

The invariance problem of Matkowski means under reduced regularity

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Motivation

Throughout the whole talk let I, J be nonempty open intervals of the real line.

Matkowski means

Let $\varphi_1, \varphi_2 : J \longrightarrow \mathbb{R}$ be continuous functions which are strictly monotone in the same sense. Then the mean

$$A_{\varphi_1, \varphi_2}(u, v) := (\varphi_1 + \varphi_2)^{-1}(\varphi_1(u) + \varphi_2(v)) \quad (x, y \in J)$$

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This concept was introduced in 2010 by J. Matkowski [3].

The invariance problem and its reformulation

If $\emptyset \neq J \subseteq \mathbb{R}$ is an open interval, then we are looking for those Matkowski means $M, N, K : J \times J \longrightarrow J$ which fulfill the invariance equation

$$M(N(u, v), K(u, v)) = M(u, v) \quad (u, v \in J).$$

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Reformulation – T. Kiss [1]

The equation above is equivalent to

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Here the six unknown functions F, f_1, f_2, g_1, g_2, G are constructed from the original generators of the three unknown Matkowski means K, M, N .

We will turn our attention towards the equivalent invariance equation

$$F\left(\frac{x+y}{2}\right) + f_1(x) + f_2(y) = G(g_1(x) + g_2(y)) \quad (x, y \in I)$$

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From the original invariance equation it follows that WLOG we may assume that F is strictly decreasing and the other five functions are strictly increasing.

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Of course, the most natural and elegant assumption would be the continuity and strict monotonicity for F, f_1, f_2, g_1, g_2 and G . However, in [1] the general approach to the solution was differentiation.

Reducing regularity

If F, f_1, f_2, G, g_1, g_2 are $\mathcal{C}^1$ then every unknown function in (*) is continuous, and it is known from the literature that the "interesting" solutions are C^∞ , provided that F is **nowhere affine**.

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In the recent paper [2] by T. Kiss and PT it has been shown that the system $(*)$ has the same family of solutions even if we suppose the mere differentiability of the unknown functions instead of continuous differentiability.

In [2] the complete characterization of the solutions of the invariance equation is supplied assuming only differentiability (and no additional technical conditions).

The main objective

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Main objective

The structure of the equation suggests that occasionally the regularity of one particular unknown function is *inherited* by the others.

Our goal is to determine the *smooth solutions* of the invariance equation assuming the differentiability of

- F and f_1, f_2 ;
- only F .

Smooth solutions in our context

Intervals of affinity of F

The intervals where F is affine play a central role during the discussion of the solution process.

One needs fundamentally different tools in the following 3 cases:

- 1 F is affine on the whole domain I .
- 2 F is affine on a proper open subinterval $J \subsetneq I$;
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In the first case the solutions can be determined without any differentiability conditions, relying on the Radó–Baker Theorem.

The investigation of the second case is based on the first and third one. Thus we consider the third case.

Regularity preservation for quasiums

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Theorem. (PT 2025. [4])

Let $n \in \mathbb{N}$, $n \geq 2$ and suppose that $\emptyset \neq I_k \subseteq \mathbb{R}$ is an open interval for $k = 1, \dots, n$. Moreover, let $\varphi_k : I_k \rightarrow \mathbb{R}$ be continuous, strictly monotone for $k = 1, \dots, n$ and let

$\psi : \varphi_1(I_1) + \dots + \varphi_n(I_n) \rightarrow \mathbb{R}$ be also continuous, strictly monotone. Consider the quasium $Q : I_1 \times \dots \times I_n \rightarrow \mathbb{R}$ defined by

$$Q(x_1, \dots, x_n) = \psi(\varphi_1(x_1) + \dots + \varphi_n(x_n)) \quad (x_1 \in I_1, \dots, x_n \in I_n).$$

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If Q is partially differentiable, then $\varphi_1, \dots, \varphi_n, \psi$ are differentiable.

Let us apply the previous Theorem for the RHS expression

$$G(g_1(x) + g_2(y)) \quad (x, y \in I).$$

Then the functions G, g_1, g_2 are differentiable, so the solutions are known from [2].

Connection to the original invariance problem

Considering the original invariance equation

$$M(N(u, v), K(u, v)) = M(u, v) \quad (u, v \in J),$$

the functions F, f_1, f_2 are derived from the generators of the means M and K via

$$F := -m_2 \circ \left(\frac{k_1 + k_2}{2} \right)^{-1} \quad f_i = m_i \circ k_i^{-1} \quad (i = 1, 2)$$

Thus no additional regularity is required for N .

F is differentiable

The situation is more difficult if only F is supposed to be differentiable. In general, this does not imply any regularity for other functions.

Theorem. (Kiss–PT 2026.)

Let $F : I \rightarrow \mathbb{R}$ be affine and $g_1, g_2 : I \rightarrow \mathbb{R}$ be continuous and strictly monotone. Then (F, f_k, G, g_k) is a solution if and only if there exist an additive function $B : \mathbb{R} \rightarrow \mathbb{R}$ and constants $\beta_1, \beta_2 \in \mathbb{R}$ such that

$$f_k = -\frac{1}{2}F + B \circ g_k + \beta_k \quad \text{and} \quad G = B + \beta_1 + \beta_2$$

hold on I and on $g_1(I) + g_2(I)$, respectively.

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Let us apply the difference operator Δ_h in the variable x to both sides, then divide by h :

$$\begin{aligned} & \frac{F\left(\frac{x+y+h}{2}\right) - F\left(\frac{x+y}{2}\right)}{h} + \frac{f_1(x+h) - f_1(x)}{h} \\ &= \frac{G(g_1(x+h) + g_2(y)) - G(g_1(x) + g_2(y))}{(g_1(x+h) + g_2(y)) - (g_1(x) + g_2(y))} \cdot \frac{g_1(x+h) - g_1(x)}{h} \end{aligned}$$

for all $x, y, h \in \mathbb{R}$ such that $x, y, x + h, y + h \in I$ and $h \neq 0$.

$\implies G$ is differentiable

Applying Lebesgue's Differentiation Theorem for g_1^{-1} , we get that it is differentiable almost everywhere. In particular, $g_1'(x) \in (0, +\infty]$ on a dense subset of I .

if $g_1' > 0$

On the other hand, if $g_1'(x_0) \in (0, +\infty)$ at some point, then we may choose $y \in I$ such that

G is differentiable at $g_1(x_0) + g_2(y)$,

as g_2 is strictly monotone, continuous. Taking the limit $h \rightarrow 0$ in the equation for the differences, we obtain that $f_1'(x_0) \in \mathbb{R}$ exists. But then G' exists on the interval $g_1(x_0) + g_2(I)$.

if $g'_1 = +\infty$

If $g'_1(x_0) = +\infty$ at some point, then again choose $y \in I$ such that G is differentiable at $g_1(x_0) + g_2(y)$. Then we divide by the ratio $\frac{g_1(x_0+h) - g_1(x_0)}{h}$, take the limit $h \rightarrow 0$ and obtain

$$\lim_{h \rightarrow 0} \frac{f_1(x_0 + h) - f_1(x_0)}{g_1(x_0 + h) - g_1(x_0)} = G'(g_1(x_0) + g_2(y)).$$

Since the limit of the LHS is finite and does not depend on y , we got that G' exists on the interval $g_1(x_0) + g_2(I)$.

Either way, G' exists on $g_1(x_0) + g_2(I)$ whenever $g'_1(x_0) \in (0, +\infty]$, thus G is differentiable on its whole domain.

$\implies g_1$ is differentiable

Let $x \in I$ be arbitrary, and let us assume that g is not differentiable at x . Then

$$\begin{aligned} 0 \leq \alpha &:= \liminf_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &< \limsup_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} =: \beta \leq +\infty. \end{aligned}$$

Clearly, $\beta \neq +\infty$. Now take the $\limsup_{h \rightarrow 0}$ and $\liminf_{h \rightarrow 0}$ in

$$\begin{aligned} &\frac{F\left(\frac{x+y+h}{2}\right) - F\left(\frac{x+y}{2}\right)}{h} + \frac{f_1(x+h) - f_1(x)}{h} \\ &= \frac{G(g_1(x+h) + g_2(y)) - G(g_1(x) + g_2(y))}{(g_1(x+h) + g_2(y)) - (g_1(x) + g_2(y))} \cdot \frac{g_1(x+h) - g_1(x)}{h} \end{aligned}$$

respectively, for any $y \in I$.

Then

$$\frac{1}{2}F'\left(\frac{x+y}{2}\right) + \liminf_{h \rightarrow 0} \frac{f_1(x+h) - f_1(x)}{h} = G'(g_1(x) + g_2(y)) \alpha$$
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The two limits must be finite, thus the system of two linear equations has a unique solution, independent from y . This would mean that F' is constant on a subinterval, contradicting that F is nowhere affine.

The same argument shows that g_2 is differentiable. The differentiability of f_1, f_2 is then straightforward.

Thus we have sketched the proof of the following statement.

Theorem.

Suppose that the functions $F, f_1, f_2, g_1, g_2 : I \rightarrow \mathbb{R}$ and $G : g_1(I) + g_2(I) \rightarrow \mathbb{R}$ solve the functional equation

$$F\left(\frac{x+y}{2}\right) + f_1(x) + f_2(y) = G(g_1(x) + g_2(y)) \quad (x, y \in I).$$

Assume that F is differentiable and nowhere affine. Then f_1, f_2, g_1, g_2, G are also differentiable.

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