

Constructing twisted sums of $C(K)$ -spaces with $c_0(X)$ -spaces using trees

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Theory

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Question 1

Let (X, Y) be a pair of Banach spaces.

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It is enough to take a space $X \oplus Y$.

Consider the following question: **Does there exist an extension of X such that Z/X is isomorphic to Y and there is no continuous projection $P : Z \rightarrow X$?**

Theorem

Let Y be a separable Banach space. The short exact sequence

$$0 \longrightarrow c_0 \longrightarrow Z \longrightarrow Y \longrightarrow 0. \quad (1)$$

splits, i.e., the answer to Question II is negative for pairs (c_0, Y) where Y is separable.

Definition

We say that a tree $\kappa^{\leq m}$ is realized in $(C(K), P(K))$ below a continuous function $g : K \rightarrow [0, 1]$ if there is a mapping

$$T \ni t \mapsto (g_t, \mu_t) \in (C(K), P(K))$$

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- ① $g_\emptyset = g$;
- ② every g_t has norm 1 and is valued in $[0, 1]$;
- ③ $g_t \cdot g_s = g_t$ whenever $s \leq t$;
- ④ $g_t \cdot g_s = 0$ for incomparable t, s ;

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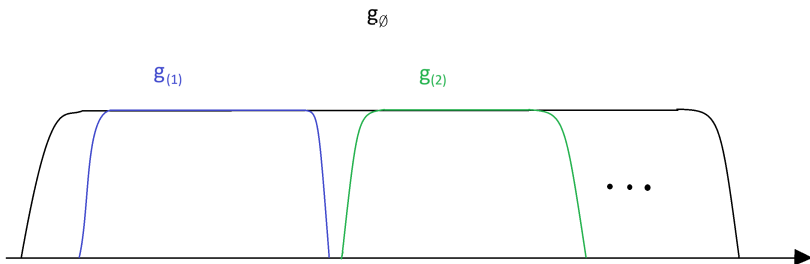
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- ⑤ $\mu_t(g_t) = 1$ for every $t \in \kappa^{\leq m}$;
- ⑥ $\lim_n \mu_{t \frown n} = \mu_t$ whenever $t \in T$ is a non-terminal element.



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Fix any partition of $\kappa = \bigcup_{k \in \omega} I_k$ into cofinal sets and write

$$\varphi_{t,k} = \sum_{i \in I_k} g_{t \smallfrown i}$$

for every $k \in \omega$ and $t \in \kappa^{\leq m}$ which is not terminal. Finally, let Φ be the family of all such functions $\varphi_{t,k}$.

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By $C(K)[\Phi]$ we mean a closed subspace of $l_\infty(K)$ generated by $C(K)$ and Φ .

Theorem

If the tree $\kappa^{\leq m}$ is realized in $(C(K), P(K))$ and Φ is obtained by the construction above then every bounded projection

$P : C(K)[\Phi] \rightarrow C(K)$ satisfies $\|P\| \geq 2m + 1$.

Consequently, for $\kappa^{< \omega}$ we get a family Φ such that $C(K)$ is not complemented in $C(K)[\Phi]$.

Corollary

Suppose that there is a sequence of norm-one functions $f_m \in C(K)$ with pairwise disjoint supports such that the tree $\kappa^{\leq m}$ can be realized in $(C(K), P(K))$ below f_m for every m .

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In particular the Question II has a positive answer for $(C(K), c_0(\kappa))$, if the tree $\kappa^{<\omega}$ can be realized in $(C(K), P(K))$.

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Suppose that $\mathfrak{A} \subset \mathfrak{B}$. We say that $\mathfrak{B}$ is a minimal extension of $\mathfrak{A}$ if $\mathfrak{C} = \mathfrak{B}$ for every subalgebra $\mathfrak{C} = \mathfrak{B}$ containing properly $\mathfrak{A}$.

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We say that $\mathfrak{B}$ is minimally generated over $\mathfrak{A}$ if there are an ordinal α and a chain of Boolean algebras $(\mathfrak{A}_\beta)_{\beta < \alpha}$ such that $\mathfrak{B} = \bigcup_{\beta < \alpha} \mathfrak{A}_\beta$, where

- i) $\mathfrak{A}_0 = \mathfrak{A}$;
- ii) $\mathfrak{A}_{\beta+1}$ is a minimal extension of $\mathfrak{A}_\beta$ for every $\beta < \alpha$
- iii) $\mathfrak{A}_\beta = \bigcup_{\gamma < \beta} \mathfrak{A}_\gamma$ for every limit $\beta < \alpha$.

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Theorem (CH)

There is an Efimov space K such that $\text{Clop}(K)$ is minimally generated over $\text{Clop}(2^\omega)$.

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Let $K = (\mathfrak{B})$, where $\mathfrak{B}$ is minimally generated over the algebra $\text{Clop}(2^\omega)$. Then the answer to Question II for $(C(K), c_0)$.

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Corollary

For such K the answer to Question II for $(C(K), c_0)$ is positive.

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