Extended finite automata and decision problems for matrix semigroups

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Introduction

Aim: Make a connection between extended finite automata over matrix semigroups and decision problems for matrix semigroups

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Let M be a monoid. M-automaton is defined analogously.

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- If S is not a monoid nor a group, then only the empty string can be accepted.

Let S be a matrix semigroup finitely generated by a generating set of square matrices F. The **membership problem** *is to decide whether or not a given matrix Y belongs to the matrix semigroup S.*

Background
Membership problem for matrix semigroups

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Given: $F = \{Y_1, Y_2, \ldots, Y_n\}$ *and a matrix* Y
Problem: *Determine if there exist an integer* $k \geq 1$ *and* $i_1, i_2, \ldots, i_k \in \{1, \ldots, n\}$ *such that* $Y_{i_1} Y_{i_2} \cdots Y_{i_k} = Y$.

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The subgroup membership problem can be seen as a special case of the (semigroup) membership problem.

Previous work

	Identity problem	**Membership problem**
$\mathbb{Z}^{2\times 2}$	decidable	decidable
H	decidable	?
$SL(3,\mathbb{Z})$	?	?
$\mathbb{Z}^{3\times 3}$	?	undecidable
$SL(4,\mathbb{Z})$	undecidable	undecidable

The emptiness problem and subgroup membership problem

Theorem

Let H be a finitely generated subgroup of G. If the emptiness problem for G-automata is decidable, then the subgroup membership problem for H in G is decidable.

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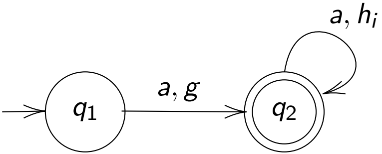
- We are going to show that $g \in H$ iff $L(V)$ is nonempty.

The emptiness problem and subgroup membership problem



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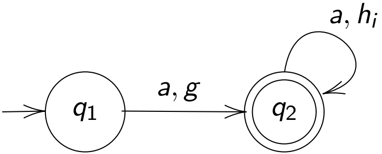

Proof.

- $\{h_1, \ldots, h_n\}$ generates H

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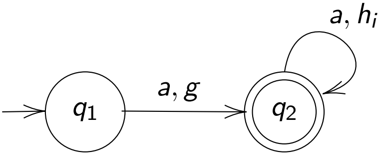

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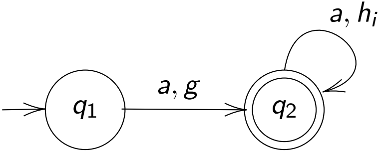

Proof.

- $g \in H \implies g^{-1} \in H$

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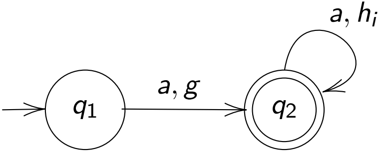

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- $g \in H \implies g^{-1} \in H$
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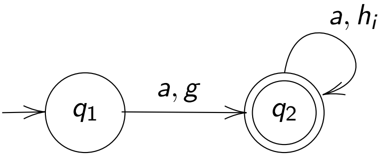

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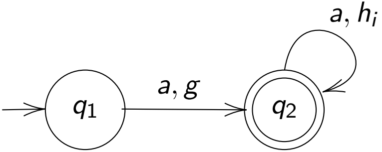

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- $gg^{-1} = 1 \implies a^{k+1} \in L(V)$
- $L(V)$ is nonempty

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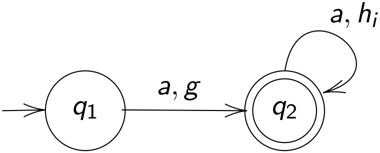

Proof.

- Suppose $L(V)$ is nonempty

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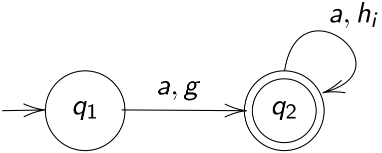

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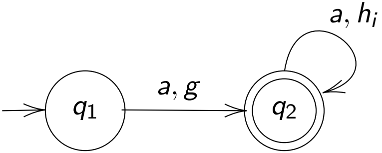

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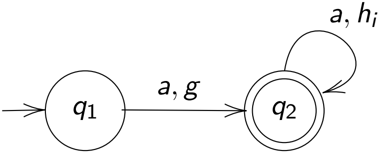

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$\implies$ subgroup membership problem for H is decidable .

Decidability of the subgroup membership problem for $\mathbb{Z}^{2\times 2}$

Theorem

Given a matrix Y *from* $\mathbb{Z}^{2\times 2}$ *and a subgroup* H *of* $\mathbb{Z}^{2\times 2}$*, it is decidable whether* Y *belongs to* H.

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- V is a $GL(2,\mathbb{Z})$-automaton

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Lemma

Let G be a finitely generated group and let H be a subgroup of finite index. Any G-automaton can be converted into an H-automaton recognizing the same language.

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Any $\mathbf{F}_2$-automaton can be converted into a pushdown automaton recognizing the same language.

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Theorem

Given a matrix Y from $\mathbb{Z}^{2\times 2}$ and a subgroup H of $\mathbb{Z}^{2\times 2}$, it is decidable whether Y belongs to H.

Proof.

- A pushdown automaton recognizing $L(V)$ can be constructed
  - $\mathbf{F}_2$ has finite index in $GL(2, \mathbb{Z})$
  - $\mathbf{F}_2$-automaton recognizing $L(V)$ can be constructed
  - $\mathbf{F}_2$-automaton can be converted to a pushdown automaton
- Emptiness problem for pda is decidable $\implies$ Emptiness problem for $\mathbb{Z}^{2\times 2}$-automata is decidable

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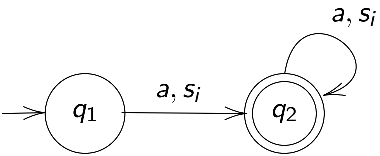

Undecidability of the emptiness problem for $\mathbb{Z}^{4\times 4}$-automata

Fact

Given a semigroup S generated by eight 4×4 integer matrices, determining whether the identity matrix belongs to S is undecidable. [KNP17]

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Corollary

Let S be a subsemigroup of $\mathbb{Z}^{4\times 4}$ generated by eight matrices. The emptiness problem for S-automaton is undecidable.

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Proof.

We know that the identity problem for S is undecidable. By the above theorem the result follows.

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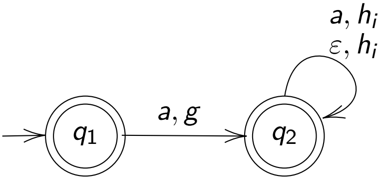
- Construct G-automaton V such that $g \in H$ iff $L(V) = \Sigma^*$.



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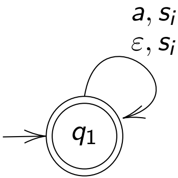

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  - Universe problem for pushdown automata is undecidable
- $\mathbf{F}_2$ is a subgroup of $SL(2, \mathbb{Z})$ and the membership and identity problems for $SL(2, \mathbb{Z})$ are decidable

Conclusion

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  - Undecidability of the emptiness problem for $\mathbb{Z}^{4\times 4}$-automata
- Universe problem for S-automata

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We make a connection between the decidability of the subgroup membership and identity problems and the universe and emptiness problems for extended finite automata

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Identity and membership problems for 3×3 integer matrix groups are open.

Thank You!