

Gateaux differentiability of convex locally Lipschitz functions defined on locally convex spaces

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The next well known theorem is quoted from Phelps' book [P89]¹

Theorem 1.20 (Mazur, 1933)

If E is a separable Banach space and f is a continuous convex function defined on a convex open subset D of E , then the set of points x where $df(x)$ exists is a dense G_δ set in D .

Therefore, separable Banach spaces are weak Asplund spaces.

Rainwater (in [R88]²) says: "In an interesting new approach, Verona [12] recently generalized Mazur's theorem to sets C which may have empty interior. She did so by substituting the set $N(C)$ of nonsupport points of C for the interior of C ."

¹[P89] RR Phelps: *Convex functions, monotone operators and differentiability*, Springer-Verlag, Berlin, 1989, 1993.

²[R88] J. Rainwater: *Yet more on the differentiability of convex functions*. Proc. Amer. Math. Soc., 103 (1988), 773–778.

Verona's result (see [V88]³) is

THEOREM 2

Let X be a separable Banach space, C be a closed, convex subset of X and $f : C \rightarrow \mathbb{R}$ be convex and locally Lipschitz on $NS(C)$. Then f is smooth on a residual subset of $NS(C)$.

Above, f is smooth at $x \in C$ if $\partial f(x)$ is a singleton; notice that Rainwater (in [R88]) says that f is Gateaux differentiable at x in this case.

In fact, [V88, Th. 2] does not extend completely Mazur's theorem. Take, for example, $f :]-\tilde{1}, 1[\rightarrow \mathbb{R}$, $f(x) := 1/(1 - x^2)$ which has not a convex extension $\tilde{f} : [-1, 1] \rightarrow \mathbb{R}$.

Ergin & Sarver obtained the next generalization of Verona's theorem in [ES10]⁴.

³[V88] ME Verona: *More on the differentiability of convex functions*. Proc. Amer. Math. Soc., 103 (1988), 137–140.

⁴[ES10] H Ergin, T Sarver: *The unique minimal dual representation of a convex function*, J. Math. Anal. Appl., 370 (2010), 600–606

Theorem 2.4

Suppose X , C , and f satisfy (i)–(iii) in Theorem 1.2. Then, the set of points x where $\partial f(x)$ is a singleton is a dense G_δ set (in the relative topology) in C .

Theorem 1.2

Suppose (i) X is a separable Banach space, (ii) C is a convex subset of X that is a Baire space (when endowed with the relative topology) such that $\text{aff}(C)$ is dense in X , and (iii) $f : C \rightarrow \mathbb{R}$ is locally Lipschitz continuous and convex. Then, for any weak*-closed $\mathcal{M} \subset X^*$, the following are equivalent:

(1) $\mathcal{M}_f \subset \mathcal{M}$. (2) For all $x \in C$, the maximization problem $\max_{x^* \in \mathcal{M}} [\langle x, x^* \rangle - f^*(x^*)]$ has a solution and the maximum value is equal to $f(x)$.

Above, $\mathcal{M}_f = \text{cl}_{w^*} \mathcal{N}_f$, where $\mathcal{N}_f = \{x^* \in X^* : x^* \in \partial f(x) \text{ for some } x \in C_f\}$ with $C_f = \{x \in C : \partial f(x) \text{ is a singleton}\}$.

It is our aim to extend these results for convex locally Lipschitz functions defined on locally convex spaces with some additional properties. Our main results are the following:

Proposition M

Let X be a separable H.l.c.s. and $f \in \Lambda(X)$ be such that $C := \text{qi}(\text{dom } f) \neq \emptyset$ and $f|_C$ is locally Lipschitz, where C is endowed with its relative topology. Then there exists a family $(O_n)_{n \geq 1}$ of dense open subsets of C such that $C_f = \bigcap_{n \geq 1} O_n$. Moreover, if X is a Fréchet space and $\text{dom } f$ is either closed or open, then C_f is a dense G_δ subset of C .

Proposition M extends [V88, Th. 2] and provides an extension of Mazur's theorem to Fréchet spaces.

Proposition E-S1

Let X be a separable and metrizable l.c.s., and let $f \in \Lambda(X)$ be such that $f|_C$ is locally Lipschitz, where $C := \text{dom } f$ is endowed with its relative topology. Then there exists a family $(O_n)_{n \geq 1}$ of dense open subsets of C such that $C_f = \bigcap_{n \geq 1} O_n$. If, moreover, C is a Baire space, then C_f is a dense G_δ subset of C .

Proposition E-S2

Let X be a separable and metrizable l.c.s., and let $f \in \Lambda(X)$ be such that $f|_C$ is locally Lipschitz, where $C := \text{dom } f$. Assume that C (endowed with the relative topology) is a Baire space and $\mathcal{M} \subset X^*$ is a w^* -closed set; then $\mathcal{M}_f := \text{cl}_{w^*} \partial f(C_f) \in \mathcal{S}_f$ and $\mathcal{M} \in \mathcal{S}_f$ if and only if $\mathcal{M}_f \subset \mathcal{M}$, where $\mathcal{S}_f := \{\mathcal{M} \subset X^* \mid \mathcal{M} = \text{cl}_{w^*} \mathcal{M} \text{ and } \partial f(x) \cap \mathcal{M} \neq \emptyset \text{ for all } x \in \text{dom } \partial f\}$.

Propositions E-S1, E-S2 provide extensions of [ES10, Ths. 2.4, 2.1] to separable and metrizable l.c.s., respectively. 

Preliminary notions and results

- X is a nontrivial real Hausdorff locally convex space (H.l.c.s.), X^* is its topological dual endowed with its weak* topology $w^* := \sigma(X^*, X)$; one has $(X^*, w^*)^* = X$;
- $\langle x, x^* \rangle := x^*(x)$ for $x \in X$ and $x^* \in X^*$;
- $\text{cl } A$, $\text{int } A$ are the *closure* and *interior* of $A (\subset X)$;
- $\text{span } A$, $\text{aff } A$, $\text{conv } A$ are the *linear*, *affine* and *convex hulls* of A , respectively;
- $\overline{\text{span}} A := \text{cl}(\text{span } A)$, $\overline{\text{aff}} A := \text{cl}(\text{aff } A)$, $\overline{\text{conv}} A := \text{cl}(\text{conv } A)$;
- $\text{lin}_0 A := \text{aff}(A - a) = \text{span}(A - a)$ for some (any) $a \in A$, the *linear (sub)space parallel* to $\text{aff } A$; $\overline{\text{lin}}_0 A := \text{cl}(\text{lin}_0 A)$;
- $A + \emptyset := \emptyset + A := \emptyset$ for $A \subset E$ with E a vector space E ;
- $\mathbb{R}_+ := [0, \infty[$, $\mathbb{P} :=]0, \infty[$, $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ with $\infty := +\infty$;

- $\text{qi } C := \{x \in C \mid \text{cl}(\mathbb{R}_+(C - x)) = X\}$, the *quasi interior* of the **convex** set $C \subset X$;
- $\text{qri } C := \{x \in C \mid \text{cl}(\mathbb{R}_+(C - x)) \text{ is a linear space}\}$, the *quasi relative interior* of C ;
- $\text{qi } C = \text{qri } C$ if $\overline{\text{aff } C} = X$, $\text{qi } C = \emptyset$ if $\overline{\text{aff } C} \neq X$;
- $x \in \text{qri } C$, $y \in C$, $\lambda \in [0, 1[\Rightarrow (1 - \lambda)x + \lambda y \in \text{qri } C$;
- $S(C) := C \setminus \text{qi } C$ is the set of *support points* of C ;
- $N(C) := C \setminus S(C)$ ($= \text{qi } C$) is the set of *nonsupport points* of C ;
- $\mathcal{N}_0 := \mathcal{N}_0^X$ the class of all *symmetric convex neighborhoods* of $0 \in X$; $\mathcal{N}_x := \{x + V \mid V \in \mathcal{N}_0\}$ for $x \in X$;
- $N_A(x) := \{x^* \in X^* \mid \langle x' - x, x^* \rangle \leq 0 \quad \forall x' \in A\}$, the *normal cone* of $A \subset X$ at $x \in A$;
- $A^\circ := \{x^* \in X^* \mid \langle x, x^* \rangle \leq 1 \quad \forall x \in A\}$, the *polar* of $\emptyset \neq A \subset X$;
- $V^\circ (= -V^\circ)$ for $V \in \mathcal{N}_0$; V° is w^* -compact for every $V \in \mathcal{N}_0$ (by Alaoglu-Bourbaki theorem);
- $x \in \text{qi } C \Leftrightarrow N_C(x) = \{0\}$, $x \in \text{qri } C \Leftrightarrow N_C(x) = -N_C(x)$.

Consider the function $f : X \rightarrow \overline{\mathbb{R}}$:

- $\text{dom } f := \{x \in X \mid f(x) < \infty\}$ is the *domain* of f ; f is *proper* if $\text{dom } f \neq \emptyset$ and $f(x) \neq -\infty$ for $x \in X$;
- $\text{epi } f := \{(x, t) \in X \times \mathbb{R} \mid f(x) \leq t\}$ is the *epigraph* of f ; f is *convex* if $\text{epi } f$ is convex;
- $f^* : X^* \rightarrow \overline{\mathbb{R}}$, $f^*(x^*) := \sup_{x \in \text{dom } f} [\langle x, x^* \rangle - f(x)]$ is the *conjugate* of f , where $\sup \emptyset := -\infty$;
- $\partial f(\bar{x}) := \{x^* \in X^* \mid \langle x - \bar{x}, x^* \rangle \leq f(x) - f(\bar{x}) \quad \forall x \in X\}$ if $f(\bar{x}) \in \mathbb{R}$, $\partial f(\bar{x}) := \emptyset$ otherwise, is the *subdifferential* of f at $\bar{x}$;
- $x^* \in \partial f(x) \Leftrightarrow f(x) = \max_{x^* \in X^*} [\langle x, x^* \rangle - f^*(x^*)] \in \mathbb{R}$;
- $\iota_A : E \rightarrow \overline{\mathbb{R}}$ defined by $\iota_A(x) := 0$ if $x \in A$, $\iota_A(x) := \infty$ if $x \in E \setminus A$, is the *indicator function* of $A \subset E$;
- $N_A(\bar{x}) = \partial \iota_A(\bar{x})$ for $\bar{x} \in A \subset X$;
- $\partial f(x) = \partial f(x) + N_{\text{dom } f}(x) \quad \forall x \in \text{dom } f$;

- $\Lambda(X)$ is the class of *proper convex functions* on X ;
- $f'_+(\bar{x}, \cdot) : X \rightarrow \overline{\mathbb{R}}$ is the *directional derivative* of $f \in \Lambda(X)$ at $\bar{x} \in \text{dom } f$, where
 - $f'_+(\bar{x}, u) := \lim_{t \downarrow 0} t^{-1} [f(\bar{x} + tu) - f(\bar{x})]$;
 - $f'_+(\bar{x}, 0) = 0$, $f'_+(\bar{x}, tu) = tf'_+(\bar{x}, u) \quad \forall u \in X, t \in \mathbb{P}$;
 - $f'_+(\bar{x}, u) = \inf_{t \in]0, \delta[} t^{-1} [f(\bar{x} + tu) - f(\bar{x})] \quad \forall \delta \in]0, \infty], u \in X$
because the mapping $0 < t \mapsto t^{-1} [f(\bar{x} + tu) - f(\bar{x})] \in \mathbb{R}_\infty$ is nondecreasing;
 - $\partial f(\bar{x}) = \partial f'_+(\bar{x}, \cdot)(0) = \overline{\partial f'_+(0, \cdot)}(0)$, where $\overline{\varphi}$ is the *lower semicontinuous envelope* of $\varphi : X \rightarrow \overline{\mathbb{R}}$ (that is, $\text{epi } \overline{\varphi} = \text{cl}(\text{epi } \varphi)$);
 - $\text{dom } f'_+(\bar{x}, \cdot) = \mathbb{R}_+(\text{dom } f - \bar{x})$.

Proposition 1

Let X be an H.l.c.s., $f \in \Lambda(X)$ and $\bar{x} \in \text{dom } f$.

(i) If $V \in \mathcal{N}_{\bar{x}}$, then $f'_+(\bar{x}, \cdot) = (f + \iota_V)'_+(\bar{x}, \cdot)$, and so $\partial f(\bar{x}) = \partial f|_{V_{\bar{x}} \cap \text{dom } f}(\bar{x})$; moreover, if $\bar{x} \in \text{qri}(\text{dom } f)$, then $f'_+(\bar{x}, \cdot) = (f + \iota_{\text{qri}(\text{dom } f)})'_+(\bar{x}, \cdot)$, and so $\partial f(\bar{x}) = \partial f|_{\text{qri}(\text{dom } f)}(\bar{x})$.

(ii) The following assertions are equivalent:

(a) $\partial f(\bar{x}) \neq \emptyset$;

(b) there exists a continuous seminorm $p : X \rightarrow \mathbb{R}$ such that $f(\bar{x}) - p(x - \bar{x}) \leq f(x)$ for every $x \in \text{dom } f$;

(c) $f'_+(\bar{x}, \cdot)$ is bounded from below on some $V \in \mathcal{N}_0$;

(d) $f'_+(\bar{x}, \cdot)$ is lower semicontinuous at 0.

Moreover, if $\bar{x} \in \text{qri}(\text{dom } f)$, then $(b) \Leftrightarrow (b')$, where (b') is obtained from (b) replacing $\text{dom } f$ with $\text{qri}(\text{dom } f)$.

The equivalence (a) $\Leftrightarrow$ (c) from Proposition 11(ii) is provided in [H75, Th. C, p. 85].⁵

From the implication (b) $\Rightarrow$ (a) in Proposition 11(ii), one obtains immediately [ES15, Lem. 3]⁶ (attributed to [G67]⁷), which states “Suppose C is a convex subset of a Banach space X . If $f : C \rightarrow \mathbb{R}$ is Lipschitz continuous and convex, then $\partial f(x) \neq \emptyset$ for all $x \in C$ ”.

⁵[H75] RB Holmes: *Geometric functional analysis and its applications*. Springer-Verlag, New York-Heidelberg, 1975.

⁶[ES15] H Ergin, T Sarver: *Hidden actions and preferences for timing of resolution of uncertainty*. Theor. Econ., 10 (2015), 489–541.

⁷[G67] D Gale: *A geometric duality theorem with economic applications*. Rev. Econ. Studies, 34 (1967), 19–24.

Proposition 2

Let X be an H.l.c.s., $f \in \Lambda(X)$, $\bar{x} \in \text{dom } f$ and $p : X \rightarrow \mathbb{R}$ a continuous seminorm. Consider $\rho := \inf B \in \overline{\mathbb{R}}$, where

$$\begin{aligned} B &:= \{\beta \in \mathbb{R} \mid f(\bar{x}) \leq f(x) + \beta p(x - \bar{x}) \ \forall x \in X\} \\ &= \{\beta \in \mathbb{R} \mid 0 \in \partial(f + \beta p(\cdot - \bar{x}))(\bar{x})\} \end{aligned} \quad (1)$$

and $\inf \emptyset := \infty$. Set $V := [p \leq 1] [\Rightarrow V^\circ = \partial p(0)]$. Then the following equivalences hold:

$$\rho = \infty \iff \partial f(\bar{x}) \cap (\mathbb{R}_+ V^\circ) = \emptyset, \quad (2)$$

$$\rho \in \mathbb{P} \iff \rho = \min\{\beta \in \mathbb{P} \mid \partial f(\bar{x}) \cap (\beta V^\circ) \neq \emptyset\}, \quad (3)$$

$$\rho = 0 \iff 0 \in \partial f(\bar{x}) \wedge [\forall \beta \in \mathbb{P} : \beta V^\circ \not\subset \partial f(\bar{x})], \quad (4)$$

$$\rho \in -\mathbb{P} \iff \rho = -\max\{\beta \in \mathbb{P} \mid \beta V^\circ \subset \partial f(\bar{x})\}, \quad (5)$$

$$\rho = -\infty \iff \mathbb{R}_+ V^\circ \subset \partial f(\bar{x}). \quad (6)$$

Applying Proposition 2 for $\rho := \|\cdot\|$ one gets the following result.

Corollary 3

Let $(X, \|\cdot\|)$ be a n.v.s., $f \in \Lambda(X)$ and $\bar{x} \in \text{dom } f \neq \{\bar{x}\}$. Consider

$$\rho := \sup_{x \in X \setminus \{\bar{x}\}} \frac{f(\bar{x}) - f(x)}{\|x - \bar{x}\|} \in \mathbb{R}_\infty.$$

Then the following assertions hold:

- (i) If $\rho \in -\mathbb{P}$ then $0 \in \partial f(\bar{x})$ and $\rho = -\text{dist}(0, X^* \setminus \partial f(\bar{x}))$; in particular, $0 \in \text{int } \partial f(\bar{x})$.
- (ii) If $\rho = 0$ then $0 \in \partial f(\bar{x}) \setminus \text{int } \partial f(\bar{x})$.
- (iii) If $\rho \in \mathbb{P}$ then $0 \notin \partial f(\bar{x}) \neq \emptyset$ and $\rho = \text{dist}(0, \partial f(\bar{x}))$; moreover, there exists $\bar{x}^* \in \partial f(\bar{x})$ such that $\rho = \|\bar{x}^*\|$.
- (iv) If $\rho = \infty$ then $\partial f(\bar{x}) = \emptyset$.

From Corollary 3 (iii) $\wedge$ (iv) one deduces Gale's Duality Theorem (see [G67, p. 20]), which, in the case $0 \notin \partial f(\bar{x})$ ($\Leftrightarrow \rho > 0$), (equivalently) states that $\rho \in \mathbb{P} \Leftrightarrow \partial f(\bar{x}) \neq \emptyset$ and, moreover, $\rho = \text{dist}(0, \partial f(\bar{x}))$ if $\rho \in \mathbb{P}$.

Notice that $\partial f(\bar{x}) = X^*$ when $\text{dom } f = \{\bar{x}\}$.

Proposition 4

Let X be an H.l.c.s., $f \in \Lambda(X)$ and $\bar{x} \in \text{dom } f$.

(i) Assume that $\partial f(\bar{x})$ is nonempty and w^* -bounded; then $\bar{x} \in \text{qi}(\text{dom } f)$. In particular, $\bar{x} \in \text{qi}(\text{dom } f)$ whenever $\text{card } \partial f(\bar{x}) = 1$.

(ii) Assume that $\bar{x} \in \text{qi}(\text{dom } f)$, $V \in \mathcal{N}_{\bar{x}}$, and the continuous seminorm $p : X \rightarrow \mathbb{R}$ is such that $|f(x) - f(\bar{x})| \leq p(x - \bar{x})$ for all $x \in V \cap \text{qi}(\text{dom } f)$; then $\partial f(\bar{x})$ is nonempty and w^* -compact.

Proposition 5

Let X be an H.l.c.s. and let $f \in \Lambda(X)$ be such that $f|_{\text{dom } f}$ is Lipschitz. Then there exists $F : X \rightarrow \mathbb{R}$ a Lipschitz convex function such that $f = F + \iota_{\text{dom } f}$. Moreover:

- (a) there exists $V \in \mathcal{N}_0$ such that $\emptyset \neq \partial F(x) \subset V^\circ$ for every $x \in X$, and so $\text{cl}_{w^*} \partial F(X)$ is w^* -compact;
- (b) $\partial f(x) = \partial F(x) + N_{\text{dom } f}(x) \supset \partial F(x) (\neq \emptyset)$ for all $x \in \text{dom } f$.
- (c) Take $x \in \text{dom } f$. Then $\partial f(x)$ is w^* -bounded $\Leftrightarrow x \in \text{qi}(\text{dom } f) \Leftrightarrow \partial f(x)$ is nonempty and w^* -compact; in particular, $C_f = C_F \cap \text{qi}(\text{dom } f)$, where $C_f := \{x \in \text{dom } f \mid \text{card } \partial f(x) = 1\}$ and $C_F := \{x \in X \mid \text{card } \partial F(x) = 1\}$.

Proof (sketch). By our hypothesis, there exists $p : X \rightarrow \mathbb{R}$ a continuous seminorm such that $|f(x) - f(x')| \leq p(x - x')$ for all $x, x' \in C := \text{dom } f$, that is, $f|_C$ is p -Lipschitz on C .

Consider $\phi := f \square p : X \rightarrow \overline{\mathbb{R}}$; hence

$$\begin{aligned}\phi(x) &:= \inf\{f(y) + p(x - y) \mid y \in X\} \\ &= \inf\{f(y) + p(x - y) \mid y \in C\} \quad (< \infty),\end{aligned}$$

and so $\text{dom } \phi = X$, and ϕ is convex, f, p being so.

As in the case $p = L \|\cdot\|$, one gets easily that $\phi(x) \in \mathbb{R}$ for $x \in X$, $\phi(x) = f(x)$ for $x \in C$ and that ϕ is p -Lipschitz. The function $F : X \rightarrow \mathbb{R}$ defined by $F(x) := \phi(x)$ does the job.

Hence $f = F + \iota_{\text{dom } f}$.

When X is a normed vector space, the existence and convexity of F from Proposition 5 is obtained by in [CM78, Th. 1].⁸

⁸[CM78] S Cobzaş, C Mustăţa: *Norm-preserving extension of convex Lipschitz functions*. J. Approx. Theory, 24 (1978), 236–244.

An immediate consequence of Proposition 5 is

Corollary 6

Let X be an H.l.c.s. and let $f \in \Lambda(X)$ be such that $f|_{\text{dom } f}$ is Lipschitz. The following assertions hold:

- (a) One has $\text{dom } f = \text{dom } \partial f$. Moreover, there exists a selection $\psi : \text{dom } \partial f \rightarrow X^*$ of ∂f such that $\text{cl}_{w^*}(\text{Im } \psi)$ is w^* -compact; in particular, $\text{Im } \psi$ is w^* -bounded.
- (b) There exists $V \in \mathcal{N}_0$ such that $\partial f(x) \subset V^\circ$ for every $x \in \text{qi}(\text{dom } f)$; consequently, $\text{cl}_{w^*}[\partial f(\text{qi}(\text{dom } f))]$ is w^* -compact, and so $\partial f(\text{qi}(\text{dom } f))$ is w^* -bounded.

Corollary 7

Let X be an H.l.c.s. and let $f \in \Lambda(X)$ be such that $f|_C$ is locally Lipschitz, where $C := \text{dom } f$. The following assertions hold:

(a) For every $x \in C$, there exist an open $V_x \in \mathcal{N}_x$ and $F_x : X \rightarrow \mathbb{R}$ a convex Lipschitz function such that $f|_{V_x \cap C} = F_x|_{V_x \cap C}$; moreover, $\partial f(x') = \partial F_x(x') + N_C(x')$ for $x' \in C \cap V_x$; consequently $\text{cl}_{w^*}[\partial f(V_x \cap \text{qi } C)]$ is w^* -compact.

(b) One has $\text{dom } f = \text{dom } \partial f$; moreover, if $\text{qi}(\text{dom } f) \neq \emptyset$, then $\partial f|_{\text{qi}(\text{dom } f)}$ is locally w^* -bounded, and so any selection $\psi : \text{qi}(\text{dom } f) \rightarrow X^*$ of $\partial f|_{\text{qi}(\text{dom } f)}$ is locally w^* -bounded.

The main results

In the sequel we are interested by the structure of the set of smooth points (in the sense of Verona [V88]) of convex functions which are locally Lipschitz on their domains, or on the quasi interior of their domains; as observed above, there are not such smooth points if the quasi interior of the domain is empty. In this context one has the next result which helps to extend Mazur theorem to locally convex spaces; its proof follows the lines of [P89, Th. 1.20].

Lemma 9

Let X be an H.l.c.s. and let $f : X \rightarrow \mathbb{R}$ be a convex Lipschitz function. Consider $u \in X$, $\alpha > 0$ and the set

$$A_{u,\alpha}^f := \{x \in X \mid \exists x^*, y^* \in \partial f(x) : |\langle u, x^* - y^* \rangle| \geq \alpha\}.$$

Then $A_{u,\alpha}^f$ is closed and the set $J_x := \{t \in \mathbb{R} \mid x + tu \in A_{u,\alpha}^f\}$ is at most countable for every $x \in X$; in particular, $\text{int } A_{u,\alpha}^f = \emptyset$.

Following the lines of the proof of [ES10, Lem. 2.9] one gets the next result.

Lemma 10

Let X be an H.l.c.s. and $f_0 \in \Lambda(X)$ be such that $C := \text{dom } f_0 = \text{dom } \partial f_0$. For $u \in \text{lin}_0 C$ and $\alpha > 0$, consider the set $A_{u,\alpha}^{f_0} := \{x \in C \mid \exists x^*, y^* \in \partial f_0(x) : |\langle u, x^* - y^* \rangle| \geq \alpha\}$, where $\text{lin}_0 C$ is the linear space parallel to $\text{aff } C$. Then $C \setminus A_{u,\alpha}^{f_0}$ is dense in C ($\Leftrightarrow \text{int}_C A_{u,\alpha}^{f_0} = \emptyset$).

Corollary 11

Let X be an H.l.c.s. and $f_0 \in \Lambda(X)$ be such that $f_0|_C$ is Lipschitz, where $C := \text{qi}(\text{dom } f_0)$. Then the set $A_{u,\alpha}^{f_0}$ defined in Lemma 10 is closed in C endowed with its relative topology for all $u \in X$ and $\alpha > 0$; moreover the set $\{t \in \mathbb{R} \mid x + tu \in A_{u,\alpha}^{f_0}\}$ is at most countable for every $x \in C$; in particular, $\text{int}_C A_{u,\alpha}^{f_0} = \emptyset$ ($\Leftrightarrow C \setminus A_{u,\alpha}^{f_0}$ is dense in C).

Corollary 12

Let X be an H.l.c.s. and $f_0 \in \Lambda(X)$ be such that $f_0|_C$ is Lipschitz, where $C := \text{dom } f_0$. For $V \in \mathcal{N}_0$, $u \in X$ and $\alpha > 0$, consider $A_{V,u,\alpha}^{f_0} := \{x \in C \mid \exists x^*, y^* \in V^\circ \cap \partial f_0(x) : |\langle u, x^* - y^* \rangle| \geq \alpha\}$. Then $A_{V,u,\alpha}^{f_0}$ is closed in C endowed with its relative topology; moreover, if $u \in \text{lin}_0 C$, then $\text{int}_C A_{V,u,\alpha}^{f_0} = \emptyset$ ($\Leftrightarrow C \setminus A_{V,u,\alpha}^{f_0}$ is dense in C).

Proposition 13 (alias M)

Let X be a separable H.l.c.s. and let $f \in \Lambda(X)$ such that $C := \text{qi}(\text{dom } f) \neq \emptyset$ and $f|_C$ is locally Lipschitz, where C is endowed with its relative topology. Then

$C_f := \{x \in \text{dom } f \mid \text{card } \partial f(x) = 1\}$ is a residual subset of $\text{qi } C$, that is, there exists a family $(O_n)_{n \geq 1}$ of dense open subsets of C such that $C_f = \bigcap_{n \geq 1} O_n$. If, moreover, X is a Fréchet space and $\text{dom } f$ is either closed or open, then C_f is a dense G_δ subset of C .

Proposition M extends [V88, Th. 2] and provides an extension of Mazur's theorem to Fréchet spaces.

Proposition 14 (alias E-S1)

Let X be a separable and metrizable l.c.s., and let $f \in \Lambda(X)$ be such that $f|_C$ is locally Lipschitz, where $C := \text{dom } f$ is endowed with its relative topology. Then C_f is residual subset of C . If, moreover, C is a Baire space, then C_f is a dense G_δ subset of C .

Proposition 16 (alias E-S2)

Let X be a separable and metrizable l.c.s., and let $f \in \Lambda(X)$ be such that $f|_C$ is locally Lipschitz, where $C := \text{dom } f$. Assume that C (endowed with the relative topology) is a Baire space and $\mathcal{M} \subset X^*$ is a w^* -closed set; then $\mathcal{M}_f := \text{cl}_{w^*} \partial f(C_f) \in \mathcal{S}_f$ and $\mathcal{M} \in \mathcal{S}_f$ if and only if $\mathcal{M}_f \subset \mathcal{M}$, where $\mathcal{S}_f := \{\mathcal{M} \subset X^* \mid \mathcal{M} = \text{cl}_{w^*} \mathcal{M} \text{ and } \partial f(x) \cap \mathcal{M} \neq \emptyset \text{ for all } x \in \text{dom } \partial f\}$.

Propositions E-S1, E-S2 provide extensions of [ES10, Ths. 2.4, 2.1] to separable and metrizable l.c.s., respectively.