

On sets preserving upper porosity under union

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The main goal of my talk is to characterize families of $[r, s]$ -upper superporous subsets of $\mathbb{R}$. $[r, s]$ -upper superporosity generalize notions of superporosity and strong superporosity of subsets of $\mathbb{R}$ introduced by L. Zajíček and V. Kelar.

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The main goal of my talk is to characterize families of $[r, s]$ -upper superporous subsets of $\mathbb{R}$. $[r, s]$ -upper superporosity generalize notions of superporosity and strong superporosity of subsets of $\mathbb{R}$ introduced by L. Zajíček and V. Kellar. Next we focus on relationships between $[r, s]$ -superporous sets for different indices $[r, s]$. Furthermore we compare $[r, s]$ -superporosity to superporosity and strong superporosity.

Definition of porosity [E. P. Dolženko, L. Zajíček]

For a set $A \subset \mathbb{R}$ and an interval $I \subset \mathbb{R}$, let $\Lambda(A, I)$ denote the length of the largest open subinterval of I having an empty intersection with A . Then the right-handed (upper) porosity of the set A at $x \in \mathbb{R}$ is defined as

$$p^+(A, x) = \limsup_{h \rightarrow 0^+} \frac{\Lambda(A, (x, x + h))}{h},$$

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Definition of porosity [E. P. Dolženko, L. Zajíček]

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$$p^+(A, x) = \limsup_{h \rightarrow 0^+} \frac{\Lambda(A, (x, x + h))}{h},$$

the left-handed (upper) porosity of the set A at x is defined as

$$p^-(A, x) = \limsup_{h \rightarrow 0^+} \frac{\Lambda(A, (x - h, x))}{h},$$

Definition of porosity [E. P. Dolženko, L. Zajíček]

For a set $A \subset \mathbb{R}$ and an interval $I \subset \mathbb{R}$, let $\Lambda(A, I)$ denote the length of the largest open subinterval of I having an empty intersection with A . Then the right-handed (upper) porosity of the set A at $x \in \mathbb{R}$ is defined as

$$p^+(A, x) = \limsup_{h \rightarrow 0^+} \frac{\Lambda(A, (x, x+h))}{h},$$

the left-handed (upper) porosity of the set A at x is defined as

$$p^-(A, x) = \limsup_{h \rightarrow 0^+} \frac{\Lambda(A, (x-h, x))}{h},$$

and the (upper) porosity of A at x is defined as

$$p(A, x) = \max \{p^-(A, x), p^+(A, x)\}.$$

Definition of porosity [E. P. Dolženko, L. Zajíček]

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We say that the set A is porous at $x \in \mathbb{R}$ if $p(A, x) > 0$. The set A is called porous if A is porous at each point $x \in A$.

Definition of porosity [E. P. Dolženko, L. Zajíček]

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We say that the set A is porous at $x \in \mathbb{R}$ if $p(A, x) > 0$. The set A is called porous if A is porous at each point $x \in A$.

We say that A is strongly porous at x if $p(A, x) = 1$ and A is called strongly porous if A is strongly porous at each $x \in A$.

Definition of porosity [E. P. Dolženko, L. Zajíček]

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We say that A is strongly porous at x if $p(A, x) = 1$ and A is called strongly porous if A is strongly porous at each $x \in A$.

Obviously, every strongly porous set is porous and every porous set is nowhere dense. Moreover, none of the reverse inclusions is true.

Definition of lower porosity [E. P. Dolženko, L. Zajíček]

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In a similar way, taking lower limit instead of upper limit, we can define lower porosity $\underline{p}(A, x)$.

Definition of superporosity [L. Zajíček]

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We say that $E \subset \mathbb{R}$ is superporous at $x \in \mathbb{R}$ if porosity of $E \cup F$ is positive for every $F \subset \mathbb{R}$ with positive porosity at x .

Definition of superporosity [L. Zajíček]

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We say that $E \subset \mathbb{R}$ is superporous at $x \in \mathbb{R}$ if porosity of $E \cup F$ is positive for every $F \subset \mathbb{R}$ with positive porosity at x . The set E is superporous if it is superporous at every of its point.

Definition of strong superporosity [L. Zajíček]

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Similarly, we say that $E \subset \mathbb{R}$ is strongly superporous at $x \in \mathbb{R}$ if porosity of $E \cup F$ is equal to 1 for every $F \subset \mathbb{R}$ with porosity 1 at x .

Definition of strong superporosity [L. Zajíček]

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Similarly, we say that $E \subset \mathbb{R}$ is strongly superporous at $x \in \mathbb{R}$ if porosity of $E \cup F$ is equal to 1 for every $F \subset \mathbb{R}$ with porosity 1 at x . The set E is strongly superporous if it is strongly superporous at every of its point.

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Let $0 < r \leq s < 1$. We say that a set $E \subset \mathbb{R}$ is $[r, s]$ -superporous at $x \in E$ if for every $F \subset \mathbb{R}$ containing x if $p(F, x) \geq s$ then $p(E \cup F, x) \geq r$.

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We say that E is $[r, s]$ -superporous if it is $[r, s]$ -superporous at each of its point. The family of all $[r, s]$ -superporous subsets of $\mathbb{R}$ we denote by $S(r, s)$.

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We say that E is $[r, s]^*$ -superporous if it is $[r, s]^*$ -superporous at each of its point. The family of all $[r, s]^*$ -superporous subsets of $\mathbb{R}$ we denote by $S^*(r, s)$.

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Proposition

- ① $S(r_2, s) \subset S(r_1, s)$ and $S^*(r_2, s) \subset S^*(r_1, s)$ for every $0 < r_1 < r_2 < 1$ and $s \geq r_2$.

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Proposition

- ❶ $S(r_2, s) \subset S(r_1, s)$ and $S^*(r_2, s) \subset S^*(r_1, s)$ for every $0 < r_1 < r_2 < 1$ and $s \geq r_2$.
- ❷ $S(r, s_1) \subset S(r, s_2)$ and $S^*(r, s_1) \subset S^*(r, s_2)$ for every $0 < s_1 < s_2 < 1$ and $r \leq s_1$.

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Proposition

- 1 $S(r_2, s) \subset S(r_1, s)$ and $S^*(r_2, s) \subset S^*(r_1, s)$ for every $0 < r_1 < r_2 < 1$ and $s \geq r_2$.
- 2 $S(r, s_1) \subset S(r, s_2)$ and $S^*(r, s_1) \subset S^*(r, s_2)$ for every $0 < s_1 < s_2 < 1$ and $r \leq s_1$.

Theorem

If $E \subset \mathbb{R}$ and $x \in E$ is an accumulation point of $E \cap (x, \infty)$ then for every $s \in (0, 1)$ there exists $F \subset \mathbb{R}$ such that $p(F, x) = s$ and $1 - s = (1 - p(E \cup F, x))^2$, i.e. $p(E \cup F, x) = 1 - \sqrt{1 - s}$.

Set $\mathbf{A}$

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Theorem

Let $0 < r \leq s < 1$. For every $x \in \mathbb{R}$ there exists $\mathbf{A} \subset \mathbb{R}$ such that

- $x \in \mathbf{A}$ is an accumulation point of $\mathbf{A} \cap (x, \infty)$,
- $\mathbf{A} \in S(r, s)$ iff $s \geq 1 - (1 - r)^2$,
- $\mathbf{A} \in S^*(r, s)$ iff $s \geq 1 - (1 - r)^2$.

Characterization of $S(r, s)$ and $S^*(r, s)$

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Theorem

Let $0 < r \leq s < 1$.

- 1 If $s < 1 - (1 - r)^2$, i.e. $r > 1 - \sqrt{1 - s}$ then $S(r, s)$ and $S^*(r, s)$ are trivial, i.e. they consist of sets with empty derivative.

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Theorem

Let $0 < r \leq s < 1$.

- 1 If $s < 1 - (1 - r)^2$, i.e. $r > 1 - \sqrt{1 - s}$ then $S(r, s)$ and $S^*(r, s)$ are trivial, i.e. they consist of sets with empty derivative.
- 2 If $s \geq 1 - (1 - r)^2$, i.e. $r \leq 1 - \sqrt{1 - s}$ then $S(r, s)$ and $S^*(r, s)$ are nontrivial, i.e. they contain sets with nonempty derivative.

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Theorem

Let $0 < r \leq s < 1$. For every $\alpha \in (0, 1)$ there exists $B_\alpha \subset \mathbb{R}$ such that

- $0 \in B_\alpha$ is an accumulation point of $B_\alpha \cap (0, \infty)$,
- $B_\alpha \in S(r, s)$ iff $r \leq \alpha$ and $s \geq 1 - (1 - r)^2$,
- $B_\alpha \in S^*(r, s)$ iff $r < \alpha$ and $s \geq 1 - (1 - r)^2$.

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Theorem

$S^*(r, s) \subsetneq S(r, s)$ for every $0 < r < 1$ and $s \geq 1 - (1 - r)^2$.

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Theorem

$S^*(r, s) \subsetneq S(r, s)$ for every $0 < r < 1$ and $s \geq 1 - (1 - r)^2$.

Theorem

- ① If $r_1 < r_2$ and $s_1 \in (1 - (1 - r_1)^2, 1)$ then $S(r_1, s_1) \not\subset S(r_2, s_2)$ and $S^*(r_1, s_1) \not\subset S^*(r_2, s_2)$ for every $s_2 \in [r_2, 1)$.

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$S^*(r, s) \subsetneq S(r, s)$ for every $0 < r < 1$ and $s \geq 1 - (1 - r)^2$.

Theorem

- 1 If $r_1 < r_2$ and $s_1 \in (1 - (1 - r_1)^2, 1)$ then $S(r_1, s_1) \not\subset S(r_2, s_2)$ and $S^*(r_1, s_1) \not\subset S^*(r_2, s_2)$ for every $s_2 \in [r_2, 1)$.
- 2 If $r_1 < r_2$ and $s \in (1 - (1 - r_1)^2, 1)$ then $S(r_1, s) \not\subset S(r_2, s)$ and $S^*(r_1, s) \not\subset S^*(r_2, s)$.

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Theorem

Let $0 < r \leq s < 1$. For every $\beta \in (0, 1)$ there exists $C_\beta \subset \mathbb{R}$ such that

- $0 \in C_\beta$ is an accumulation point of $C_\beta \cap (0, \infty)$,
- $C_\beta \in S(r, s)$ iff $s \geq 1 - (1 - \beta)(1 - r)^2$,
- $C_\beta \in S^*(r, s)$ iff $s \geq 1 - (1 - \beta)(1 - r)^2$.

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Theorem

Let $0 < r \leq s < 1$. For every $\beta \in (0, 1)$ there exists $C_\beta \subset \mathbb{R}$ such that

- $0 \in C_\beta$ is an accumulation point of $C_\beta \cap (0, \infty)$,
- $C_\beta \in S(r, s)$ iff $s \geq 1 - (1 - \beta)(1 - r)^2$,
- $C_\beta \in S^*(r, s)$ iff $s \geq 1 - (1 - \beta)(1 - r)^2$.

Theorem

If $r \in (0, 1)$ and $s_2 > s_1 \geq 1 - (1 - r)^2$ then $S(r, s_1) \subsetneq S(r, s_2)$ and $S^(r, s_1) \subsetneq S^*(r, s_2)$.*

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Theorem

Let $0 < r \leq s < 1$, $E \subset \mathbb{R}$ and $x \in E$. If $\underline{p}(E, x) \geq 1 - s + r$ then E is $[r, s]^$ -superporous at x .*

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Theorem

Let $0 < r \leq s < 1$, $E \subset \mathbb{R}$ and $x \in E$. If $\underline{p}(E, x) \geq 1 - s + r$ then E is $[r, s]^$ -superporous at x .*

Theorem

Let $0 < r \leq s < 1$, $E \subset \mathbb{R}$ and $x \in E$. If E is $[r, s]$ -superporous at x then $\underline{p}(E, x) > 0$.

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There is no any relationship between $S(r, s)$, $S^*(r, s)$, superporous sets and strongly superporous sets.

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Fix any $0 < r \leq s < 1$, where $s \geq 1 - (1 - r)^2$.

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Fix any $0 < r \leq s < 1$, where $s \geq 1 - (1 - r)^2$.

For any $\alpha \in (r, 1)$ the set $\mathbf{B}_\alpha$ belongs to $S(r, s) \cap S^*(r, s)$ and is not strongly superporous.

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For any $\alpha \in (r, 1)$ the set $\mathbf{B}_\alpha$ belongs to $S(r, s) \cap S^*(r, s)$ and is not strongly superporous.

For any $\alpha \in (0, r)$ the set $\mathbf{B}_\alpha$ is superporous and belongs neither to $S(r, s)$ nor $S^*(r, s)$.

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For any $\alpha \in (0, r)$ the set $\mathbf{B}_\alpha$ is superporous and belongs neither to $S(r, s)$ nor $S^*(r, s)$.

For any $\beta \in (s, 1)$ the set $\mathbf{C}_\beta$ is strongly superporous and does not belong to $S(r, s) \cup S^*(r, s)$.

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For any $\alpha \in (0, r)$ the set $\mathbf{B}_\alpha$ is superporous and belongs neither to $S(r, s)$ nor $S^*(r, s)$.

For any $\beta \in (s, 1)$ the set $\mathbf{C}_\beta$ is strongly superporous and does not belong to $S(r, s) \cup S^*(r, s)$.

For any $\beta > 0$ the set $\mathbf{C}_\beta$ belongs to $S(r, s) \cap S^*(r, s)$ (for sufficiently large s) and is not superporous.

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The sets $S(r, s)$ and $S^*(r, s)$ can be used to describe the adders and multipliers of the families of porouscontinuous functions defined by J. Borsík, J. Holos.

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Thank you for your attention!