

holds for all nonnegative nonincreasing functions f and k_1, $k_2 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, are positive kernels if and only if

$$\left[\int_0^\infty u(x)\left(\int_0^s k_1(x,t)dt\right)^q dx\right]^{\frac{1}{q}} \leq C\left[\int_0^\infty v(x)\left(\int_0^s k_2(x,t)dt\right)^p dx\right]^{\frac{1}{p}},$$

for all $s > 0$.

In 2000, Barza et al. [2] pointed that if $0 < p \leq q < \infty$, $1 \leq q < \infty$ and u, v are positive functions, then the inequality

$$\left[\int_0^\infty u(x) f^q(x)dx\right]^{\frac{1}{q}} \leq C\left[\int_0^\infty v(x)\left(\int_0^\infty k_2(x,y)f(y)dy\right)^p dx\right]^{\frac{1}{p}}, \tag{16}$$

holds for all nonnegative nonincreasing functions f and $k_2 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a positive kernel, where the constant $C > 0$ in (16) is given by

$$C = \sup_{t>0}\left[\int_0^t u(x)dx\right]^{\frac{1}{q}}\left[\int_0^\infty v(x)\left(\int_0^t k_2(x,y)dy\right)^p dx\right]^{\frac{-1}{p}} < \infty.$$

Time scales calculus

Definition ([4])

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.

For example, the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the nonnegative integers $\mathbb{N}_0$ and the quantum calculus $q^{\mathbb{N}_0}$ for $q > 1$.

Definition ([4])

Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ as

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

We note that $\sigma(t) \geq t$ for any $t \in \mathbb{T}$.

Definition ([4])

Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we define the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ as

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

We note that $\rho(t) \leq t$ for any $t \in \mathbb{T}$.

Definition ([4])

Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we define the graininess function $\mu : \mathbb{T} \to \mathbb{R}^+ = [0, \infty)$ as

$$\mu(t) = \sigma(t) - t.$$

Example ([4])

1. If $\mathbb{T} = \mathbb{R}$, then we have that $\sigma(t) = \rho(t) = t$ and $\mu(t) = 0$ for all $t \in \mathbb{T}$.
2. If $\mathbb{T} = \mathbb{N}$, then we have that $\sigma(t) = t + 1, \rho(t) = t - 1$ and $\mu(t) = 1$ for all $t \in \mathbb{T}$.
3. If $\mathbb{T} = h\mathbb{N}, h > 0$, then we have $\sigma(t) = t + h, \rho(t) = t - h$ and $\mu(t) = h$ for all $t \in \mathbb{T}$.
4. If $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, then we have $\sigma(t) = qt, \rho(t) = t/q$ and $\mu(t) = (q-1)t$ and for all $t \in \mathbb{T}$.

Classification of points

$t \in \mathbb{T}$ is called right-scattered if $t < \sigma(t)$
$t \in \mathbb{T}$ is called right dense if $t = \sigma(t)$
$t \in \mathbb{T}$ is called left-scattered if $t > \rho(t)$
$t \in \mathbb{T}$ is called left dense if $t = \rho(t)$
$t \in \mathbb{T}$ is called isolated if $\rho(t) < t < \sigma(t)$
$t \in \mathbb{T}$ is called dense if $\rho(t) = t = \sigma(t)$

Differentiation

Definition (The Delta Derivative [4])

Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}$. We define $f^{\Delta}(t)$ to be the number, provided it exists, as follows: for any $\varepsilon > 0$ there is a neighborhood U of t, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$, such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

We say $f^{\Delta}(t)$ the delta (or Hilger) derivative of f at t. We say that f is delta (or Hilger) differentiable on $\mathbb{T}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}$.

Example (Delta derivative [4])

1. If $\mathbb{T} = \mathbb{R}$, then for $f : \mathbb{R} \to \mathbb{R}$, we get

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f^{'}(t), \text{ for all } t \in \mathbb{T},$$

   where $f^{'}$ is the usual derivative.

2. If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then we have $\sigma(t) = t + h$, $\mu(t) = h$ and

$$f^{\Delta}(t) = \Delta_h f(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+h) - f(t)}{h}.$$

3. If $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, then we have $\sigma(t) = qt$, $\mu(t) = (q-1)t$ and

$$f^{\Delta}(t) = \Delta_q f(t) = \frac{f(q\,t) - f(t)}{(q-1)\,t}.$$

Theorem ([4])

Assume that $f, g : \mathbb{T} \to \mathbb{R}$ *are differentiable at* $t \in \mathbb{T}$*, then*

- *The sum* $f + g : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with*

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

- *For any constant* α*,* $\alpha f : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with*

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

- *The product* $fg : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

- *If* $g(t)g(\sigma(t)) \neq 0$*, then* f/g *is differentiable at* t *and*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Chain Rule

Definition ([4])

A function $f:\mathbb{R}\to\mathbb{R}$ is called *continuously differentiable if it is continuous and its* derivative is continuous.

Theorem (Chain Rule [4])

Assume $g:\mathbb{R}\to\mathbb{R}$ is continuous, $g:\mathbb{T}\to\mathbb{R}$ is delta differentiable on $\mathbb{T}^{\kappa}$, and $f:\mathbb{R}\to\mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[t,\sigma(t)]$ with

$$(f\circ g)^{\Delta}(t)=f'(g(c))\,g^{\Delta}(t). \tag{17}$$

where

$$\mathbb{T}^{k}=\begin{cases}\mathbb{T}\backslash(\rho(\sup\mathbb{T}),\,\sup\mathbb{T}], & \text{if } \sup\mathbb{T}<\infty,\\ \mathbb{T}, & \text{otherwise.}\end{cases}$$

Definition of rd-continuous functions

Definition ([4])

A function $f : \mathbb{T} \to \mathbb{R}$ is called $rd-$continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of $rd-$continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by C_{rd} or $C_{rd}(\mathbb{T})$ or $C_{rd}(\mathbb{T}, \mathbb{R})$.

Integration

Definition ([4])

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^{\Delta}(t) = f(t), \quad \text{holds for all} \quad t \in \mathbb{T}.$$

In this case, the Cauchy integral of f is defined by

$$\int_r^s f(t)\Delta t = F(s) - F(r), \quad \text{for all} \;\; r,\, s \in \mathbb{T}.$$

Theorem ([4])

Every rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then

$$\left(\int_{t_0}^t f(\tau)\Delta\tau\right)^{\Delta} = f(t), \quad \textit{for} \quad t \in \mathbb{T}.$$

The interval $[a, b]_{\mathbb{T}}$ is defined by $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$.

Theorem ([4])

Let $a, b \in \mathbb{T}$ and $f \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$.

- *If $\mathbb{T} = \mathbb{R}$, then*
$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt.$$
- *If $\mathbb{T} = \mathbb{N}$, $\sigma(t) = t + 1$ and $\mu(t) = 1$, then*
$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)} \mu(t)f(t) = \sum_{t=a}^{b-1} f(t).$$
- *If $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, $\sigma(t) = qt$, $\rho(t) = t/q$ and $\mu(t) = (q-1)\,t$, then*
$$\int_a^b f(t)\Delta t = \sum_{t=a}^{b/q} (q-1)\,tf(t)$$

Lemma (Integration by Parts [4])

If $a, b \in \mathbb{T}$ *and* $u, v \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$, *then*

$$\int_a^b u(t)v^{\Delta}(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^{\Delta}(t)v(\sigma(t))\Delta t. \tag{18}$$

Lemma (Hölder's Inequality [4])

If $a, b \in \mathbb{T}$ *and* $f, g \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$, *then*

$$\int_a^b |f(t)g(t)|\Delta t \leq \left[\int_a^b |f(t)|^{\gamma}\Delta t\right]^{\frac{1}{\gamma}} \left[\int_a^b |g(t)|^{\nu}\Delta t\right]^{\frac{1}{\nu}}, \tag{19}$$

where $\gamma, \nu > 1$ *such that* $1/\gamma + 1/\nu = 1$. *The inequality (19) is reversed for* $0 < \gamma < 1$ *or* $\gamma < 0$.

In 2005, Rehak [14] established the time scale version of Hardy-type inequalities (1) and (2).

Theorem

Assume that $\mathbb{T}$ *is a time scale with* $a \in \mathbb{T}$, $1 < \alpha < \infty$ *and* f *is a nonnegative function such that* $\int_a^\infty f^\alpha(t)\Delta t$ *exists as a finite number. Then*

$$\int_a^\infty \left(\frac{F^\sigma(t)}{\sigma(t)-a}\right)^\alpha \Delta t \leq \left(\frac{\alpha}{\alpha-1}\right)^\alpha \int_a^\infty f^\alpha(t)\Delta t, \tag{20}$$

where $F(t) = \int_a^t f(x)\Delta x$. *In addition, if* $\mu(t)/t \to 0$ *as* $t \to \infty$, *then the constant* $(\alpha/(\alpha-1))^\alpha$ *is the best possible.*

The discrete inequality is given by

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a(i)\right)^{p} \le \left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}a^{p}(n), \quad p>1, \tag{21}$$

where $a(n) \geq 0$ for $n \geq 1$, $a(n) \in l^p(\mathbb{N})$ (i.e. $\sum_{n=1}^{\infty} a^p(n) < \infty$). The continuous version states that for $f \geq 0$ and integrable over any finite interval $(0, x)$, where $x \in (0, \infty)$ and $f \in L^p(0, \infty)$ and $p > 1$, then

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^{x}f(t)dt\right)^{p}dx \le \left(\frac{p}{p-1}\right)^{p}\int_0^{\infty}f^{p}(x)dx. \tag{22}$$

The constant $(p/\,(p-1))^p$ in (21) and (22) is the best possible.

Proof. Denote $\varphi(t) = F(t)/(t-a)$. Then for $t \geq a$,

$$\begin{aligned}
&[\varphi^{\sigma}(t)]^{\alpha} - \frac{\alpha}{\alpha-1}[\varphi^{\sigma}(t)]^{\alpha-1} f(t) \\
&= [\varphi^{\sigma}(t)]^{\alpha} - \frac{\alpha}{\alpha-1}[\varphi^{\sigma}(t)]^{\alpha-1}[(t-a)\varphi(t)]^{\Delta} \\
&= [\varphi^{\sigma}(t)]^{\alpha} - \frac{\alpha}{\alpha-1}[\varphi^{\sigma}(t)]^{\alpha-1}(t-a)\varphi^{\Delta}(t) - \frac{\alpha}{\alpha-1}[\varphi^{\sigma}(t)]^{\alpha} \\
&= \frac{-1}{\alpha-1}[\varphi^{\sigma}(t)]^{\alpha} - \frac{\alpha}{\alpha-1}[\varphi^{\sigma}(t)]^{\alpha-1}(t-a)\varphi^{\Delta}(t). \qquad (23)
\end{aligned}$$

Applying the chain rule formula (17), we have that

$$[\varphi^{\alpha}(t)]^{\Delta} = \alpha\varphi^{\alpha-1}(c)\varphi^{\Delta}(t) \text{ for some } c \in [t, \sigma(t)].$$

Since $\varphi^{\Delta}(t) > 0$ and $c \leq \sigma(t)$, then φ is increasing function and $\varphi(c) \leq \varphi^{\sigma}(t)$, thus

$$[\varphi^{\alpha}(t)]^{\Delta} \leq \alpha[\varphi^{\sigma}(t)]^{\alpha-1}\varphi^{\Delta}(t),$$

and then (23) gives

$$\begin{aligned}
&[\varphi^{\sigma}(t)]^{\alpha} - \frac{\alpha}{\alpha-1}[\varphi^{\sigma}(t)]^{\alpha-1} f(t) \\
&\leq \frac{-1}{\alpha-1}[\varphi^{\sigma}(t)]^{\alpha} - \frac{1}{\alpha-1}[\varphi^{\alpha}(t)]^{\Delta}(t-a) \\
&= \frac{-1}{\alpha-1}[(t-a)\varphi^{\alpha}(t)]^{\Delta}. \qquad (24)
\end{aligned}$$

Integrating the last inequality over t from a to s, we get

$$\begin{aligned}
&\int_a^s [\varphi^{\sigma}(t)]^{\alpha}\Delta t - \frac{\alpha}{\alpha-1}\int_a^s [\varphi^{\sigma}(t)]^{\alpha-1} f(t)\Delta t \\
\leq\ & \frac{-1}{\alpha-1}\int_a^s [(t-a)\varphi^{\alpha}(t)]^{\Delta}\Delta t \\
=\ & \frac{-1}{\alpha-1}(s-a)\varphi^{\alpha}(s) \leq 0,
\end{aligned}$$

so

$$\int_a^s [\varphi^{\sigma}(t)]^{\alpha}\Delta t \leq \frac{\alpha}{\alpha-1}\int_a^s [\varphi^{\sigma}(t)]^{\alpha-1} f(t)\Delta t. \qquad (25)$$

Applying Hölder's inequality on the right hand side of (25) with indices $\alpha > 1$ and $\alpha/(\alpha-1)$, we obtain

$$\int_a^s [\varphi^\sigma(t)]^\alpha \Delta t \le \frac{\alpha}{\alpha-1} \left(\int_a^s f^\alpha(t) \Delta t \right)^{\frac{1}{\alpha}} \left(\int_a^s [\varphi^\sigma(t)]^\alpha \Delta t \right)^{\frac{\alpha-1}{\alpha}},$$

then

$$\int_a^s [\varphi^\sigma(t)]^\alpha \Delta t \le \left(\frac{\alpha}{\alpha-1} \right)^\alpha \int_a^s f^\alpha(t) \Delta t$$

i.e.

$$\int_a^s \left(\frac{F^\sigma(t)}{\sigma(t)-a} \right)^\alpha \Delta t \le \left(\frac{\alpha}{\alpha-1} \right)^\alpha \int_a^s f^\alpha(t) \Delta t,$$

when $s \to \infty$, we can get the dynamic Hardy inequality (20).

To prove that $(\alpha/(\alpha-1))^{\alpha}$ is the best possible, we assume that

$$f(t)=\begin{cases} 0 \ \text{ for } t\in[a,\acute{a}) \\ (t-a)^{-1/\alpha} \ \text{ for } t\in[\acute{a},b] \\ 0 \ \text{ for } t\in(b,\infty), \end{cases}$$

where $a<\acute{a}<b$. Then $\int_a^{\infty} f^{\alpha}(t)\Delta t=\int_{\acute{a}}^{\sigma(b)}(\Delta t/(t-a))$ and

$$\begin{aligned} F^{\sigma}(t) &= \int_a^{\sigma(t)} f(s)\Delta s=\int_{\acute{a}}^{\sigma(t)}\frac{\Delta s}{(s-a)^{1/\alpha}} \\ &\geq \int_{\acute{a}}^{t}\frac{ds}{(s-a)^{1/\alpha}}=\frac{\alpha}{\alpha-1}\left[(t-a)^{\frac{\alpha-1}{\alpha}}-(\acute{a}-a)^{\frac{\alpha-1}{\alpha}}\right], \end{aligned}$$

for $t\in[\acute{a},b]$. Hence

$$\frac{F^{\sigma}(t)}{t-a}\geq\frac{\alpha}{\alpha-1}\frac{1-((\acute{a}-a)\ /\ (t-a))^{(\alpha-1)/\alpha}}{(t-a)^{1/\alpha}},$$

which implies

$$\left(\frac{F^{\sigma}(t)}{t-a}\right)^{\alpha} \geq \left(\frac{\alpha}{\alpha-1}\right)^{\alpha} \frac{1-\varepsilon_t}{t-a},$$

for $t \in [\acute{a}, b]$, where $\varepsilon_t \longrightarrow 0$ as $t \longrightarrow \infty$.
Consequently,

$$\begin{aligned} & \int_a^{\infty} \left(\frac{F^{\sigma}(t)}{\sigma(t)-a}\right)^{\alpha} \Delta t \\ = & \int_{\acute{a}}^{\sigma(b)} \left(\frac{F^{\sigma}(t)}{t+\mu(t)-a}\right)^{\alpha} \Delta t \\ = & \int_{\acute{a}}^{\sigma(b)} \left(\frac{F^{\sigma}(t)}{t-a}\right)^{\alpha} \left(\frac{t-a}{t+\mu(t)-a}\right)^{\alpha} \Delta t \\ \geq & \left(\frac{\alpha}{\alpha-1}\right)^{\alpha} (1-\delta_b) \int_a^{\infty} f^{\alpha}(t)\, \Delta t, \end{aligned}$$

where $\delta_b \longrightarrow 0$ as $b \longrightarrow \infty$.

Hence any inequality of the type

$$\int_a^\infty \left(\frac{F^\sigma(t)}{\sigma(t)-a}\right)^\alpha \Delta t < \left(\frac{\alpha}{\alpha-1}\right)^\alpha (1-\varepsilon) \int_a^\infty f^\alpha(t)\,\Delta t,$$

with $\varepsilon > 0$, fails to hold if f is chosen as above and b is sufficiently large.

Application to Hardy's inequality

Theorem ([15])

Let $\mathbb{T}$ *be a time scale with* $a, b \in \mathbb{T}$, $1 < p \leq q < \infty$, $f \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ *and let* u, v *be positive rd-continuous functions on* $(a,b)_{\mathbb{T}}$. *Then*

$$\left(\int_a^b u(x)\left(\int_a^{\sigma(x)} f(t)\Delta t\right)^q \Delta x\right)^{1/q} \leq C\left(\int_a^b v(x) f^p(x)\Delta x\right)^{1/p}, \tag{26}$$

holds if and only if

$$B = \sup_{a<x<b}\left(\int_x^b u(t)\Delta t\right)^{1/q}\left(\int_a^{\sigma(x)} v^{1-p^*}(t)\Delta t\right)^{1/p^*} < \infty,$$

where $p^* = p/(p-1)$.

Moreover, the estimate for the constant $C > 0$ in (26) is given by

$$B \leq C \leq \left(1 + \frac{q}{p^*}\right)^{1/q} \left(1 + \frac{p^*}{q}\right)^{1/p^*} B.$$

The authors employed the weighted Hardy inequality (26) to the following equation in order to examine its oscillatory properties

$$\left(r(t)\varphi_\alpha(x^\Delta)\right)^\Delta + s(t)\varphi_\alpha(x^\sigma) = 0, \tag{27}$$

where $1 < \alpha < \infty$, $\varphi_\alpha(y) = |y|^{\alpha-2} y$ and $r, s \in C_{rd}\left([a,b]_{\mathbb{T}}, \mathbb{R}\right)$ with $r(t) \neq 0$ for $t \in [a,b]_{\mathbb{T}}$.
They showed that the equation (27) is nonoscillatory if and only if the weighted Hardy inequality (26) holds for $p = q = \alpha$ and the constant $C = 1$.

Aims of PhD-Thesis

Hölder's inequality

- Generalized dynamic Hölder's inequality on time scales
- Reversed dynamic Hölder's inequality using Specht's ratio on time scales

Minkowski's inequality

- Generalized dynamic Minkowski's inequality on time scales
- Reversed dynamic Minkowski's inequality using Specht's ratio on time scales

Hilbert-type inequality

- Dynamic inequalities of Hilbert-type on time scales nabla calculus
- Reverse dynamic Hilbert-type inequalities using the mean inequality
- Novel Dynamic inequalities of Hilbert-Pachpatte-type for a class of non-homogeneous kernels on time scales
- Recent generalized inequalities of Hilbert-type for a class of homogeneous kernels on time scales

Hardy-type inequality

- Some new generalizations of inequalities involving Hardy-type operator on time scales.
- Dynamic Hardy-type inequalities involving a single negative parameter on time scales.
- Generalized dynamic inequalities similar to Hardy's inequality involving a convex function
- New formulation of dynamic Hardy-type inequalities on time scales such that the inequality holds when the parameter $p = 1$
- Characterizations of the weighted functions for Hardy's inequality with general kernel on time scales

- Generalized Hardy-type inequalities through conformable fractional time scale calculus.
- New properties of weighted Muckenhoupt and Gehring classes on time scales such that the characterization of weight for Muckenhoupt class represents the validity of condition of Hardy's inequality and the characterization of weight for Gehring class represents reverse Hölder's inequality.
- Weighted Lorentz spaces and equivalent relations between ℓ_p-classes by using Hardy operator.

References

1— M. A. Ariňo and B. Muckenhoupt, Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions, Trans. Amer. Math. Soc. 320 (1990), 727-735.

2— S. Barza, L. E. Persson and J. Soria, Sharp weighted multidimensional integral inequalities for monotone functions. Mathematische Nachrichten, 210(1) (2000), 43-58.

3— G. Bennett and K-G. Erdmann, Weighted Hardy inequalities for decreasing sequences and functions, Mathematische Annalen 334(3) (2006), 489-531.

4— M. Bohner and A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Boston, Mass, USA, 2001.

5— J. S. Bradley, Hardy inequalities with mixed norms, Canad. Math. Bull. 21 (1978), 405-408.

6— P. Gao, On weighted Hardy inequalities for nonincreasing sequences, arXiv preprint arXiv:1401.7156 (2014).

7− G. H. Hardy, Notes on a theorem of Hilbert, Math. Z. 6 (1920), 314-317.
8− G. H. Hardy, Notes on some points in the integral calculus, LX. An inequality between integrals, Mess. Math. 54 (1925), 150-156.
9− G. H. Hardy, Notes on some points in the integral calculus (LXIV), Mess. Math. 57 (1928), 12-16.
10− H. Heinig and L. Maligranda, Weighted inequalities for monotone and concave functions, Studia Math. 116 (1995), 133-165.
11− J. E. Littlewood and G. H. Hardy, Elementary theorems concerning power series with positive coefficients and moment constants of positive functions, Walter de Gruyter, Berlin/New York Berlin, New York, (1927).
12− B. Muckenhoupt, Hardy's inequality with weights, Sudia Math. 44 (1972), 31-38.
13− B. Opic and A. Kufner, *Hardy-Type Inequalities*, Pitman Research Notes in Mathematics Series, Longman Scientific and Technical, Harlow, 1990.

14− P. Řehák, Hardy inequality on time scales and its application to half-linear dynamic equations. Journal of Inequalities and Applications, 2005, 1-13, (2005).
15− S. H. Saker, R. R. Mahmoud and A. Peterson, Weighted Hardy-type inequalities on time scales with applications, Mediterr. J. Math. 13(2) (2016), 585-606.
16− S. H. Saker, J. Alzabut, A. I. Saied and D. O'Regan, New characterizations of weights on dynamic inequalities involving a Hardy operator. Journal of Inequalities and Applications, 2021(1), 73, (2021).
17− G. J. Sinnamon, Weighted Hardy and Opial-type inequalities, J. Math. Anal. Appl. 160 (1991), 434-445.

Thanks for your attention!